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EXPONENTIAL ATTRACTORS FOR SEMILINEAR WAVE EQUATIONS WITH DAMPING

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I. Introduction

1. In this lecture we propose to present some results on the existence of exponential attractors for the semilinear damped wave equation

$$u_{tt} + u_t - \Delta u + g(u) = f \tag{1}$$

that we have recently obtained in cooperation with Dr. Alp Eden of Arizona State University; these results are contained in [4] and [5]. Typical examples of nonlinearities we consider for (1) are

$$g(u) = \sin u\,, \quad \text{in any space dimension}\,,$$
$$g(u) = u^3 + p(u)\,, \quad p \text{ quadratic}, \quad \text{in three space dimensions}\,.$$

Our goal is to describe the long time behavior of solutions to (1), by means of a set, called exponential attractor or inertial set, which is finite-dimensional, contains the global attractor and attracts the solutions of (1) at an exponential rate.

2. The recently developed theory of exponential attractors has revealed itself to be an extremely powerful tool for the description of the asymptotic behavior of infinite-dimensional dynamical systems; it retains many aspects of both the theories of global attractors and inertial manifolds (refer e.g. to Hale [6] or Temam [8]), while requiring, so to speak, less stringent conditions for its construction.

The main difference between exponential attractors and global attractors lies in the fact that all solutions converge to the exponential attractor at a uniform exponential rate, once they are in an absorbing ball. Thus, the exponential attractor contains the global attractor, and those stable manifolds where the rate of convergence is only polynomial. However, in contrast to inertial manifolds, which also have finite dimension and attract solutions exponentially, exponential attractors are not required to have a manifold structure; indeed, a simple way of constructing an exponential attractor would be to restrict the inertial manifold to an absorbing set.

3. We briefly recall the main definitions concerning exponential attractors. Let X be a separable Hilbert space on which a dynamical system, typically arising from an initial value problem for a dissipative differential equation such as (1), is described by a solution operator $S(t) : X \to X$, S being a continuous semigroup. The first step consists in showing the existence of a bounded *absorbing set* for $S(t)$ (usually a ball), that is, a bounded invariant set $B \subseteq X$ into which all solutions eventually enter; namely,

$$\begin{aligned} &\text{(i)} \quad \forall t \geq 0\,, \qquad S(t)B \subseteq B\,, \\ &\text{(ii)} \quad \forall u_0 \in X \ \exists t_0 = t_0(\|u_0\|_X) \ \forall t \geq t_0\,, \qquad S(t)u_0 \in B\,. \end{aligned} \tag{2}$$

4. The next step consists in showing that the ω-limit set of B is the *global attractor* of $S(t)$; namely, if

$$A = \omega(B) \equiv \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}$$

then A is compact, invariant, finite-dimensional and attracts all solutions, i.e.

$$\forall u_0 \in X\,, \qquad \lim_{t \to +\infty} d_X(S(t)u_0, A) = 0\,. \tag{3}$$

Although the global attractor completely describes the asymptotic behavior of the dynamical system, it is in general quite difficult, in concrete examples, to describe its geometrical and differential structure; also, the rate of convergence of solutions in (3) may be quite slow and, finally, the available estimates on the dimension of the attractor may be extremely rough.

5. To counter these problems, one looks for the existence of *inertial manifolds*, that is, of sets which are Lipschitz manifolds of finite dimension, flow invariant and exponentially attracting. Namely, X is decomposed into an N-dimensional component $P_N X$ and its orthogonal complement $Q_N X$, and a Lipschitz function $\phi : P_N X \to Q_N X$ is sought such that if M is its graph, then

$$\begin{aligned} &\text{(i)} \quad S(t)M \subseteq M\,, \\ &\text{(ii)} \quad \forall u_0 \in B \ \exists c_0, c_1 > 0 \ \exists t_0 \ \forall t \geq t_0\,, \qquad d_X(S(t)u_0, M) \leq c_1 e^{-c_0 t}\,. \end{aligned} \tag{4}$$

