

**THE FIRST BOUNDARY VALUE PROBLEM  
FOR THE VIBRATING STRING EQUATION  
IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARY**

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**0.** We investigate the solvability in  $\mathbf{W}_2^1(\Omega)$  of the boundary value problem

$$(1) \quad u_{xy} = f, \quad f \in \mathbf{L}_2(\Omega), \quad (x, y) \in \Omega,$$

$$(2) \quad u|_{\partial\Omega} = \varphi(x, y), \quad (x, y) \in \partial\Omega,$$

where  $\Omega \subset \mathbb{R}^2$  is a convex bounded domain with piecewise smooth boundary  $\Gamma = \bigcup_{j=1}^n \Gamma_j$ ,  $\Gamma_j \subset \mathbf{C}^2$ . Assume that each  $\Gamma_j$  is either part of a straight line, or has strictly positive curvature at each point of  $\Gamma_j$ . Also,  $\Gamma$  is assumed to have at most two common points with every line parallel to a coordinate axis.

This problem was first studied by J. Hadamard [2]. He noted that it is not well-posed. The history and references can be found in [1, 3, 5].

We shall look for a solution  $u(x, y)$  in the form

$$u(x, y) = \int_0^x \int_0^y f(\xi, \eta) d\xi d\eta + v(x, y)$$

( $f(x, y) = 0$  for every  $(x, y) \notin \Omega$ ). Then the problem (1), (2) can be written in the form

$$(3) \quad v_{xy} = 0, \quad (x, y) \in \Omega,$$

$$(4) \quad v = h(x, y), \quad (x, y) \in \partial\Omega.$$

It is well known that every generalized solution  $v \in \mathbf{W}_2^1(\Omega)$  of (3) can be written in the form

$$v(x, y) = p(x) + q(y)$$

where  $p, q$  are arbitrary functions in  $\mathbf{W}_2^1(\Omega)$ . So the problem (3), (4) can be

reformulated as follows: find  $p, q \in \mathbf{W}_2^1(\Omega)$  which satisfy

$$(5) \quad p(x) + q(y) = h(x, y), \quad (x, y) \in \partial\Omega.$$

Following [3], we call the points of  $\Gamma$  at which there is a line of support parallel to a coordinate axis the *vertices* of  $\Gamma$ . It is evident that  $\Gamma$  has either two, three or four vertices.

We shall consider here the cases of two and three vertices. In those cases, necessary and sufficient conditions for the solvability of the problem (3), (4) in  $\mathbf{W}_2^1(\Omega)$  and explicit formulae for solutions will be obtained. Uniqueness of solutions was proved in [4].

1. Let  $\Gamma$  have two vertices  $P_0, P_1$ . We shall consider a regular case: the one-sided tangents to  $\Gamma$  at  $P_0, P_1$  are not parallel to coordinate axes. Let  $s$  be a natural parametrization of  $\Gamma$ :  $\Gamma = \{(x(s), y(s)) \mid 0 \leq s < l\}$ ,  $(x(0), y(0)) = P_0$ ,  $(x(s_1), y(s_1)) = P_1$ . Define

$$(6) \quad H(s) = h(x(s), y(s)),$$

$$(7) \quad P(s) = p(x(s)), \quad Q(s) = q(y(s)).$$

It is not hard to prove the following statement.

LEMMA 1.1.  $p, q \in \mathbf{W}_2^1(\Omega)$  if and only if  $P, Q \in \mathbf{W}_{2,\varrho}^1(0, l)$ , where  $\varrho(s) = s(l-s)|s_1-s|$ ,  $s \in (0, l)$ .

Define onto functions  $f^+, f^- : (0, l) \rightarrow (0, l)$  to be the nontrivial solutions of the equations

$$(8) \quad x(s) = x(f^+(s)), \quad y(s) = y(f^-(s)), \quad s \in (0, l).$$

Then  $P, Q$  defined by (7) satisfy

$$(9) \quad P(s) = P(f^+(s)), \quad Q(s) = Q(f^-(s)), \quad s \in (0, l).$$

So, the problem (3), (4) can be reformulated as follows: find  $P, Q \in \mathbf{W}_{2,\varrho}^1(0, l)$  satisfying (9) and

$$(10) \quad P(s) + Q(s) = H(s), \quad s \in (0, l).$$

Define onto functions  $f_k : (0, l) \rightarrow (0, l)$ ,  $k \in \mathbb{Z}$ , by

$$f_0(s) = s, \quad f_1(s) = f^-(f^+(s)), \quad f_{-1}(s) = f^+(f^-(s)), \quad f_{k+1}(s) = f_1(f_k(s)).$$

The properties of the functions  $f_k$  were described in [4]. Choose  $\psi_0 \in (0, s_1)$  and define a system of intervals  $M_k$ ,  $k \in \mathbb{Z}$ , by

$$M_0 = \begin{cases} (f_1(\psi_0), \psi_0], & f_1(\psi_0) < \psi_0, \\ (\psi_0, f_1(\psi_0)], & f_1(\psi_0) > \psi_0, \end{cases}$$

$$M_{2k} = \{f_k(s) \mid s \in M_0\}, \quad M_{2k+1} = [f^+(f_k(s)) \mid s \in M_0\}.$$

The following lemma was proved in [4].

