PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 27
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1992

HARMONIC MORPHISMS AND NON-LINEAR POTENTIAL THEORY

ILPO LAINE

Department of Mathematics, University of Joensuu
P.O. Box 111, SF-80101 Joensuu, Finland

Originally, harmonic morphisms were defined as continuous mappings $\varphi : X \to X'$ between harmonic spaces such that $h' \circ \varphi$ remains harmonic whenever h' is harmonic, see [1], p. 20. In general linear axiomatic potential theory, one has to replace harmonic functions h' by hyperharmonic functions u' in this definition, in order to obtain an interesting class of mappings, see [3], Remark 2.3. The modified definition appears to be equivalent with the original one, provided X' is a Bauer space, i.e. a harmonic space with a base consisting of regular sets, see [3], Theorem 2.4. To extend the linear proof of this result directly into the recent non-linear theories fails, even in the case of semi-classical non-linear considerations [6]. The aim of this note is to give a modified proof which settles such difficulties in the quasi-linear theories [4], [5].

1. Preliminaries. We assume that X, X' are quasi-linear harmonic spaces in the sense of [4]. Therefore, the axioms of quasi-linearity, resolutivity, quasi-linear positivity, completeness and Bauer convergence hold, see [4], pp. 340–342. Moreover, we assume that the axiom of MP-sets holds; see [5], p. 123. Notations and results from [4] and [5] will be applied, as well as standard notations from [2]. In particular, recall that an open set $U \subset X$ is *sufficiently small* (see [4], p. 344) if $\operatorname{cl} U$ is contained in an open set V such that there exists a strictly positive harmonic function h on V which belongs to the linear subsheaf $\mathcal{V}(V)$, see [4], p. 340. Finally, unless otherwise specified, we assume that X' is a Bauer space, i.e. a quasi-linear harmonic space with a base consisting of regular sets. Therefore, the Poisson modification $P(u', U')$ of a hyperharmonic function u' on a regular, relatively compact and sufficiently small set U' takes the form

$$P(u',U') = \begin{cases} u' & \text{on } X' \setminus U', \\ \underline{H}_{u'}^{U'} & \text{on } U'. \end{cases} \tag{1.1}$$

In fact, let $\{f'_\alpha\}_{\alpha\in I}$ denote the upper directed family of continuous minorants of u' on $\partial U'$. By regularity of U',

$$\liminf_{U'\ni x'\to y'} \underline{H}_{u'}^{U'}(x') \geq \liminf_{U'\ni x'\to y'} H_{f'_\alpha}^{U'}(x') = f'_\alpha(y')$$

for all $y' \in \partial U'$, hence

$$\liminf_{U'\ni x'\to y'} \underline{H}_{u'}^{U'}(x') \geq \sup_{\alpha\in I} f'_\alpha(y') = u'(y')\,.$$

By [5], Lemma 4.2, $P(u', U')$ is hyperharmonic.

Next, we give non-linear versions of two well-known lemmas from the standard linear theory.

LEMMA 1.1. *Let $\mathcal{W}'$ be a neighbourhood base of $x' \in X'$, consisting of sufficiently small, relatively compact, regular neighbourhoods of x', and let s' be hyperharmonic on a neighbourhood V' of x'. Then*

$$s'(x') = \sup_{W'\in\mathcal{W}'} \underline{H}_{s'}^{W'}(x')\,.$$

Proof. See the proof of [3], Lemma 2.1. We only have to take the strictly positive harmonic function h' used in that proof from the corresponding linear subsheaf. ■

LEMMA 1.2. *Let u' be superharmonic on a sufficiently small open set in a Bauer space X'. Then u' is the supremum of its finitely continuous superharmonic minorants.*

Proof. Clearly, u' is the supremum of its finitely continuous minorants, say f'_α. By [5], Lemma 4.2, and the reasoning used in the proof of [5], Proposition 6.2, $Rf'_\alpha \leq u'$ is superharmonic and finitely continuous. Obviously, $u' = \sup_\alpha Rf'_\alpha$. ■

2. Harmonic morphisms

DEFINITION 2.1. A continuous mapping $\varphi : X \to X'$ is called a *harmonic morphism* provided $u' \circ \varphi$ is hyperharmonic on $\varphi^{-1}(U') \neq \emptyset$ whenever $U' \subset X'$ is open and u' is hyperharmonic on U'.

THEOREM 2.2. *Let X' be a Bauer space. Then a continuous mapping $\varphi : X \to X'$ is a harmonic morphism if and only if $h' \circ \varphi$ is harmonic on $\varphi^{-1}(U') \neq \emptyset$ whenever $U' \subset X'$ is open and h' is harmonic on U'.*

Proof. By the sheaf property of hyperharmonic functions, we may assume that U' in Definition 2.1 is sufficiently small. Let u' be hyperharmonic on U'. Then

$$u' = \sup_{n\in\mathbb{N}}(\inf(u', nh'_0))\,,$$

where $h'_0 \in \mathcal{V}(V')$ for a neighbourhood V' of U'. By this and Lemma 1.2, we may assume that u' is superharmonic and finitely continuous.