If such a ϕ exists, M is called an inertial manifold; it has finite dimension not greater than N, and $A \subseteq M \cap B$. In particular, the decomposition

$$u = P_N u + \phi(P_N u)$$

shows that the evolution of the system can be described by a finite number of ordinary differential equations. Clearly, the geometrical and differential structure of M is completely described by ϕ; at present, however, the existing theory of inertial manifolds hinges heavily on a condition on the growth of the eigenvalues $\{\lambda_n\}$ of the operator in the evolution equation, called the *gap condition*. This is a quite strong condition on the divergence of the difference $\lambda_{n+1} - \lambda_n$ as $n \to \infty$, and is extremely difficult to verify in concrete examples (for instance, it is not known if it holds for the 2-dimensional Navier–Stokes equations). Moreover, here again the available estimates on the dimension of the inertial manifolds are too crude.

6. To counter this other set of problems, it is expedient to introduce a smaller set Z, the *exponential attractor*. This set is compact and contains the attractor; like the inertial manifold, it is finite-dimensional, flow invariant and attracts solutions exponentially, but with a uniform rate. Namely,

$$(5)\quad \begin{aligned} &\text{(i)}\quad S(t)Z \subseteq Z\,, \\ &\text{(ii)}\quad \exists c_0, c_1 > 0\ \forall u_0 \in B\ \forall t \geq 0\,, \qquad d_X(S(t)u_0, Z) \leq c_1 e^{-c_0 t}\,. \end{aligned}$$

We remark that in general, when all these sets exist, they are related by the following set inclusions:

$$A \subseteq Z \subseteq M \cap B \subseteq B\,.$$

7. As we have mentioned, whenever an inertial manifold M exists, the set $Z = M \cap B$ is an exponential attractor; however, in light of the problems described above, it is expedient to resort to a different approach to construct the exponential attractors directly. This can be obtained by adding to the global attractor a certain set of points that fail to satisfy a condition, called the *discrete squeezing property*. One way of describing this property is the following:

DEFINITION. The solution operator $S(t)$ satisfies the *discrete squeezing property* on B if there exist $t_* > 0$ and an orthogonal projection P of finite rank N_0 such that, $\forall u$, $\forall v \in B$, either

$$\|S_* u - S_* v\|_X \leq \tfrac{1}{8}\|u - v\|_X\,,$$

where $S_* = S(t_*)$, or

$$\|(I - P)(S_* u - S_* v)\|_X \leq \|P(S_* u - S_* v)\|_X\,.$$

In other words, if the infinite-dimensional part dominates the finite one, the map S_* is actually a contraction or, to describe the property in another way, if $w_* = S_* u - S_* v$,

$$\|w_*\|_X > \sqrt{2}\|P w_*\|_X \ \Rightarrow\ \|w_*\|_X \leq \tfrac{1}{8}\|u - v\|_X\,.$$

8. The importance of the discrete squeezing property for the construction of exponential attractors is based on the following

THEOREM 1. *If the solution operator $S(t)$ satisfies the discrete squeezing property on a bounded absorbing set B, then there exists an exponential attractor $Z \subseteq B$ satisfying* (5) *and whose* (*fractal*) *dimension is of the order of N_0.*

Proof. See Eden, Foiaş, Nicolaenko and Temam [3].

Our goal is thus to show that the solution operator associated with equation (1) does satisfy the discrete squeezing property and, therefore, possesses an exponential attractor. Before this, however, we conclude this introduction by recalling several models of equations for which exponential attractors have been shown to exist in this way; namely, the 2-dimensional Navier–Stokes equations, the Kuramoto–Sivashinsky equations with periodic boundary conditions

$$u_t + u_{xxxx} + u_{xx} + uu_x = f\,,$$

the Chaffee–Infante equations with Dirichlet boundary conditions

$$u_t - \Delta u + u^3 - u = f\,,$$

and the original Burger's equations

$$\begin{cases} U_t = R - U - \int\limits_0^1 |v|^2\,dx\,, \\ v_t = Uv + v_{xx} - (v^2)_x \end{cases}$$

(see Eden–Foiaş–Nicolaenko–Temam [3] and Eden [2]).