LEMMA 1.2. 1)  $M_k \cap M_m = \emptyset$ ,  $k \in \mathbb{Z}$ ,  $k \neq m$ .

2)  $\bigcup_{k=-\infty}^{+\infty} M_{2k} = (0, s_1)$ ,  $\bigcup_{k=-\infty}^{\infty} M_{2k+1} = (s_1, l)$ .

Define  $w(s)$  for  $s \in (0, l) \setminus \{s_1\}$  as follows: if  $s \in (0, s_1)$ , then

$$(11) \quad w(s) = \begin{cases} 0, & s \in M_0, \\ \sum_{j=-k+1}^0 H(f_j(s)) - H(f^-(f_j(s))), & s \in M_{2k}, k \in \mathbb{N}, \\ -\sum_{j=1}^k H(f_j(s)) - H(f^-(f_j(s))), & s \in M_{-2k}, k \in \mathbb{N}; \end{cases}$$

if  $s \in (s_1, l)$ , then

$$(12) \quad w(s) = w(f^+(s)).$$

The following theorem is the main result of this section.

**THEOREM 1.1.** *There exist  $P, Q \in \mathbf{W}_{2,\varrho}^1(0, l)$  satisfying (9), (10) if and only if the following conditions hold:*

- 1) for all  $s \in (0, s_1)$ 
  - (a)  $w(f_k(s)) - w(f_{-k}(s)) - w(f_k(\psi_0)) + w(f_{-k}(\psi_0)) \rightarrow 0$  as  $k \rightarrow \infty$ ;
  - (b)  $w(f_k(s)) - w(f_{k+1}(s)) \rightarrow 0$  as  $k \rightarrow -\infty$ ;
- 2) (a) for all  $s \in (0, l) \setminus \{s_1\}$  the following limits exist:

$$(13) \quad w_1(s) = \begin{cases} \lim_{k \rightarrow \infty} (w(f_k(s)) - w(f_k(\psi_0))), & s \in (0, s_1), \\ \lim_{k \rightarrow \infty} (w(f_k(s)) - w(f_k(f^+(\psi_0)))), & s \in (s_1, l); \end{cases}$$

$$(b) \quad w - w_1 \in \mathbf{W}_{2,\varrho}^1(0, l), H - w + w_1 \in \mathbf{W}_{2,\varrho}^1(0, l).$$

If conditions 1), 2) hold, then all solutions  $P, Q \in \mathbf{W}_{2,\varrho}^1(0, l)$  of (9), (10) have the form

$$(14) \quad P(s) = w(s) - w_1(s) + c, \quad Q(s) = H(s) - w(s) + w_1(s) - c, \quad c \in \mathbb{R}.$$

**2.** Let  $\Gamma$  have three vertices  $P_0, P_1, P_2$ . Moreover, suppose that neither of the lines  $x = \text{const}$ ,  $y = \text{const}$  through  $P_0$  intersects  $\Omega$ , and the line  $x = \text{const}$  (resp.  $y = \text{const}$ ) through  $P_1$  (resp.  $P_2$ ) does not meet  $\Omega$ . We shall consider a regular case: the one-sided tangents to  $\Gamma$  at  $P_0$  are not parallel to coordinate axes. Let  $s$  be a natural parametrization of  $\Gamma$ :  $\Gamma = \{(x(s), y(s)) \mid 0 \leq s < l\}$ ,  $(x(0), y(0)) = P_0$ ,  $(x(s_j), y(s_j)) = P_j$ ,  $j = 1, 2$ , where  $s_1, s_2 \in (0, l)$ .

Define  $\mu_j^\pm \in \mathbb{R} \cup \{\pm\infty\}$ ,  $j = 1, 2$ , by

$$\mu_j^\pm = \lim_{s \rightarrow s_j \pm 0} y'(s)/x'(s), \quad j = 1, 2.$$

Define  $d_j(s), \varrho_j(s)$ ,  $j = 1, 2$ ,  $s \in (0, l)$ , by

$$d_j(s) = \begin{cases} 1, & \mu_j^+ = \mu_j^-, \\ |s_j - s|, & \mu_j^+ \neq \mu_j^-, 0 < |\mu_j^\pm| < \infty, \\ \sqrt{|s_j - s|} + \theta(s - s_j), & 0 < |\mu_j^-| < \infty, |\mu_j^+| \in \{0, \infty\}, \\ \sqrt{|s_j - s|} + \theta(s_j - s), & 0 < |\mu_j^+| < \infty, |\mu_j^-| \in \{0, \infty\}, \end{cases}$$

$$\varrho_j(s) = s(l - s)d_j(s),$$

where  $\theta(s) = (1 + \text{sign}(s))/2$ .