Let $U \subset \varphi^{-1}(U') \neq \emptyset$ be a non-empty set relatively compact in $\varphi^{-1}(U')$; hence $\varphi(\operatorname{cl} U) \subset U'$ is compact and non-empty. Let $\mathfrak{V}'$ be the collection of all finite open covers $\mathcal{V}'$ of $\varphi(\operatorname{cl} U)$ by regular sets which are sufficiently small and relatively compact in U'. Let us fix such an open cover $\mathcal{V}'$. Given $V' \in \mathcal{V}'$, consider the Poisson modification

$$P(u', V') = \begin{cases} u' & \text{on } U' \setminus V', \\ H_{u'}^{V'} & \text{on } V', \end{cases} \tag{2.1}$$

defined on U' (see (1.1)). As noted above, $P(u', V')$ is hyperharmonic on U'. We now define

$$P(u', \mathcal{V}') := \inf_{V' \in \mathcal{V}'} P(u', V').$$

Since $\mathcal{V}'$ is a finite collection of sets, $P(u', \mathcal{V}')$ is hyperharmonic. By (2.1), we have

$$P(u', \mathcal{V}') = \begin{cases} u' & \text{on } U' \setminus \bigcup \mathcal{V}', \\ \inf_{V' \in \mathcal{V}'} H_{u'}^{V'} & \text{on } \bigcup \mathcal{V}'. \end{cases}$$

Clearly, $(P(u', \mathcal{V}')) \circ \varphi$ is lower semicontinuous on U and

$$(P(u', \mathcal{V}')) \circ \varphi = (\inf_{V' \in \mathcal{V}'} H_{u'}^{V'}) \circ \varphi = \inf_{V' \in \mathcal{V}'} (H_{u'}^{V'} \circ \varphi).$$

Given $x \in U$, there are finitely many $V' \in \mathcal{V}'$ such that $\varphi(x) \in V'$. Since u' is superharmonic, $H_{u'}^{V'} \circ \varphi$ is harmonic on $\varphi^{-1}(V')$, hence $(P(u', \mathcal{V}')) \circ \varphi$ is hyperharmonic on a neighbourhood $\bigcap \{\varphi^{-1}(V') \mid \varphi(x) \in V',\ V' \in \mathcal{V}'\}$ of x. By the sheaf property, $(P(u', \mathcal{V}')) \circ \varphi$ is hyperharmonic on U.

Next, we have to prove that $\{(P(u', \mathcal{V}')) \circ \varphi \mid \mathcal{V}' \in \mathfrak{V}'\}$ is an upper directed family. Let $P(u', \mathcal{V}_1')$ and $P(u', \mathcal{V}_2')$ be given, and construct a new cover $\mathcal{W}' \in \mathfrak{V}'$ of $\varphi(\operatorname{cl} U)$ as follows: Given $x' \in \varphi(\operatorname{cl} U)$, there are finitely many sets $V' \in \mathcal{V}_1' \cup \mathcal{V}_2'$ such that $x' \in V'$. Let now $W' := W_{x'}'$ be a regular set such that $x' \in W'$ and that $\operatorname{cl} W' \subset \bigcap \{V' \mid x' \in V',\ V' \in \mathcal{V}_1' \cup \mathcal{V}_2'\}$. For every such $V' \in \mathcal{V}_1' \cup \mathcal{V}_2'$, we have

$$H_{u'}^{V'} \le u'$$

on $\partial W'$. By [4], Proposition 3.3,

$$H_{u'}^{V'} = H_{H_{u'}^{V'}}^{W'} \le H_{u'}^{W'}$$

holds on W', hence

$$\sup(P(u', \mathcal{V}_1'), P(u', \mathcal{V}_2')) = \sup(\inf_{V' \in \mathcal{V}_1'} H_{u'}^{V'}, \inf_{V' \in \mathcal{V}_2'} H_{u'}^{V'}) \le H_{u'}^{W'}$$

on W'. Now, we may choose a finite cover $\mathcal{W}' \in \mathfrak{V}'$ of $\varphi(\operatorname{cl} U)$, using finitely many of the above sets $W_{x'}'$. Then obviously

$$\sup((P(u', \mathcal{V}_1')) \circ \varphi, (P(u', \mathcal{V}_2')) \circ \varphi) \le (P(u', \mathcal{W}')) \circ \varphi.$$