II. Statement of results

1. We now turn to a concrete example of equation (1), concentrating on the quantum mechanics nonlinearity $g(u) = u^3 - u$. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open domain with a smooth boundary $\partial\Omega$; we consider the semilinear initial-boundary value problem

$$\begin{cases} \varepsilon u_{tt} + u_t - \Delta u + u^3 - u = f(x,t)\,, \\ u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,, \quad u|_{\partial\Omega} = 0\,, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$. We consider (2.1) as an evolution equation for the pair $\{u(t), u_t(t)\} = S^\varepsilon(t)\{u_0, u_1\}$, on the Hilbert space $E_0 = V \times H$, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$; setting also $Y = H^2(\Omega) \cap H_0^1(\Omega)$ and $E_1 = Y \times V$, the existence of the solution operator, and its regularity properties, are assured for all $\varepsilon > 0$ by the following

THEOREM 2. (i) $\forall f \in C_{\mathrm{b}}(\mathbb{R}^+; H)$ $\forall \{u_0, u_1\} \in E_0$ $\exists! u \in C_{\mathrm{b}}(\mathbb{R}^+; V) \cap C_{\mathrm{b}}^1(\mathbb{R}^+; H)$, *solution of* (2.1).

(ii) *If moreover* $f \in C_{\mathrm{b}}^1(\mathbb{R}^+; H)$ *and* $\{u_0, u_1\} \in E_1$, *then* $u \in C_{\mathrm{b}}(\mathbb{R}^+; Y) \cap C_{\mathrm{b}}^1(\mathbb{R}^+; V) \cap C_{\mathrm{b}}^2(\mathbb{R}^+; H)$.

Proof. See Temam [8] or Babin–Vishik [1].

The existence of attractors and inertial manifolds for $S^\varepsilon(t)$, when f is independent of t, is described by

THEOREM 3. (i) $\forall \varepsilon > 0$, *there exists a compact attractor* A_ε *for* $S^\varepsilon(t)$ *in* E_0.

(ii) *If* $\varepsilon \gg 1$, *there exists no* C^1 *inertial manifold for* $S^\varepsilon(t)$.

(iii) *If* $\varepsilon \ll 1$, *there exists a* C^1 *inertial manifold for* $S^\varepsilon(t)$, *at least in one space dimension.*

Proof. For (i), see Babin–Vishik [1] and also Eden–Milani [4] if ε is small; for (ii) and (iii), see Mora–Solà-Morales [7].

We now come to our result on the existence of exponential attractors for equation (2.1):

THEOREM 4. *Assume* $f \in C^1_{\mathrm{b}}(\mathbb{R}^+; H)$. *Then* $\forall \varepsilon > 0$, $S^\varepsilon(t)$ *admits an absorbing ball* $B_1 \subseteq E_1$ *over which it satisfies the discrete squeezing property.*

Proof. See the next section.

2. We consider the following norms in E_0 and E_1:

$$\|\{u,v\}\|^2_{E_0} = \varepsilon|v|^2 + |\nabla u|^2\,, \qquad \|\{u,v\}\|^2_{E_1} = \varepsilon|\nabla v|^2 + |\Delta u|^2\,,$$

where $|\cdot|$ is the norm in $L^2(\Omega)$; we recall that these norms are equivalent to the standard ones, because of Poincaré's inequalities; in particular,

$$\exists p > 0\ \forall u \in Y\,, \qquad |\nabla u|^2 \le p|\Delta u|^2\,. \tag{2.2}$$

