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NEUMANN PROBLEM FOR ONE-DIMENSIONAL NONLINEAR THERMOELASTICITY

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Abstract. The global existence theorem of classical solutions for one-dimensional nonlinear thermoelasticity is proved for small and smooth initial data in the case of a bounded reference configuration for a homogeneous medium, considering the Neumann type boundary conditions: traction free and insulated. Moreover, the asymptotic behaviour of solutions is investigated.

1. Introduction. The equations of one-dimensional nonlinear thermoelasticity have been investigated in the case of a bounded reference configuration for a homogeneous medium by Slemrod in 1981 (see [4]). He proved the global existence of smooth solutions for small data, considering the boundary conditions: traction free and constant temperature, or rigidly damped and insulated. The cases of Dirichlet boundary conditions: rigidly damped and constant temperature, and of Neumann boundary conditions: traction free and insulated, remained open for several years after Slemrod's work. In 1990, Racke and Shibata [3] proved the global existence of smooth solutions for small and smooth data in the case of Dirichlet boundary conditions. As is well known, in proving the existence theorem of smooth solutions for at least small and smooth data, the main step is to show the decay properties of solutions to linearized equations. In [3], Racke and Shibata used spectral analysis to the reduced stationary problem to get the decay properties, which was a completely different approach from Slemrod's work.

In this paper, the global existence of smooth solutions for small and smooth data is proved in the case of Neumann boundary conditions. Our approach here is principally the same as in Racke and Shibata [3], but more delicate discussions are needed, because of the Neumann boundary conditions.

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Now, let us recall the equations of one-dimensional nonlinear thermoelasticity. Let $\Omega = (0,1)$ be the unit interval in $\mathbb{R}$, which is identified with the reference configuration $\mathcal{R}$. The thermoelastic motion is described mathematically by the deformation map $\Omega \ni x \mapsto X(t,x) \in \mathbb{R}$ and the absolute temperature $T(t,x) \in \mathbb{R}$ of the material point of coordinate $X(t,x)$, where t denotes the time variable. Then, the equations of balance of linear momentum and of balance of energy are given by (cf. Carlson [1])

$$\varrho_{\mathcal{R}} X_{tt} = \widetilde{S}_x + \varrho_{\mathcal{R}} b\,, \tag{B.M}$$

$$(\widetilde{\varepsilon} + (\varrho_{\mathcal{R}}/2)X_t^2)_t = (\widetilde{S}X_t)_x + \varrho_{\mathcal{R}} b X_t + \widetilde{q}_x + \varrho_{\mathcal{R}} r\,, \tag{B.E}$$

where we use the following notation: The subscripts t and x denote differentiations with respect to t and x, respectively. $\varrho_{\mathcal{R}}$ is the material density and it is assumed to be 1 in the sequel. The b and r are specific body force and heat supply, respectively. We assume that $b = r = 0$ below. $\widetilde{\varepsilon}$ is the specific internal energy. $\widetilde{q}$ is the heat flux. $\widetilde{S}$ is the Piola–Kirchhoff stress tensor. According to the 2nd Law of Thermodynamics and Coleman's theorem [2], we make throughout the following assumptions.

Assumptions. (1) There exists a so-called Helmholtz energy function $\psi(F,T)$, which is real-valued and in $C^\infty(G(B))$, such that

$$\widetilde{S} = S(X_x(t,x), T(t,x)) \quad \text{and} \quad \widetilde{\varepsilon} = \varepsilon(X_x(t,x), T(t,x)) \quad \text{where} \tag{A.1}$$

$$S(F,T) = (\partial\psi/\partial F)(F,T)\,, \quad \varepsilon(F,T) = \psi(F,T) - T(\partial\psi/\partial T)(F,T) \quad \text{and} \quad F = X_x\,; \tag{A.2}$$

$$G(B) = \{(F,T) \in \mathbb{R}^2 \mid |F-1| + |T-T_0| < B\,,\ T > T_0/2\}\,;$$

T_0 is a positive constant denoting the natural temperature of the reference body $\mathcal{R}$ and B is another positive constant. Moreover, we assume that

$$(\partial^2\psi/\partial F^2)(F,T) > 0\,, \quad (\partial^2\psi/\partial T^2)(F,T) < 0\,, \quad (\partial^2\psi/\partial F\partial T)(F,T) \neq 0 \quad \text{for } (F,T) \in G(B)\,; \tag{A.3}$$

$$S(1,T_0) = 0\,. \tag{A.4}$$

(2) There exists a positive function $Q(F,T) \in C^\infty(G(B))$ such that

$$\widetilde{q} = Q(X_x(t,x), T(t,x))T_x(t,x)\,. \tag{A.5}$$

The purpose of this paper is to prove the global existence of smooth solutions to the following problem:

$$X_{tt} = S(X_x,T)_x \quad \text{in } (0,\infty)\times\Omega\,, \tag{1.1}$$

$$(\varepsilon(X_x,T) + \tfrac{1}{2}X_t^2)_t = (S(X_x,T)X_t)_x + (Q(X_x,T)T_x)_x \quad \text{in } (0,\infty)\times\Omega\,, \tag{1.2}$$

$$S(X_x,T) = T_x = 0 \quad \text{on } (0,\infty)\times\partial\Omega\,, \tag{1.3}$$

$$X(0,x) = x + u_0(x)\,, \quad X_t(0,x) = u_1(x)\,, \quad T(0,x) = T_0 + \theta_0(x) \quad \text{in } \Omega\,, \tag{1.4}$$

where $\partial\Omega$ denotes the boundary of Ω, i.e. $\partial\Omega = \{0\} \cup \{1\}$, and u_0, u_1 and θ_0 are given functions.

Now, let us discuss the equilibrium state. In view of (A.4), $X = x$ and $T = T_0$ are solutions for the initial data $u_0 = u_1 = \theta_0 = 0$. Integrating (1.2) on $(0,t) \times \Omega$, we have

$$\int_0^1 \{\varepsilon(X_x(t,x), T(t,x)) + \tfrac{1}{2}X_t^2(t,x)\}\,dx = \int_0^1 \{\varepsilon(1 + u_0'(x), T_0 + \theta_0(x)) + \tfrac{1}{2}u_1(x)^2\}\,dx\,, \tag{1.5}$$

$u_0'(x) = (du_0/dx)(x)$, as long as the solutions exist. If we expect that $X_t \to 0$ and $(X_x, T) \to (X_\infty, T_\infty)$ (other constant states) as $t \to \infty$, in view of (1.3) and (1.5), X_∞ and T_∞ satisfy

$$\varepsilon(X_\infty, T_\infty) = \int_0^1 \{\varepsilon(1 + u_0'(x), T_0 + \theta_0(x)) + \tfrac{1}{2}u_1(x)^2\}\,dx\,; \tag{1.6.a}$$

$$S(X_\infty, T_\infty) = 0\,; \tag{1.6.b}$$

$$(X_\infty, T_\infty) \in G(B)\,. \tag{1.6.c}$$

On the other hand, if we consider the map $G(B) \ni (F,T) \mapsto (\varepsilon(F,T), S(F,T)) \in \mathbb{R}^2$, by (A.2), (A.3) and (A.4) we see that the Jacobian $\partial(\varepsilon, S)/\partial(F,T)$ of this map at $(F,T) = (1, T_0)$ is equal to $-T_0(\partial^2\psi/\partial T^2)(1,T_0)(\partial^2\psi/\partial F^2)(1,T_0) + T_0(\partial^2\psi/\partial F\partial T)(1,T_0)^2 > 0$. The inverse mapping theorem gives the unique existence of (X_∞, T_∞) satisfying (1.6) provided that $|u_0'(x)|$, $|u_1(x)|$ and $|\theta_0(x)|$ are sufficiently small for $x \in [0,1]$.