It is not hard to prove the following result.

LEMMA 2.1.  $p \in \mathbf{W}_2^1(\Omega)$  (resp.  $q \in \mathbf{W}_2^1(\Omega)$ ) if and only if

$$P(s) = p(x(s)) \in \mathbf{W}_{2,\varrho_2}^1(0,l) \quad (\text{resp. } Q(s) = q(y(s)) \in \mathbf{W}_{2,\varrho_1}^1(0,l)).$$

So, the problem (3), (4) in this case can be reformulated as follows: find  $P \in \mathbf{W}_{2,\varrho_2}^1(0,l)$ ,  $Q \in \mathbf{W}_{2,\varrho_1}^1(0,l)$  satisfying (9), (10) for all  $s \in (0,l) \setminus \{s_1, s_2\}$ .

Define a system of open intervals  $M_k$ ,  $k \in \mathbb{Z}$ , by

$$M_0 = \begin{cases} (s_1, s_2), & s_1 < s_2, \\ (s_2, s_1), & s_1 > s_2, \end{cases}$$

$$M_{2k} = \{f_k(s) \mid s \in M_0\}, \quad M_{2k+1} = \{f^+(f_k(s)) \mid s \in M_0\}.$$

The following lemma was proved in [4].

LEMMA 2.2. 1)  $M_k \cap M_m = \emptyset$ ,  $k, m \in \mathbb{Z}$ ,  $k \neq m$ .

2)  $\bigcup_{k=-\infty}^{\infty} \bar{M}_k = (0,l)$ .

Define  $w(s)$  for  $s \in \bigcup_{k=-\infty}^{\infty} M_k$  by formula (11) if  $s \in \bigcup_{k=-\infty}^{\infty} M_{2k}$ , and by (12) if  $s \in \bigcup_{k=-\infty}^{\infty} M_{2k+1}$ .

The following theorem is the main result of this section.

THEOREM 2.1. *There exist  $P \in \mathbf{W}_{2,\varrho_2}^1(0,l)$ ,  $Q \in \mathbf{W}_{2,\varrho_1}^1(0,l)$  satisfying (9), (10) for all  $s \in (0,l) \setminus \{s_1, s_2\}$  if and only if the following conditions hold:*

- 1) for all  $s \in \bigcup_{k=-\infty}^{\infty} M_k$ ,
  - (a)  $w(f_k(s)) - w(f_{-k}(s)) \rightarrow 0$  as  $k \rightarrow \infty$ ;
  - (b)  $w(f_k(s)) - w(f_{k+1}(s)) \rightarrow 0$  as  $k \rightarrow -\infty$ ;
- 2) (a) for all  $s \in \bigcup_{k=-\infty}^{\infty} M_k$  the limit

$$(15) \quad w_1(s) = \lim_{k \rightarrow \infty} (w(f_k(s)) - w(f_k(\psi_0)))$$

exists, where  $\psi_0 \in M_0$  is arbitrary;

- (b)  $(w(s) - w_1(s)) \in \mathbf{W}_{2,\varrho_2}^1(0,l)$ ;
- $(H(s) - w(s) + w_1(s)) \in \mathbf{W}_{2,\varrho_1}^1(0,l)$ .

If conditions 1), 2) hold, then all solutions  $P \in \mathbf{W}_{2,\varrho_2}^1$ ,  $Q \in \mathbf{W}_{2,\varrho_1}^1$  of (9), (10) have the form (14).

Remark 1. Conditions 1), 2) of Theorems 1.1, 2.1 are in fact the conditions of orthogonality of the right sides of (1), (2) to  $\mathbf{L}_2(\Omega)$  generalized solutions of the homogeneous problem (1), (2).

Remark 2. Irregular cases for two and three vertices can be handled similarly.

Remark 3. Consider the family of differential operators  $L_\lambda u = (1 - \lambda)u_{xx} - \lambda u_{yy}$ ,  $\lambda \in (0,1)$ . Then there are bounded convex domains with piecewise smooth boundary such that for every  $\lambda \in (0,1)$  except some set  $\{\lambda_1, \dots, \lambda_k\}$  the boundary has less than four vertices in the coordinates  $\xi = \sqrt{\lambda} \cdot x + \sqrt{1 - \lambda} \cdot y$ ,  $\eta = \sqrt{\lambda} \cdot x - \sqrt{1 - \lambda} \cdot y$ . A general form of such domains was described in [4].

**References**

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