For the sake of simplicity, we limit ourselves to the case $\varepsilon \le 1$; defining then, for $w = \{u, v\} \in E_1$, the function

$$N_1(w) = \varepsilon|\nabla v|^2 + \varepsilon(\nabla u, \nabla v) + \tfrac{1}{2}|\nabla u|^2 + |\Delta u|^2 + 3(u^2\nabla u, \nabla u)\,,$$

a simple application of Schwarz' inequality yields the inequality

$$\forall \{u,v\} = w \in E_1\,, \qquad \|w\|^2_{E_1} \le 2N_1(w)\,. \tag{2.3}$$

We can now start the proof of Theorem 4. After Temam [8] we can assume the existence of an absorbing ball $B_0 \subseteq E_0$ for $S^\varepsilon(t)$, that is, if $\{u(t), u_t(t)\} = S^\varepsilon(t)\{u_0, u_1\}$,

$$\exists R_0 > 0\ \exists T_0 = T(R_0)\ \forall t \ge T_0\ \forall \{u_0, u_1\} \in E_0\,, \quad \varepsilon|u_t(t)|^2 + |\nabla u(t)|^2 \le R_0^2\,. \tag{2.4}$$

We claim then

PROPOSITION 1. $\exists R_1 > 0\ \exists T_1 = T(R_1, T_0, R_0) \ge T_0\ \forall t \ge T_1\ \forall \{u_0, u_1\} \in E_1$,

$$\varepsilon|\nabla u_t(t)|^2 + |\Delta u(t)|^2 \le R_1^2\,. \tag{2.5}$$

Proof. We multiply the equation of (2.1) in H by $-\Delta u$ and $-2\Delta u_t$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\{\varepsilon|\nabla u_t|^2 + \varepsilon(\nabla u, \nabla u_t) + \tfrac{1}{2}|\nabla u|^2 + |\Delta u|^2 + 3(u^2\nabla u, \nabla u) + 2(f, \Delta u)\}\\
&\qquad + (2-\varepsilon)|\nabla u_t|^2 + |\Delta u|^2 + 3(u^2\nabla u, \nabla u) + 2\mu(f, \Delta u)\\
&\quad = (2\mu - 1)(f, \Delta u) + 2(f_t, \Delta u) + 6(u\nabla u, \nabla u) + 2(\nabla u, \nabla u_t) + |\nabla u|^2 \equiv r\,,
\end{aligned}$$

where $\mu = \min[1/3, 1/(2(p+1))]$, with p defined in (2.2).

Recalling then (2.4), we estimate r for $t \ge T_0$ as follows:

$$\begin{aligned}
r &\le \tfrac{3}{2}|2\mu - 1|\,|f|^2 + \tfrac{1}{6}|\Delta u|^2 + 6|f_t|^2 + \tfrac{1}{6}|\Delta u|^2\\
&\quad + 36|\nabla u|^4 + \tfrac{1}{6}|\Delta u|^2 + 2|\nabla u|^2 + \tfrac{1}{2}|\nabla u_t|^2 + |\nabla u|^2\\
&\le \tfrac{3}{2}|f|^2 + 6|f_t|^2 + 36R_0^4 + 3R_0^2 + \tfrac{1}{2}|\Delta u|^2 + \tfrac{1}{2}|\nabla u_t|^2\,;
\end{aligned}$$

thus, there exists a constant $\gamma_1 > 0$, depending only on R_0 and the norm of f in $C^1_{\mathrm{b}}(\mathbb{R}^+; H)$, such that, if $t \geq T_0$,

$$\frac{d}{dt}\{N_1(u,t) + 2(f, \Delta u)\} + \tfrac{3}{2}\varepsilon|\nabla u_t|^2 + \tfrac{1}{2}|\Delta u|^2 + 3(u^2\nabla u, \nabla u) + 2\mu(f, \Delta u) \leq \gamma_1 . \tag{2.6}$$

It is now easy to see that, if $\varepsilon \leq 1$,

$$\mu\{N_1(u,t) + 2(f, \Delta u)\} \leq \tfrac{3}{2}\varepsilon|\nabla u_t|^2 + \tfrac{1}{2}|\Delta u|^2 + 3(u^2\nabla u, \nabla u) + 2\mu(f, \Delta u) ,$$

so that (2.6) implies

$$\frac{d}{dt}\{N_1(u,t) + 2(f, \Delta u)\} + \mu\{N_1(u,t) + 2(f, \Delta u)\} \leq \gamma_1$$

from which we deduce that for $t \geq T_0$

$$\{N_1(u,u_t) + 2(f, \Delta u)\}(t) \leq e^{-\mu(t-T_0)}\{N_1(u,u_t) + 2(f, \Delta u)\}(T_0) + \gamma_1\mu^{-1} . \tag{2.7}$$