To find the energy conservation (1.5) and other constant states (X_∞, T_∞) at $t = \infty$, (1.2) is quite important, but the form of (1.2) is rather complicated. So, once we know (1.5) and (1.6), using the entropy

$$N(F,T) = -(\partial\psi/\partial T)(F,T)\,, \tag{1.7}$$

we rewrite (1.2) as follows:

$$TN(X_x, T)_t = (Q(X_x,T)T_x)_x \quad \text{in } (0,\infty) \times \Omega\,. \tag{1.2$'$}$$

In fact, multiplying (1.1) by X_t implies that $\frac{1}{2}(X_t^2)_t = S_x X_t$. Using the constitutive relations (A.2) and (1.7), we have $\varepsilon(X_x,T)_t = TN(X_x,T)_t + S(X_x,T)X_{tx}$. Since $(S(X_x,T)X_t)_x = S(X_x,T)_x X_t + S(X_x,T)X_{tx}$, (1.2)$'$ follows from (1.1) and (1.2). Obviously, (1.2) also follows from (1.1) and (1.2)$'$. From now on, we shall solve the problem (1.1), (1.2)$'$, (1.3) and (1.4) instead of the problem (1.1), (1.2), (1.3) and (1.4).

Now, we discuss the initial conditions and compatibility conditions. To do this, assume for a moment the existence of solutions X and T satisfying

$(X_x(t,x), T(t,x)) \in G(B)$. Put

$$u_{i+2}(x) = (\partial_t^{i+2} X)(0,x) \quad \text{and} \quad \theta_{i+1}(x) = (\partial_t^{i+1} T)(0,x) \quad \text{for } i \geq 0\,. \tag{1.8}$$

In fact, u_{i+2} and θ_{i+1} are determined successively from u_0, u_1 and θ_0 by differentiating (1.1) and (1.2)$'$ with respect to t at $t = 0$. We would like to show the existence of solutions satisfying the conditions

$$X \in \bigcap_{j=0}^{L+2} C^j([0,\infty), H^{L+2-j})\,, \tag{1.9.a}$$

$$T \in C^{L+1}([0,\infty), L^2) \cap \bigcap_{j=0}^{L} C^j([0,\infty), H^{L+2-j})\,, \tag{1.9.b}$$

$$(X_x(t,x), T(t,x)) \in G(B) \quad \text{for } (t,x) \in [0,\infty) \times [0,1] \tag{1.9.c}$$

where the notation is summarized at the end of this section. Therefore, we must assume that

$$\begin{gathered} u_i \in H^{L+2-i} \quad (0 \leq i \leq L+1)\,, \qquad \theta_i \in H^{L+2-i} \quad (0 \leq i \leq L)\,, \\ (1 + u_0'(x), T_0 + \theta_0(x)) \in G(B) \quad \text{for } x \in [0,1]\,. \end{gathered} \tag{1.10}$$

Note that the fact that u_{L+2} and θ_{L+1} belong to L^2 follows from (1.10) if we differentiate (1.1) and (1.2)$'$ L times with respect to t at $t = 0$.

Moreover, differentiating the boundary condition (1.3) with respect to t at $t = 0$, we have

$$\partial_t^i S(X_x, T)|_{t=0} = \theta_{ix} = 0 \quad \text{for } x = 0,1 \text{ and } i = 0,1,\ldots,L\,, \tag{1.11}$$

because $\partial_t^i S(X_x, T)$ and $\partial_t^i T_x$ belong to H^{L+1-i} for $t \geq 0$. Note that (1.11) are conditions imposed on u_0, u_1 and θ_0. We shall say that u_0, u_1 and θ_0 satisfy the *compatibility condition of order* L if (1.11) is satisfied.

The purpose of this paper is to prove

THEOREM 1.1. *Let* $0 < \tau < 1/16$ *and* K *and* L *be integers such that*

$$K \geq 3 \quad \text{and} \quad L \geq \frac{8K^2 + 15K - (1+\tau)}{K - (1+\tau)}\,. \tag{1.12}$$

Let u_0, u_1 *and* θ_0 *in* (1.4) *be given and let* u_{i+2} *and* θ_{i+1} $(0 \leq i \leq L-1)$ *be the functions defined by* (1.8). *Assume that* (1.10) *holds true and that* u_0, u_1 *and* θ_0 *satisfy the compatibility condition of order* L. *In addition, assume that*

$$\int_0^1 u_1(x)\,dx = 0\,. \tag{1.13}$$

Put

$$E = \sum_{i=0}^{L+1} \|u_i\|_{L+1-i} + \sum_{i=0}^{L-1} \|\theta_i\|_{L+1-i} + \|\theta_L\|\,. \tag{1.14}$$

Then there exists a $\delta > 0$ *such that if* $E \le \delta$, *then the problem* (1.1)–(1.4) *admits a unique solution* X, T *satisfying* (1.9). *Moreover*, *the asymptotic behaviour of* X *and* T *is given by* $Y(t) \le 1$ *for* $t \ge 0$, *where*

$$
\begin{aligned}
&Y(t) = Y_1(t) + Y_2(t)\,, \qquad (1.15)\\
&Y_1(t) = \|V\|_{t,K,0} + \|(T_{tx}, T_{txx})\|_{t,K,0}\,,\\
&Y_2(t) = \|V\|_{t,0,L-1} + \Big\{ \int_0^t \|\overline{\partial}_s^L T_x(s,\cdot)\|^2\,ds \Big\}^{1/2},\\
&V = (X_t, X_x - X_\infty, X_{tt}, X_{tx}, X_{xx}, T - T_\infty, T_t, T_x, T_{xx})\,.
\end{aligned}
$$

Finally, we explain the notation used throughout. All the functions are assumed to be real-valued, except for paragraphs 2.2 and 2.3. The i stands for $\sqrt{-1}$ in paragraphs 2.2, 2.3 and 2.4 only, otherwise it is used as an index. We denote the usual L^2 space on $(0,1)$, its inner product and its norm by $L^2 = H^0$, $(\ ,\ )$ and $\|\cdot\|$, respectively. It will be clear from the context whether (a,b) denotes an open interval in $\mathbb{R}$ or the inner product. Put $H^m = \{u \in L^2 \mid \|u\|_m = \|\overline{d}_x^m u\| < \infty\}$, where $\overline{d}_s^m u(s) = ((d^l u/ds^l)(s),\ 0 \le l \le m)$ for $s = t, x$. By $C^L(I,B)$ we denote the set of all B-valued functions which are L times continuously differentiable in I. Put

$$
\begin{aligned}
\|v\|_{t,K,L} &= \sup\{(1+s)^K \|\overline{D}^L v(s,\cdot)\| \mid 0 \le s < t\}\,,\\
|v|_{t,K,L} &= \sup\{(1+s)^K |\overline{d}_s^L v(s)| \mid 0 \le s < t\}\,,\\
\|(v_1,\ldots,v_l)\|_{t,K,L} &= \|v_1\|_{t,K,L} + \ldots + \|v_l\|_{t,K,L}\,,\\
|(v_1,\ldots,v_l)|_{t,K,L} &= |v_1|_{t,K,L} + \ldots + |v_l|_{t,K,L}\,,\\
\langle u,v\rangle = u(1)\overline{v(1)} - u(0)\overline{v(0)}\,, &\qquad \langle u\rangle^2 = |u(1)|^2 + |u(0)|^2\,,\\
\overline{D}^L u(t,x) = (\partial_t^j \partial_x^k u(t,x)\,, 0 \le j+k \le L)\,, &\qquad \partial_t^j = \partial^j/\partial t^j\,, \qquad \partial_x^k = \partial^k/\partial x^k\,.
\end{aligned}
$$

We also write $u_x = \partial_x u$, $u_t = \partial_t u$, $u_{xx} = \partial_x^2 u$, $u_{tx} = \partial_t\partial_x u$, $u_{tt} = \partial_t^2 u$. Moreover, $\overline{\partial}_s^m u = (u, \partial_s u, \ldots, \partial_s^m u)$ for $s = t, x$. We use the same letter C to denote various positive constants and $C(A,B,\ldots)$ means that the constant depends essentially on $A, B, \ldots$ only.