We still have to observe that

$$u' = \sup_{\mathcal{V}' \in \mathfrak{V}'} P(u', \mathcal{V}') \tag{2.2}$$

holds on $\varphi(\operatorname{cl} U)$. In fact, if $x' \in \varphi(\operatorname{cl} U)$ and $\alpha < u'(x')$, we may apply Lemma 1.1 to construct a neighbourhood W' of x' such that

$$H_{u'}^{W'}(x') > \alpha\,,$$

W' being regular, sufficiently small and relatively compact in U'. Construct now a finite open cover $\mathcal{V}' \in \mathfrak{V}'$ of $\varphi(\operatorname{cl} U)$ such that $W' \in \mathcal{V}'$ and that $x' \notin \operatorname{cl} V'$ for all other sets $V' \in \mathcal{V}'$. Then

$$H_{u'}^{W'}(x') = P(u', \mathcal{V}')(x')\,,$$

and (2.2) follows.

By (2.2), we now see that

$$u' \circ \varphi = (\sup_{\mathcal{V}' \in \mathfrak{V}'} P(u', \mathcal{V}')) \circ \varphi = \sup_{\mathcal{V}' \in \mathfrak{V}'} ((P(u', \mathcal{V}')) \circ \varphi)$$

is hyperharmonic on U, hence on $\varphi^{-1}(U')$ by the sheaf property of hyperharmonic functions. ■

The following theorem may be considered as a slight non-linear improvement of [2], Theorem 2.5.

THEOREM 2.3. *If $\varphi : X \to X'$ is a homeomorphic harmonic morphism, then $\varphi^{-1} : X' \to X$ is a harmonic morphism. If X' is a Bauer space, then so is X.*

Proof. To prove the first assertion, where it is not necessary to assume that X' is a Bauer space, let h be a hyperharmonic function on an open set $U \subset X$. By the sheaf property of hyperharmonic functions, it is no restriction to assume that U is an MP-set. To prove that $h \circ \varphi^{-1}$ is hyperharmonic on $\varphi(U)$, let $V' \subset \varphi(U)$ be a resolutive set relatively compact in $\varphi(U)$ and take $v' \in \underline{\mathcal{U}}_{h\circ\varphi^{-1}}^{V'}$ arbitrarily. Since $h \circ \varphi^{-1}$ is lower semicontinuous, we see that

$$\begin{aligned}\limsup_{\varphi^{-1}(V') \ni x \to y} v' \circ \varphi(x) &= \limsup_{V' \ni x' \to \varphi(y)} v'(x') \le h \circ \varphi^{-1}(\varphi(y)) \\ &\le \liminf_{V' \ni x' \to \varphi(y)} h \circ \varphi^{-1}(x') = \liminf_{\varphi^{-1}(V') \ni x \to y} h(x)\end{aligned}$$

holds for all $y \in \partial\varphi^{-1}(V')$. The comparison principle now results in $v' \circ \varphi \le h$ and therefore $v' \le h \circ \varphi^{-1}$. Since $v' \in \underline{\mathcal{U}}_{h\circ\varphi^{-1}}^{V'}$ was arbitrary, we obtain

$$\underline{H}_{h\circ\varphi^{-1}}^{V'} \le h \circ \varphi^{-1}\,,$$

hence the assertion follows by the axiom of completeness.

Let now X' be a Bauer space, and let $U' \subset X'$ be a regular set such that $\varphi^{-1}(U')$ is a relatively compact MP-set. This may be assumed by the axioms of resolutivity and MP-sets. It now suffices to prove that $\varphi^{-1}(U')$ is regular. To this end, take $f \in \mathcal{C}(\partial\varphi^{-1}(U'))$. Then $f \circ \varphi^{-1} \in \mathcal{C}(\partial U')$; hence it has a unique continuous extension h' into $\operatorname{cl} U'$, harmonic in U'. Therefore $h := h' \circ \varphi$ is continuous on $\operatorname{cl} \varphi^{-1}(U')$, equal to f on $\partial\varphi^{-1}(U')$ and harmonic on $\varphi^{-1}(U')$. The extension h of f into $\operatorname{cl} \varphi^{-1}(U')$ is unique, since $\varphi^{-1}(U')$ is an MP-set. ■

References

[1] C. Constantinescu and A. Cornea, *Compactifications of harmonic spaces*, Nagoya Math. J. 25 (1965), 1–57.
[2] —, —, *Potential Theory on Harmonic Spaces*, Springer, 1972.
[3] I. Laine, *Covering properties of harmonic Bl-mappings III*, Ann. Acad. Sci. Fenn. Ser. AI Math. 1 (1975), 309–325.
[4] —, *Introduction to a quasi-linear potential theory*, ibid. 10 (1985), 339–348.
[5] —, *Axiomatic non-linear potential theories*, in: Lecture Notes in Math. 1344, Springer, 1988, 118–132.
[6] O. Martio, *Potential theoretic aspects of non-linear elliptic partial differential equations*, Univ. of Jyväskylä, Dept. of Math., Report 44, 1989.