From (ii) of Theorem 2 we know that there exists $\gamma_2 > 0$, depending on the norm of $\{u_0, u_1\}$ in E_1, and on that of f on $C^1_{\mathrm{b}}(\mathbb{R}^+; H)$, such that

$$\{N_1(u,u_t) + 2(f, \Delta u)\}(T_0) \leq \gamma_2 ;$$

from (2.7) we then obtain

$$\{N_1(u,u_t) + 2(f, \Delta u)\}(t) \leq 2\gamma_1\mu^{-1} \tag{2.8}$$

for $t \geq T_1$, where $T_1 = T_0 + (1/\mu)\ln(\gamma_2\mu/\gamma_1)$ $(\geq T_0)$.

We now recall (2.3) to deduce that for $t \geq T_1$, (2.8) yields

$$\|\{u,u_t\}(t)\|^2_{E_1} \leq 2N_1(u,t) \leq 2\gamma_2 - 2(f, \Delta u)(t) \leq 2\gamma_2 + 2|f(t)|^2 + \tfrac{1}{2}|\Delta u(t)|^2 ,$$

so that eventually we have (2.5) with $R_1 = 2(\gamma_2 + \|f\|_{C_{\mathrm{b}}(\mathbb{R}^+;H)})^{1/2}$. ∎

3. We now proceed to prove the discrete squeezing property for $S^\varepsilon(t)$. Let $\{\lambda_n\}$ be the sequence of the eigenvalues of $-\Delta$, and $\{w_n\}$ the corresponding sequence of eigenvectors. Let $H_N = \operatorname{span}\{w_1, \dots, w_N\}$, and $p_N : H \to H_N$, $q_N = I - p_N$ be the corresponding orthogonal projections (which are orthogonal both in V and H); clearly, we have

$$\forall u \in q_N V , \quad |u|^2 \leq \frac{1}{\lambda_{N+1}}|\nabla u|^2 . \tag{2.9}$$

Next, we define corresponding product projections in E_0, namely

$$P_N : E_0 \to (p_N V) \times (p_N H) , \quad P_N(\{u,v\}) = \{p_N u, p_N v\}, \quad Q_N = I - P_N .$$

Then, for $w = \{u,v\} \in E_0$, we define the functions

$$N_0(w) = \varepsilon|v|^2 + \varepsilon(u,v) + \tfrac{1}{2}|u|^2 + |\nabla u|^2 ,$$
$$M(w) = \|w\|^2_{E_0} + (u,v) = \varepsilon|v|^2 + (u,v) + |\nabla u|^2$$

and claim:

LEMMA 1. (i) *Let* $K = \max\{1+1/\lambda_1, 3/2\}$. *Then if* $\varepsilon \le 1$, N_0 *is an equivalent norm in* E_0; *indeed*, $\forall w = \{u, v\} \in E_0$,

$$N_0(w) \ge |\nabla u|^2\,, \qquad \tfrac{1}{2}\|w\|_{E_0}^2 \le N_0(w) \le K\|w\|_{E_0}^2\,. \tag{2.10}$$

(ii) *If* $\varepsilon \le 1$ *and* N *is so large that* $\varepsilon\lambda_{N+1} \ge 1$, *then* M *is an equivalent norm in* $Q_N E_0$; *indeed*, $\forall w \in Q_N E_0$,

$$\|w\|_{E_0}^2 \le 2M(w) \le 3\|w\|_{E_0}^2\,. \tag{2.11}$$

Proof. Consequence of Schwarz' inequality; in particular for (2.11), note that by (2.9) we have

$$(u, v) \le \frac{1}{2\varepsilon}|u|^2 + \frac{\varepsilon}{2}|v|^2 \le \frac{1}{2\varepsilon\lambda_{N+1}}|\nabla u|^2 + \frac{\varepsilon}{2}|v|^2 \le \tfrac{1}{2}\|w\|_{E_0}^2\,. \ \blacksquare$$