2. Decay rate of solutions to linearized problem. In this section, we investigate the decay of solutions to the linear problem

$$
\begin{aligned}
&(2.1)\quad u_{tt} - \alpha u_{xx} + \delta\theta_x = f_\Omega \qquad \text{in } [0,t_0] \times (0,1)\,,\\
&(2.2)\quad \beta\theta_t - \gamma\theta_{xx} + \delta u_{tx} = g_\Omega \qquad \text{in } [0,t_0] \times (0,1)\,,\\
&(2.3)\quad (\alpha u_x - \delta\theta)(t,l) = f_{\Gamma l}(t)\,, \quad \gamma\theta_x(t,l) = 0 \quad \text{for } l = 0,1 \text{ and } t \in [0,t_0]\,,
\end{aligned}
$$

where α, β and γ are positive constants and δ is a non-zero real number. The purpose of this section is to prove

THEOREM 2.1. *Let* $t_0 > 1$, $0 < \tau < 1$ *and* K *be an integer* ≥ 1. *Let* u *and* θ

satisfy (2.1)–(2.3) *and*

$$(2.4.a)\quad u(t,x)\in\bigcap_{j=0}^{2}C^{4K+6+j}([0,t_0],H^{3-j})\,,$$

$$(2.4.b)\quad \theta(t,x)\in C^{4K+8}([0,t_0],L^2)\cap C^{4K+7}([0,t_0],H^2)\cap C^{4K+6}([0,t_0],H^3)\,.$$

Then, we have the following decay estimate of solutions u and θ to (2.1)–(2.3):

$$\begin{aligned}(2.5)\quad &\|\alpha u_x-\delta\theta\|_{t,K,1}+\|\overline{\partial}_t^1\overline{\partial}_x^1\theta_x\|_{t,K,0}\\ &\le C(K)\Big\{\|\overline{\partial}_t^{4K+7}u(0,\cdot)\|_1+\|\overline{\partial}_t^{4K+6}u(0,\cdot)\|_2\\ &\quad+\|\overline{\partial}_t^{4K+7}\theta(0,\cdot)\|_2+\|\overline{\partial}_t^{4K+6}\theta(0,\cdot)\|_1+\|\overline{\partial}_t^{4K+6}f_{\Omega,x}\|_{t,K+\tau+1,0}\\ &\quad+\|\overline{\partial}_t^{4L+6}g_{\Omega}\|_{t,K+\tau+1,1}+\sum_{l=0}^{1}|f_{\Gamma l}|_{t,K+\tau+1,4K+7}\Big\}\end{aligned}$$

for $0\le t\le t_0$, *where* $\overline{\partial}_t^1\overline{\partial}_x^1\theta_x=(\theta_x,\theta_{xx},\theta_{tx},\theta_{txx})$.

We shall prove Theorem 2.1 below, dividing the proof into several paragraphs.

2.1. *Reduction of equations*. Since the Neumann boundary condition seems to be more complicated to deal with than the Dirichlet boundary condition, and since Racke and Shibata [3] developed a technique for dealing with the Dirichlet condition case, we shall reduce the problem (2.1)–(2.4) to Dirichlet problem. Put

$$(2.6)\qquad v=\alpha u_x-\delta\theta\quad\text{and}\quad\kappa=\gamma\theta_x\,.$$

Then v and κ satisfy the Dirichlet problem

$$(2.7)\qquad av_{tt}-bv_{xx}+\delta\kappa_{tx}=F_\Omega\qquad\text{in }[0,t_0]\times(0,1)\,,$$

$$(2.8)\qquad c\theta_t-d\theta_{xx}+\delta v_{tx}=G_\Omega\qquad\text{in }[0,t_0]\times(0,1)\,,$$

$$(2.9)\qquad v(t,l)=f_{\Gamma l}(t)\quad\text{and}\quad\kappa(t,l)=0\qquad\text{for }l=0,1\text{ and }t\in[0,t_0]\,,$$

where $a=\beta$, $b=\alpha\beta+\delta^2$, $c=(\alpha\beta+\delta^2)/\gamma$, $d=\alpha$, $F_\Omega=(\alpha\beta+\delta^2)f_{\Omega,x}-\delta g_{\Omega,t}$ and $G_\Omega=\alpha g_{\Omega,x}$. Indeed, by using (2.6), we easily get (2.7), so we may omit the proof.

2.2. *Spectral analysis*. To get the decay properties of solutions to (2.7)–(2.9), changing ∂_t to ik, where $i=\sqrt{-1}$ and $k\in\mathbb{C}$, we consider the following system of ordinary differential equations of second order with parameter $k\in\mathbb{C}$:

$$(2.10)\qquad bu''+ak^2u-ik\delta\theta'=f_\Omega\qquad\text{in }(0,1)\,,$$

$$(2.11)\qquad d\theta''-ikc\theta-ik\delta u'=g_\Omega\qquad\text{in }(0,1)\,,$$

$$(2.12)\qquad u(l)=f_{\Gamma l}\quad\text{and}\quad\theta(l)=0\qquad\text{for }l=0\text{ and }1\,.$$

THEOREM 2.2. *There exists a discrete set Λ in $\mathbb{C}$ and operators $R_l(k)$, $l=1,2$, $k\in\mathbb{C}-\Lambda$, with the following properties*: R_l, $l=1,2$, *is a holomorphic map from*

$\mathbb{C} - \Lambda$ to $L(H, H^2)$, where $H = L^2 \times L^2 \times \mathbb{C}^2$ and $L(H, H^2)$ is the space of all bounded linear operators from H into H^2; moreover,

$$\Lambda \cap \{k \in \mathbb{C} \mid \operatorname{Im} k \le 0\} = \emptyset\,, \tag{2.13}$$

and $u = R_1(l)U$ and $\theta = R_2(k)U$ satisfy the problem (2.10)–(2.12) *for $k \in \mathbb{C} - \Lambda$ and $U = (f_\Omega, g_\Omega, f_{\Gamma 0}, f_{\Gamma 1}) \in H$.*

Employing the same arguments as in Racke and Shibata [3, Lemmas 1.1–1.6 of §1], we can construct the operators $R_l(k)$ by using the well-known analytic Fredholm theorem. So, we may omit the proof.

Theorem 2.3. *Let $k \in \mathbb{C}$ with $|\operatorname{Re} k| \ge 1$ and $\operatorname{Im} k \le 0$. For $U = (f_\Omega, g_\Omega, f_{\Gamma 0}, f_{\Gamma 1}) \in H$, we put*

$$I(k) = |k|\,\|f_\Omega\| + |k|^{-1}\|g_\Omega\| + |k|^{5/2}(|f_{\Gamma 0}| + |f_{\Gamma 1}|)\,.$$

Then we have for $j = 0, 1, 2$,

$$\|R_1(k)U\|_j \le C|k|^{2+j} I(k) \quad \text{and} \quad \|R_2(k)U\|_j \le C|k|^{2j} I(k)$$

where $\|\cdot\|_0 = \|\cdot\|$.

Differentiating (2.10)–(2.12) with respect to k and using Theorem 2.3, by induction we easily get

Corollary 2.4. *Let $k \in \mathbb{C}$ with $|\operatorname{Re} k| \ge 1$ and $\operatorname{Im} k \le 0$. Under the same notation as in Theorem* 2.3,

$$\|(d/dk)^l R_1(k)U\|_j \le C(l)|k|^{4l+2+j} I(k)\,,$$
$$\|(d/dk)^l R_2(k)U\|_j \le C(l)|k|^{4l+2j} I(k)$$

for $j = 0, 1, 2$ and any integer $l \ge 0$.