We now estimate the difference of two solutions $u, \overline{u} \in C_{\mathrm{b}}(\mathbb{R}^+; V) \cap C^1_{\mathrm{b}}(\mathbb{R}^+; H)$ of (2.1): if $w = u - \overline{u}$ and $W = \{w, w_t\}$, so that $W \in C_{\mathrm{b}}(\mathbb{R}^+; E_0)$, we claim

LEMMA 2. *Let* K *be as in Lemma* 1. *There exists* $\alpha > 0$ *such that*

$$\forall t \ge 0\,, \qquad \|W(t)\|_{E_0}^2 \le 2Ke^{\alpha t}\|W(0)\|_{E_0}^2\,. \tag{2.12}$$

Proof. w solves the equation

$$\varepsilon w_{tt} + w_t - \Delta w = w + u^3 - v^3\,, \tag{2.13}$$

which we multiply by $2w_t$ and w to obtain

$$\begin{aligned}\frac{d}{dt}N_0(w) + (2-\varepsilon)|w_t|^2 + |\nabla w|^2 &= (w + u^3 - v^3, 2w_t + w)\\ &\le |w_t|^2 + \left(\frac{3}{\lambda_1} + 18R^4 + \frac{3R^2}{\lambda_1}\right)|\nabla w|^2 \equiv |w_t|^2 + \alpha|\nabla w|^2\,,\end{aligned}$$

so that (2.12) follows by Gronwall's inequality, recalling (2.10). $\blacksquare$

LEMMA 3. *Let* N *be such that* $\varepsilon\lambda_{N+1} \ge 1$, $q = q_N w$, $Q = Q_N W$. *Then* $\exists\beta > 0$ *such that*

$$\forall t \ge 0\,, \qquad \frac{d}{dt}M(Q(t)) + \frac{1}{2\varepsilon}M(Q(t)) \le \frac{\beta}{\lambda_{N+1}}|\nabla w(t)|^2\,. \tag{2.14}$$

Proof. We apply q_N to (2.13): since q_N and $-\Delta$ commute, we have

$$\varepsilon q_{tt} + q_t - \Delta q = \Gamma \equiv q_N(v^3 - u^3) + q\,.$$

Multiplying this by $2q_t$ and $(1/\varepsilon)q$ we obtain

$$\frac{d}{dt}M(Q) + |q_t|^2 + \frac{1}{\varepsilon}|\nabla q|^2 + \frac{1}{2\varepsilon}(q, q_t) = -\frac{1}{2\varepsilon}(q, q_t) + \left(\Gamma, 2q_t + \frac{1}{\varepsilon}q\right). \tag{2.15}$$

Recalling (2.9), we estimate

$$|\Gamma| \le \frac{1}{\lambda_{N+1}^{1/2}}|\nabla(u^3 - v^3 - w)| \le \frac{1}{\lambda_{N+1}^{1/2}}(9R_1^2 + 1)|\nabla w|\,, \tag{2.16}$$

where R_1 is the radius of the absorbing ball for $S^\varepsilon(t)$ in E_1, as provided by Proposition 1. Indeed, we have

$$\begin{aligned}|\nabla(u^3-v^3)| &\le 3|(u^2-v^2)\nabla u| + 3|v^2\nabla w| \\ &\le 3\{(|u|_{L^\infty}+|v|_{L^\infty})|\nabla u|_{L^3}|w|_{L^6} + |v|^2_{L^\infty}|\nabla w|\} \\ &\le 3\{(|\Delta u|+|\Delta v|)|\Delta u|\,|\nabla w| + |\Delta w|^2|\nabla w|\} \le 9R_1^2|\nabla w|\,.\end{aligned}$$