Proof of Theorem 2.3. Put $u = R_1(k)U$, $\theta = R_2(k)U$ and $|f_\Gamma| = |f_{\Gamma 0}| + |f_{\Gamma 1}|$. We shall use the following six inequalities:

$$\|\theta\| \le C\{|k|^{-1/2}\|f_\Omega\|^{1/2}\|u\|^{1/2} + |k|^{-1}\|g_\Omega\| + |k|^{-1/2}\langle u'\rangle^{1/2}|f_\Gamma|^{1/2}\}\,; \tag{2.14}$$

$$\langle \theta' \rangle \le C\{\|\theta'\| + |k|^{1/2}\|\theta\|^{1/2}\|\theta'\|^{1/2} + |k|^{1/2}\|u'\|^{1/2}\|\theta'\|^{1/2} + \|g_\Omega\|\}\,; \tag{2.15}$$

$$\langle u' \rangle \le C\{\|u'\| + |k|\,\|u'\|^{1/2}\|u\|^{1/2} + |k|\,\|u'\|^{1/2}\|\theta'\|^{1/2} + \|f_\Omega\|\}\,; \tag{2.16}$$

$$\|u\| \le C\{|f_\Gamma| + \|\theta\| + |k|^{-1}\|\theta'\| + |k|^{-1}\langle\theta'\rangle + |k|^{-1}\|g_\Omega\|\}\,; \tag{2.17}$$

$$\|\theta'\| \le C\{|k|^{1/2}\|u'\|^{1/2}\|\theta\|^{1/2} + \|g_\Omega\|^{1/2}\|\theta\|^{1/2}\}\,; \tag{2.18}$$

$$\begin{aligned}\|u'\| \le C\{&|k|\,\|u\| + \|\theta'\| + |\operatorname{Re} k|^{-1}\|f_\Omega\| \\ &+ |\operatorname{Re} k|^{-1/2}|k|^{1/2}(\langle u'\rangle^{1/2}|f_\Gamma|^{1/2} + \|\theta\|^{1/2}\|g_\Omega\|^{1/2})\}\,.\end{aligned} \tag{2.19}$$

Before explaining how to get (2.14)–(2.19), we shall prove the estimates of $R_l(k)$, $l = 1, 2$. Since the equations are linear, we decompose $R(k)U = (R_1(k)U, R_2(k)U)$

as follows:

$$R(k)U = \sum_{j=1}^{3} (u_j, \theta_j)$$

where $(u_1, \theta_1) = R(k)(f_\Omega, 0, 0, 0)$, $(u_2, \theta_2) = R(k)(0, g_\Omega, 0, 0)$ and $(u_3, \theta_3) = R(k)(0, 0, f_{\Gamma 0}, f_{\Gamma 1})$.

Step 1. We prove

$$\begin{aligned} \|\theta_1\| &\le C|k|\,\|f_\Omega\|\,, \qquad \|\theta_1'\| \le C|k|^3\|f_\Omega\|\,, \\ \|u_1\| &\le C|k|^3\|f_\Omega\|\,, \qquad \|u_1'\| \le C|k|^4\|f_\Omega\|\,. \end{aligned} \tag{2.20}$$

For simplicity, we write $u_1 = u$ and $\theta_1 = \theta$ in the course of the proof of (2.20). Note that $g_\Omega = 0$, $f_{\Gamma 0} = f_{\Gamma 1} = 0$ in this case. By (2.14),

$$\|\theta\| \le C|k|^{-1/2}\|f\|^{1/2}\|u\|^{1/2}\,. \tag{2.21}$$

Substituting (2.21) into (2.18), we have

$$\|\theta'\| \le C|k|^{1/4}\|f\|^{1/4}\|u'\|^{1/2}\|u\|^{1/4}\,. \tag{2.22}$$

Substituting (2.21) and (2.22) into (2.15), we have

$$\begin{aligned} \langle\theta'\rangle \le C\{&|k|^{1/4}\|f_\Omega\|^{1/4}\|u'\|^{1/2}\|u\|^{1/4} + |k|^{3/8}\|f_\Omega\|^{3/8}\|u'\|^{1/4}\|u\|^{3/8} \\ &+|k|^{5/8}\|f_\Omega\|^{1/8}\|u'\|^{3/4}\|u\|^{1/8}\}\,. \end{aligned} \tag{2.23}$$

Substituting (2.23) into (2.17) and using (2.21) and (2.22), we have

$$\begin{aligned} \|u\| \le C\{&|k|^{-1/2}\|f_\Omega\|^{1/2}\|u\|^{1/2} + |k|^{-3/4}\|f_\Omega\|^{1/4}\|u'\|^{1/2}\|u\|^{1/4} \\ &+|k|^{-5/8}\|f_\Omega\|^{3/8}\|u'\|^{1/4}\|u\|^{3/8} + |k|^{-3/8}\|f_\Omega\|^{1/8}\|u'\|^{3/4}\|u\|^{1/8}\}\,. \end{aligned} \tag{2.24}$$

Now, we use the well-known inequality

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{\varepsilon q}b^q \quad \text{for } a, b \ge 0\,,\ \varepsilon > 0\,,\ \text{and } p, q \ge 1\,,\ \frac{1}{p} + \frac{1}{q} = 1\,. \tag{2.25}$$

Then, applying (2.25) to (2.24), we have

$$\begin{aligned} \|u\| \le C\{&|k|^{-1}\|f_\Omega\| + |k|^{-1}\|f_\Omega\|^{1/3}\|u'\|^{2/3} + |k|^{-1}\|f_\Omega\|^{3/5}\|u'\|^{2/5} \\ &+|k|^{-3/7}\|f_\Omega\|^{1/7}\|u'\|^{6/7}\}\,. \end{aligned} \tag{2.26}$$

Substituting (2.26) into (2.22), we have

$$\begin{aligned} \|\theta'\| \le C\{&\|f_\Omega\|^{1/2}\|u'\|^{1/2} + \|f_\Omega\|^{1/3}\|u'\|^{2/3} + \|f_\Omega\|^{2/5}\|u'\|^{3/5} \\ &+|k|^{1/7}\|f_\Omega\|^{2/7}\|u'\|^{5/7}\}\,. \end{aligned} \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.19), we have

$$\begin{aligned} \|u'\| \le C\{&\|f_\Omega\| + \|f_\Omega\|^{1/3}\|u'\|^{2/3} + \|f_\Omega\|^{3/5}\|u'\|^{2/5} \\ &+ |k|^{4/7}\|f_\Omega\|^{1/7}\|u'\|^{6/7} + \|f_\Omega\|^{1/2}\|u'\|^{1/2} + \|f_\Omega\|^{2/5}\|u'\|^{3/5} \\ &+|k|^{1/7}\|f_\Omega\|^{2/7}\|u'\|^{5/7}\}\,. \end{aligned} \tag{2.28}$$

Applying (2.25) to (2.28), we have

$$\|u'\| \le C|k|^4\|f_\Omega\|\,. \tag{2.29}$$

Substituting (2.29) into (2.26) and (2.27), we have

$$\|u\|\,,\ \|\theta'\| \le C|k|^3\|f_\Omega\|\,. \tag{2.30}$$

Substituting (2.30) into (2.21), we have (2.20).

S t e p 2. Employing the same arguments as in Step 1, we prove

$$\begin{aligned} &\|\theta_2\| \le C|k|^{-1}\|g_\Omega\|\,, \qquad \|\theta_2'\| \le C|k|\,\|g_\Omega\|\,,\\ &\|u_2\| \le C|k|\,\|g_\Omega\|\,, \qquad \|u_2'\| \le C|k|^2\|g_\Omega\|\,. \end{aligned} \tag{2.31}$$

S t e p 3. We prove

$$\begin{aligned} &\|\theta_3\| \le C|k|^{5/2}|f_\Gamma|\,, \qquad \|\theta_3'\| \le C|k|^{9/2}|f_\Gamma|\,,\\ &\|u_3\| \le C|k|^{9/2}|f_\Gamma|\,, \qquad \|u_3'\| \le C|k|^{11/2}|f_\Gamma|\,. \end{aligned} \tag{2.32}$$

For simplicity, we put $u_3 = u$, $\theta_3 = \theta$ and $M = \langle u'\rangle$. Note that $f_\Omega = g_\Omega = 0$ in this case. By (2.14), we have

$$\|\theta\| \le C|k|^{-1/2}M^{1/2}|f_\Gamma|^{1/2}\,. \tag{2.33}$$

By (2.18) and (2.33), we have

$$\|\theta'\| \le C|k|^{1/4}\|u'\|^{1/2}M^{1/4}|f_\Gamma|^{1/4}\,. \tag{2.34}$$

Combining (2.15), (2.33) and (2.34), we have

$$\begin{aligned} \langle\theta'\rangle \le C\{&|k|^{1/4}\|u'\|^{1/2}M^{1/4}|f_\Gamma|^{1/4} + |k|^{3/8}\|u'\|^{1/4}M^{3/8}|f_\Gamma|^{3/8}\\ &+|k|^{5/8}\|u'\|^{3/4}M^{1/8}|f_\Gamma|^{1/8}\}\,. \end{aligned} \tag{2.35}$$

Substituting (2.33), (2.34) and (2.35) into (2.17), we have

$$\begin{aligned} \|u\| \le C\{&|f_\Gamma| + |k|^{-1/2}M^{1/2}|f_\Gamma|^{1/2} + |k|^{-3/4}\|u'\|^{1/2}M^{1/4}|f_\Gamma|^{1/4}\\ &+|k|^{-5/8}\|u'\|^{1/4}M^{3/8}|f_\Gamma|^{3/8} + |k|^{-3/8}\|u'\|^{3/4}M^{1/8}|f_\Gamma|^{1/8}\}\,. \end{aligned} \tag{2.36}$$

Substituting (2.34) and (2.36) into (2.19) yields

$$\begin{aligned} \|u'\| \le C\{&|k|\,|f_\Gamma| + |k|^{1/2}M^{1/2}|f_\Gamma|^{1/2} + |k|^{1/4}\|u'\|^{1/2}M^{1/4}|f_\Gamma|^{1/4}\\ &+|k|^{3/8}\|u'\|^{1/4}M^{3/8}|f_\Gamma|^{3/8} + |k|^{5/8}\|u'\|^{3/4}M^{1/8}|f_\Gamma|^{1/8}\}\,. \end{aligned} \tag{2.37}$$

Applying (2.25) to (2.37), we have

$$\|u'\| \le C|k|\{|f_\Gamma| + |k|^{3/2}M^{1/2}|f_\Gamma|^{1/2}\}\,. \tag{2.38}$$

Substituting (2.38) into (2.34) and (2.36) yields

$$\|\theta'\|\,,\ \|u\| \le C\{|f_\Gamma| + |k|^{3/2}M^{1/2}|f_\Gamma|^{1/2}\}\,. \tag{2.39}$$

Substituting (2.38) and (2.39) into (2.16) yields

$$M \le C\{|k|^{3/2}|f_\Gamma| + |k|^3M^{1/2}|f_\Gamma|^{1/2}\}\,. \tag{2.40}$$

Applying (2.25) to (2.40) implies that $M \le C|k|^6|f_\Gamma|$; substituting this into (2.33), (2.38) and (2.39), we have (2.32).

Using (2.20), (2.31) and (2.32), we have the conclusion of the theorem for $j=0$ and 1. Finally, by using (2.10) and (2.11) and the estimates for $j=0$ and 1, we have the estimates for $j=2$.

Now, we shall prove the inequalities (2.14)–(2.19). By integration by parts, we have

$$b\|u'\|^2 - ak^2\|u\|^2 + ik\delta(\theta', u) = b\langle u', u\rangle - (f_\Omega, u)\,; \tag{2.41}$$

$$d\|\theta'\|^2 + ikc\|\theta\|^2 + ik\delta(u', \theta) = d\langle\theta', \theta\rangle - (g_\Omega, \theta)\,. \tag{2.42}$$

From the real part of (2.42) it follows that

$$d\|\theta'\|^2 - (\operatorname{Im} k)c\|\theta\|^2 = \operatorname{Re}\{-ik\delta(u', \theta) + d\langle\theta', \theta\rangle - (g_\Omega, \theta)\}\,.$$

Since $\operatorname{Im} k \le 0$, we have (2.18). Taking the complex conjugate of (2.42) and using the identity $(\theta, u') = \langle\theta, u\rangle - (\theta', u)$, we have

$$d\|\theta'\|^2 - i\overline{k}c\|\theta\|^2 - i\overline{k}\delta\langle\theta, u\rangle + i\overline{k}\delta(\theta', u) = d\langle\theta, \theta'\rangle - (\theta, g_\Omega)\,.$$

Multiplying this and (2.42) by k and $\overline{k}$, respectively, we have

$$\begin{aligned} b\overline{k}\|u'\|^2 - a|k|^2k\|u\|^2 - dk\|\theta'\|^2 + i|k|^2c\|\theta\|^2 \\ = b\overline{k}\langle u', u\rangle - \overline{k}(f_\Omega, u) - i|k|^2\delta\langle\theta, u\rangle - dk\langle\theta, \theta'\rangle - k(\theta, g_\Omega)\,. \end{aligned} \tag{2.43}$$

Note that $\operatorname{Im}\overline{k} = -\operatorname{Im} k \ge 0$. (2.14) and (2.19) follow from the imaginary part and real part of (2.43), respectively. Since

$$2\operatorname{Re}(\theta'', (2x-1)\theta') = \langle\theta'\rangle^2 - 2\|\theta\|^2\,, \tag{2.44}$$

substituting (2.11) into the left-hand side of (2.44) and using Schwarz's inequality, we have (2.15). Employing the same arguments implies (2.16), too. Finally, integration of (2.11) on $(x, 1)$ yields

$$u(x) = (i\delta k)^{-1}\Big\{\int_x^1 g_\Omega(s)\,ds + i\delta k f_{\Gamma 1} + ick\int_x^1 \theta(s)\,ds - d(\theta'(1) - \theta'(0))\Big\}\,,$$

from which (2.17) follows immediately. This completes the proof of the theorem.

2.3. *Decay rate of solutions to* (2.7)–(2.9)

Theorem 2.5. *Let K be an integer ≥ 1 and $0 < \tau < 1$. Let v and κ satisfy* (2.7)–(2.9) *with $t_0 = \infty$ and the regularity condition*

$$v \in \bigcap_{j=0}^{2} C^{4K+6+j}([0,\infty), H^{2-j})\,, \qquad \kappa \in \bigcap_{j=0}^{1} C^{4K+6+j}([0,\infty), H^{2-j})\,. \tag{2.45}$$

In addition, assume that $I(4K+6, 4K+7, K) < \infty$, where

$$I(L, M, K) = \|\overline{\partial}_t^L(F_\Omega, G_\Omega)\|_{\infty, K+\tau+1, 0} + |(f_{\Gamma 0}, f_{\Gamma 1})|_{\infty, K+\tau+1, M}\,.$$

Then for any $t > 0$ we have

$$\begin{aligned} &\|v\|_{t,K,1} + \|\overline{\partial}_t^1\overline{\partial}_x^1\kappa\|_{t,K,0} \\ &\quad \le C(K)\{\|\overline{\partial}_t^{4K+6}\overline{D}^1 v(0,\cdot)\| + \|\overline{\partial}_t^{4K+6}\overline{\partial}_x^1\kappa(0,\cdot)\| + I(4K+6, 4K+7, K)\}\,. \end{aligned}$$