Thus, we see from (2.16) that the right side of (2.15) is estimated by

$$\begin{aligned}&\frac{1}{2\varepsilon}|q|\,|q_t| + \frac{1}{\lambda_{N+1}^{1/2}}(9R_1^2+1)|\nabla w|\left(2|q_t| + \frac{1}{\varepsilon}|q|^2\right) \\ &\qquad\le \frac{1}{2}|q_t|^2 + \frac{3}{8\varepsilon^2}|q|^2 + \frac{6(9R_1^2+1)}{\lambda_{N+1}}|\nabla w|^2 \\ &\qquad\le \frac{1}{2}|q_t|^2 + \frac{3}{8\varepsilon}\frac{1}{\varepsilon\lambda_{N+1}}|\nabla q|^2 + \frac{\beta}{\lambda_{N+1}}|\nabla w|^2\,,\end{aligned}$$

with $\beta = 6(9R_1^2+1)$; inserting this in (2.15) yields (2.14). ■

We are now ready to show that the discrete squeezing property holds. Let $U = \{u_0, u_1\}$ and $\overline{U}_0 = \{\overline{u}_0, \overline{u}_1\}$ be in E_0, and set $W(t) = S^\varepsilon(t)U - S^\varepsilon(t)\overline{U}$. We will show that there exist $t_* > 0$ and N_0 such that if

$$\|P_{N_0}W(t_*)\|_{E_0} \le \|Q_{N_0}W(t_*)\|_{E_0} \tag{2.17}$$

then in fact $\|W(t_*)\|_{E_0} \le \frac{1}{8}\|W(0)\|_{E_0}$ as well. By Lemmas 3 and 2 we have

$$\frac{d}{dt}M(Q) + \frac{1}{2\varepsilon}M(Q) \le \frac{\beta}{\lambda_{N+1}}|\nabla w|^2 \le \frac{\beta}{\lambda_{N+1}}\|W\|^2_{E_0} \le \frac{2K\beta}{\lambda_{N+1}}e^{\alpha t}\|W(0)\|^2_{E_0}\,,$$

from which it follows that

$$M(Q(t)) \le M(Q(0))e^{-t/(2\varepsilon)} + \frac{4K\beta\varepsilon}{\lambda_{N+1}}e^{\alpha t}\|W(0)\|^2_{E_0}$$

and, recalling (2.11), if $\varepsilon\lambda_{N+1} \ge 1$,

$$\|Q_N W(t)\|^2_{E_0} \le \left(9e^{-t/(2\varepsilon)} + \frac{4\beta\varepsilon}{\lambda_{N+1}}e^{\alpha t}\right)\|W(0)\|^2_{E_0}\,. \tag{2.18}$$

We now choose first t_* so that $18e^{-t_*/(2\varepsilon)} \le 1/128$ and then N_0 so large that $\lambda_{N_0+1} \ge 1/\varepsilon$ and

$$\frac{8K\beta\varepsilon}{\lambda_{N+1}}e^{\alpha t_*} \le \frac{1}{128}\,;$$

if for this choice of t_* and N_0, (2.17) holds, then from (2.18) we deduce that

$$\begin{aligned}\|W(t_*)\|^2_{E_0} &= \|P_N W(t_*)\|^2_{E_0} + \|Q_N W(t_*)\|^2_{E_0} \le 2\|Q_N W(t_*)\|^2_{E_0} \\ &\le (18e^{-t_*/(2\varepsilon)} + 8\beta\varepsilon\lambda_{N+1}^{-1}e^{\alpha t_*})\|W(0)\|^2_{E_0} \le \tfrac{1}{64}\|W(0)\|^2_{E_0}\,,\end{aligned}$$

that is, $\|W(t_*)\|_{E_0} \le \frac{1}{8}\|W(0)\|^2_{E_0}$, as desired. The proof of Theorem 4 is thus complete; note that $t_* = t_*(\varepsilon)$ and $N_0 = N_0(t_*, \varepsilon)$. ■

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