Proof. Let $\varphi(t)$ be a function in $C^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ for $t \geq 2$ and $= 0$ for $t \leq 1$. Put $u = \varphi v$ and $\theta = \varphi\kappa$. Then u and θ satisfy

$$au_{tt} - bu_{xx} + \delta\theta_{tx} = f_\Omega \quad \text{in } [0,\infty) \times (0,1)\,, \tag{2.46}$$

$$c\theta_t - d\theta_{xx} + \delta u_{tx} = g_\Omega \quad \text{in } [0,\infty) \times (0,1)\,, \tag{2.47}$$

$$u(t,l) = \varphi(t) f_{\Gamma l}(t)\,, \quad \theta(t,l) = 0 \quad \text{for } l = 0 \text{ and } 1, \text{ and } t \in [0,\infty)\,, \tag{2.48}$$

where

$$\begin{aligned} f_\Omega &= \varphi(t) F_\Omega(t,x) - 2a\varphi'(t) v_t(t,x) - a\varphi''(t) v(t,x) - \delta\varphi'(t)\kappa_x(t,x)\,, \\ g_\Omega &= \varphi(t) G_\Omega(t,x) - c\varphi'(t)\kappa(t,x) - \delta\kappa'(t) v_x(t,x)\,. \end{aligned} \tag{2.49}$$

Put

$$\widehat{H}_\Omega(k,x) = \int_{-\infty}^{\infty} e^{-ikt} H_\Omega(t,x)\,dt \qquad \text{for } H = F \text{ and } G\,,$$

$$\widehat{f}_{\Gamma l}(k) = \int_{-\infty}^{\infty} e^{-ikt} \varphi(t) f_{\Gamma l}(t)\,dt \quad \text{for } l = 0 \text{ and } 1\,.$$

Moreover, put

$$w(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} R_1(k) U(k)\,dk\,, \quad \xi(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} R_2(k) U(k)\,dk$$

where $U(k) = -(\widehat{F}_\Omega(k,\cdot), \widehat{G}_\Omega(k,\cdot), \widehat{f}_{\Gamma 0}(k), \widehat{f}_{\Gamma 1}(k))$. Then, employing the same arguments as in the proof of Theorem 2.1 in Racke and Shibata [3, §2], by Theorem 2.2 and Corollary 2.4 and the uniqueness of solutions to the problem (2.46)–(2.48) which will be guaranteed by the energy inequality below, we see that $w = u$ and $\xi = \theta$. Moreover, for $t \geq 2$ we have

$$\begin{aligned} &\|\overline{D}^1 v(t,\cdot)\| + \|\overline{\partial}_t^1 \overline{\partial}_x^1 \kappa(t,\cdot)\| = \|\overline{D}^1 w(t,\cdot)\| + \|\overline{\partial}_t^1 \overline{\partial}_x^1 \xi(t,\cdot)\| \\ &\quad \leq C(K)(1+t)^{-K}\{\|\overline{\partial}_t^{4K+6}(f_\Omega, g_\Omega)\|_{\infty,K+\tau+1,0} + |(f_{\Gamma 0}, f_{\Gamma 1})|_{\infty,K+\tau+1,4K+7}\} \\ &\quad \leq C(K)(1+t)^{-K}\{I(4K+6, 4K+7, K) \\ &\qquad\qquad + \max_{0\leq t\leq 2} \|\overline{\partial}_t^{4K+6}(\overline{D}^1 v(t,\cdot), \overline{\partial}_x^1 \kappa(t,\cdot))\|\} \end{aligned} \tag{2.50}$$

where in the final step of (2.50) we have used the facts that $\operatorname{supp}\varphi'(t), \operatorname{supp}\varphi''(t) \subset [1,2]$ (cf. (2.49)). To estimate the final two terms of (2.50), we give the energy estimate for the problem (2.7)–(2.9). Namely, we show that

$$\begin{aligned} &\|(v_t(t,\cdot), v_x(t,\cdot), \kappa_x(t,\cdot))\|^2 + \int_0^t \|\kappa_s(s,\cdot)\|^2\,ds \\ &\qquad \leq Ce^{Ct}\Big[\|(v_t(0,\cdot), v_x(0,\cdot), \kappa_x(0,\cdot))\|^2 \\ &\qquad\quad + \int_0^t \{\|(F_\Omega(s,\cdot), G_\Omega(s,\cdot))\|^2 + |(f'_{\Gamma 0}(s), f'_{\Gamma 1}(s))|^2\}\,ds\Big]\,. \end{aligned} \tag{2.51}$$

Once we get (2.51), differentiating (2.7)–(2.9) l times ($1 \le l \le 4K+6$) with respect to t and applying (2.51) to the resulting equations, we have

$$\text{(2.52)} \quad \begin{aligned} &\|\overline{\partial}_t^{4K+6}(v_t(t,\cdot), v_x(t,\cdot), \kappa_x(t,\cdot))\| \\ &\quad \le C[\|\overline{\partial}_t^{4K+6}(v_t, v_x, \kappa_x)\mid_{t=0}\| + I(4K+6, 4K+7, 0)] \quad \text{for } t \in [0,2]\,. \end{aligned}$$

Since $\|w\| \le C\{\|w_x\| + |w(0)|\}$, from (2.52) it follows that

$$\text{(2.53)} \quad \begin{aligned} &\max_{0\le t\le 2} \|\overline{\partial}_t^{4K+6}(\overline{D}^1 v(t,\cdot), \overline{\partial}_x^1 \kappa(t,\cdot))\| \\ &\qquad \le C\{\|\overline{\partial}_t^{4K+6}(\overline{D}^1 v, \overline{\partial}_x^1 \kappa)\mid_{t=0}\| + I(4K+6, 4K+7, 0)\}\,. \end{aligned}$$

Combining (2.53) and (2.50), we have the assertion of the theorem.

Now, let us prove (2.51). Multiplying (2.7) by v_t, we have

$$\text{(2.54)} \quad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\{a\|v_t(t,\cdot)\|^2 + b\|v_x(t,\cdot)\|^2\} + \delta(\kappa_{tx}(t,\cdot), v_t(t,\cdot)) \\ &\qquad = b\langle v_x(t,\cdot), v_t(t,\cdot)\rangle + (F_\Omega(t,\cdot), v_t(t,\cdot))\,. \end{aligned}$$

Noting that $\kappa_t(t,l) = 0$ for $l = 0$ and 1, by integration by parts and (2.8), we have

$$\text{(2.55)} \quad \begin{aligned} \delta(\kappa_{tx}(t,\cdot), v_t(t,\cdot)) &= -(\kappa_t(t,\cdot), \delta v_{tx}(t,\cdot)) \\ &= -(\kappa_t(t,\cdot), G_\Omega(t,\cdot)) + c\|\kappa_t(t,\cdot)\|^2 + \frac{d}{2}\frac{d}{dt}\|\kappa_x(t,\cdot)\|^2\,. \end{aligned}$$

Combining (2.54) and (2.55) implies that

$$\text{(2.56)} \quad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\{a\|v_t(t,\cdot)\|^2 + b\|v_x(t,\cdot)\|^2 + d\|\kappa_x(t,\cdot)\|^2\} + c\|\kappa_t(t,\cdot)\|^2 \\ &\quad = (F_\Omega(t,\cdot), v_t(t,\cdot)) + (G_\Omega(t,\cdot), \kappa_t(t,\cdot)) + b\langle v_x(t,\cdot), v_t(t,\cdot)\rangle \\ &\quad \le \tfrac{1}{2}[\|F_\Omega(t,\cdot)\|^2 + \|v_t(t,\cdot)\|^2 + \sigma\|\kappa_t(t,\cdot)\|^2 + \sigma^{-1}\|G_\Omega(t,\cdot)\|^2 \\ &\qquad + (|\delta| + b\sigma^{-1})|f'_\Gamma(t)|^2 + b\sigma\langle v_x\rangle^2] \quad \text{for any } \sigma \in (0,1)\,, \end{aligned}$$

where $|f'_\Gamma(t)|^2 = |f'_{\Gamma 0}(t)|^2 + |f'_{\Gamma 1}(t)|^2$. To estimate the boundary term $\langle v_x(t,\cdot)\rangle$, we use the identity (2.44). Then by integration by parts and by (2.7)–(2.9), we have

$$\text{(2.57)} \quad \begin{aligned} &\frac{b}{2}\langle v_x(t,\cdot)\rangle^2 + \frac{d}{2}\langle \kappa_x(t,\cdot)\rangle^2 - b\|v_x(t,\cdot)\|^2 - d\|\kappa_x(t,\cdot)\|^2 \\ &\quad = (av_{tt}(t,\cdot) + \delta\kappa_{tx}(t,\cdot) - F_\Omega(t,\cdot), (2x-1)v_x(t,\cdot)) \\ &\qquad + (c\kappa_t(t,\cdot) + \delta v_{tx}(t,\cdot) - G_\Omega(t,\cdot)\,, (2x-1)\kappa_x(t,\cdot)) \\ &\quad = a\frac{d}{dt}(v_t(t,\cdot), (2x-1)v_x(t,\cdot)) - \frac{a}{2}|f'_\Gamma(t)|^2 + a\|v_t(t,\cdot)\|^2 \\ &\qquad + (c\kappa_t(t,\cdot), (2x-1)\kappa_x(t,\cdot)) - (F_\Omega(t,\cdot), (2x-1)v_x(t,\cdot)) \\ &\qquad - (G_\Omega(t,\cdot), (2x-1)\kappa_x(t,\cdot))\,. \end{aligned}$$

Combining (2.56) and (2.57) and choosing $\sigma > 0$ sufficiently small, we have

$$\begin{aligned}(2.58)\qquad \frac{1}{2}\frac{d}{dt}[a\|v_t(t,\cdot)\|^2 + b\|v_x(t,\cdot)\|^2 + d\|\kappa_x(t,\cdot)\|^2 \\ + 2\sigma(av_t(t,\cdot) + \delta\kappa_x(t,\cdot),(2x-1)v_x(t,\cdot))] + \frac{c}{2}\|\kappa_t(t,\cdot)\|^2 \\ \le C[a\|v_t(t,\cdot)\|^2 + b\|v_x(t,\cdot)\|^2 + d\|\kappa_x(t,\cdot)\|^2 + \|F_\Omega(t,\cdot)\|^2 \\ + C(\sigma)\{\|G_\Omega(t,\cdot)\|^2 + |f_\Gamma'(t)|^2\}]\,.\end{aligned}$$

Choosing $\sigma > 0$ so small that

$$\begin{aligned}|2\sigma(av_t(t,\cdot) + \delta\kappa_x(t,\cdot),(2x-1)v_x(t,\cdot))| \\ \le \tfrac{1}{2}(a\|v_t(t,\cdot)\|^2 + b\|v_x(t,\cdot)\|^2 + d\|\kappa_x(t,\cdot)\|^2)\,,\end{aligned}$$

integrating (2.58) from 0 to t and applying Gronwall's inequality to the resulting inequality, we have (2.51), which completes the proof of the theorem.

2.4. *Proof of Theorem* 2.1. Noting (2.6), we get Theorem 2.1 immediately in case $t_0 = \infty$. Employing the same arguments as in the proof of Theorem 2.2 in Racke and Shibata [3, §2], by using the cut-off technique and Theorem 2.1 for $t_0 = \infty$, we can prove Theorem 2.1 for general $t_0 > 1$.

3. A priori estimate of solutions local in time. Let $X(t,x)$ and $T(t,x)$ satisfy the following:

$$(3.1)\qquad X_{tt} = S(X_x,T)_x \quad \text{in } [0,t_0]\times\Omega\,;$$

$$(3.2)\qquad TN(X_x,T)_t = (Q(X_x,T)T_x)_x \quad \text{in } [0,t_0]\times\Omega\,;$$

$$(3.3)\qquad S(X_x,T) = T_x = 0 \quad \text{on } [0,t_0]\times\partial\Omega\,;$$

$$(3.4)\qquad X(0,x) = x + u_0(x)\,,\ X_t(0,x) = u_1(x)\,,\ T(0,x) = T_0 + \theta_0(x) \text{ in } \Omega\,;$$

$$(3.5)\qquad (X_x(t,x),T(t,x)) \in G(B) \quad \text{for all } (t,x)\in[0,t_0]\times[0,1]\,,$$

$$(3.6)\qquad \begin{aligned} X \in \bigcap_{j=0}^{L+2} C^j([0,t_0],H^{L+2-j})\,, \\ T \in C^{L+1}([0,t_0],L^2) \cap \bigcap_{j=0}^{L} C^j([0,t_0],H^{L+2-j})\,.\end{aligned}$$

For simplicity, we shall say that X and T are solutions in $[0,t_0]$ if X and T satisfy all of (3.1)–(3.6). Put $u(t,x) = X(t,x) - X_\infty x$ and $\theta(t,x) = T(t,x) - T_0$. Note that $u_{tt} = X_{tt}$, $\theta_t = T_t$, $u_x = X_x - X_\infty$, and then from (3.1)–(3.4) we easily find the equations which u and θ should satisfy. Let V be the same as in Theorem 1.1; then $V = (u_t,u_x,u_{tt},u_{tx},u_{xx},\theta,\theta_t,\theta_x,\theta_{xx})$. Also, let $Y(t)$, $Y_1(t)$ and $Y_2(t)$ be the same as in Theorem 1.1. We use this notation throughout this section. Moreover, set

$$E_1 = \sum_{j=0}^{L+1} \|\partial_t^j u(0,\cdot)\|_{L+1-j} + \sum_{j=0}^{L-1} \|\partial_t^j \theta(0,\cdot)\|_{L+1-j} + \|\partial_t^L \theta(0,\cdot)\|\,. \tag{3.7}$$

Note that

$$E_1 \le E + (3/2)^{1/2}|1 - X_\infty| + |T_0 - T_\infty|\,. \tag{3.8}$$

Since $|1-X_\infty|$, $|T_0-T_\infty| \to 0$ as $E \to 0$, and since E will be chosen small enough, we choose $\delta > 0$ in such a way that

$$(X_\infty, T_\infty) \in G'(B) = \{(F,T) \in \mathbb{R}^n \mid |F-1| + |T-T_0| < \tfrac{3}{4}B,\ T > \tfrac{3}{4}T_0\}\,. \tag{3.9}$$

Obviously, $G'(B) \subset G(B)$. By (A.3), Assumption (2) and (1.7), we see that

$$\alpha_0 \le \frac{\partial S}{\partial F}(F,T),\ \frac{\partial N}{\partial T}(F,T),\ Q(F,T),\ \left|\frac{\partial^2 \psi}{\partial F \partial T}(F,T)\right| \le \alpha_1 \tag{3.10}$$

for $(F,T) \in G'(B)$ with some positive constants α_0 and α_1.

Our purpose in this section is to prove the following a priori estimate for solutions in $[0,t_0]$.

THEOREM 3.1. *Let X and T be solutions in $[0,t_0]$. Assume that* (1.13) *is valid. Then there exists a $\delta > 0$ such that if $E \le \delta$ then $Y(t) \le 1$ for all $t \in [0,t_0]$.*

To prove Theorem 3.1, we shall essentially use the following.

THEOREM 3.2. *Let X and T be solutions in* $[0,t_0]$. *Assume that* (1.13) *is valid and that $E_1 \le 1$. Then there exists a $\sigma > 0$ such that*

$$Y(t) \le C\{\exp CY(t)\}\{E_1 + (1+Y(t))^{L-1}Y(t)^2\} \tag{3.11}$$

provided that $|V(t,x)| \le \sigma$ for all $(t,x) \in [0,t_0] \times [0,1]$. Here, C is a positive constant independent of X, T, t_0 and σ.

Proof of Theorem 3.1. We assume that Theorem 3.2 is valid. In view of (3.8), we choose $\delta > 0$ in such a way that $E_1 \le 1$. Let $\delta' \in (0,1]$, to be determined in the course of the proof. Put $I = \{t \in [0,t_0] \mid Y(s) \le \delta'$ for $0 \le s \le t\}$. Our task is to prove that $I = [0,t_0]$ under the suitable choice of δ and δ'. Since $Y(0) \le 2E_1$, in view of (3.8), we choose $\delta > 0$ so small that $E_1 < \frac{1}{2}\delta'$ provided that $E \le \delta$. Then $Y(0) < \delta'$ if $E \le \delta$. By the continuity of $Y(s)$, this implies that I is a non-empty set. The continuity of $Y(s)$ also implies that I is closed, so it suffices to prove that I is open. Let $t \in I$, namely, $Y(t) \le \delta'$ (≤ 1). Since $Y(s)$ is monotonically increasing and continuous, it is sufficient to prove that $Y(t) < \delta'$. Let $\sigma > 0$ be the same constant as in Theorem 3.2. By Sobolev's inequality, we know that $|V(s,x)| \le c_1 Y(s)$ for $(s,x) \in [0,t_0] \times [0,1]$ with some constant $c_1 > 0$. Choose $\delta' > 0$ in such a way that $c_1\delta' \le \sigma$. Then $|V(s,x)| \le \sigma$ for $(s,x) \in [0,t] \times [0,1]$. Replacing t_0 by t, we can apply Theorem 3.2. Then from (3.11) we see that $Y(t) \le c_2\{E_1 + Y(t)^2\}$ where $c_2 = 2^{L-1}Ce^C$, where we have used the fact that $Y(t) \le 1$. We choose $\delta > 0$ so small that $c_2E_1 < \delta'/2$ provided that $E \le \delta$. Moreover, we choose δ' in such a way that $c_2\delta' < 1/2$. Then we have $Y(t) \le c_2 E_1 + c_2(\delta')^2 < \delta'/2 + \delta'/2 = \delta'$, which completes the proof of Theorem 3.1.

Proof of Theorem 3.2. Choose $\sigma > 0$ so small that

$$(X_\infty + u_x(t,x), T_\infty + \theta(t,x)) \in G'(B) \quad \text{for all } (t,x) \in [0,t_0] \times [0,1] .$$

We begin with L^2 estimates of higher order derivatives. Put

$$E_l(t)^2 = \|\partial_t^l(u_t(t,\cdot), u_x(t,\cdot), \theta(t,\cdot))\|^2 + \int_0^t \|\partial_s^l \theta_s(s,\cdot)\|^2\, ds .$$

First, we shall estimate $E_L(t)$. Differentiating (3.1)–(3.3) L times with respect to t, we have

$$\partial_t^2 v - (S_F v_x + S_T \xi + F_L^1)_x = 0 \quad \text{in } [0,t_0] \times \Omega , \tag{3.12}$$

$$N_T \xi_t + N_F v_{tx} - T^{-1}(Q\xi_x + \partial_t^L Q \cdot \theta_x)_x = F_L^2 + G_L \quad \text{in } [0,t_0] \times \Omega , \tag{3.13}$$

$$S_F v_x + S_T \xi + F_L^1 = \xi_x = 0 \quad \text{on } [0,t_0] \times \partial\Omega , \tag{3.14}$$

where
(3.15)

$$\begin{aligned}
&v = \partial_t^L u , \quad \xi = \partial_t^L \theta ,\\
&R_G = R_G(t) = (\partial R/\partial G)(X_\infty + u_x, T_\infty + \theta) \quad \text{for } R = N, S, \text{and } G = F, T ;\\
&F_L^1 = F_L^1(t) = \partial_t^L S - (S_F \partial_t^L u_x + S_T \partial_t^L \theta) ;\\
&F_L^2 = F_L^2(t) = \{\partial_t^L N - (N_T \partial_t^L \theta + N_F \partial_t^L u_x)\}_t + (N_T)_t \partial_t^L \theta + (N_F)_t \partial_t^L u_x ;\\
&G_L = G_L(t) = \partial_t^L\{T^{-1}(Q\theta_x)_x\} - T^{-1}(Q\partial_t^L \theta_x + \partial_t^L Q \cdot \theta_x)_x .
\end{aligned}$$

Note that $-S_T = N_F$. Multiplying (3.12) and (3.13) by $\partial_t v$ and ξ, respectively, integrating the resulting equations on Ω and using (3.14), we have

$$\begin{aligned}
0 = {} & \frac{1}{2}\frac{d}{dt}\{\|\partial_t v\|^2 + (S_F v_x, v_x) + (N_T \xi, \xi) + 2(F_L^1, v_x)\} - ((F_L^1)_t, v_x)\\
& - \tfrac{1}{2}((N_T)_t \xi, \xi) - \tfrac{1}{2}((S_F)_t v_x, v_x) + (T^{-1}(Q\xi_x + \partial_t^L Q \cdot \xi_x), \xi_x)\\
& + ((T^{-1})_x Q \xi_x, \xi) - (F_L^2 + G_L, \xi) .
\end{aligned} \tag{3.16}$$

Now, we use the following trick:

$$\begin{aligned}
|(F_L^1, v_x)| &\le (\alpha_0/4)\|v_x\|^2 + \alpha_0^{-1}\|F_L^1\|^2\\
&\le (\alpha_0/4)\|v_x\|^2 + 2\alpha_0^{-1}\|F_L^1(0)\|^2 + 2\alpha_0^{-1}\Big(\int_0^t \|\partial_s F_L^1(s)\|\, ds\Big)^2\\
&\le (\alpha_0/4)\|v_x\|^2 + 2\alpha_0^{-1}\|F_L^1(0)\|^2 + 2\alpha_0^{-1}\tau^{-2}\|(F_L^1)_t\|_{t,1+\tau,0}^2 ;
\end{aligned} \tag{3.17.a}$$

$$\begin{aligned}
\Big|\int_0^t ((F_L^1)_t, v_x)\, ds\Big| \le {} & \frac{1}{2}\int_0^t (1+s)^{-(1+\tau)}\|v_x(s,\cdot)\|^2\, ds\\
& + (2\tau^2)^{-1}\|(F_L^1)_t\|_{t,1+\tau,0}^2 ;
\end{aligned} \tag{3.17.b}$$

$$\left| \int_0^t (F_L^2 + G_L, \xi)\, ds \right| \le \frac{1}{2} \int_0^t (1+s)^{-(1+\tau)} \|\xi(s,\cdot)\|^2\, ds + \tau^{-2}(\|F_L^2\|^2_{t,1+\tau,0} + \|G_L\|^2_{t,1+\tau,0})\,; \tag{3.17.c}$$

$$\begin{aligned} &\int_0^t (T^{-1}(Q\xi_x + \partial_t^L Q \cdot \theta_x), \xi_x)\, ds \\ &\quad \ge \frac{4\alpha_0}{3T_0} \int_0^t \|\xi_x(s,\cdot)\|^2\, ds - \int_0^t \|\partial_t^L Q(s,\cdot) \cdot \theta_x(s,\cdot)\|\, \|\xi_x(s,\cdot)\|\, ds \\ &\quad \ge \frac{2\alpha_0}{3T_0} \int_0^t \|\xi_x(s,\cdot)\|^2\, ds - C\tau^{-2} \|\partial_t^L Q \cdot \theta_x\|^2_{t,(1+\tau)/2,0}\,; \end{aligned} \tag{3.17.d}$$

$$\begin{aligned} &\left| \int_0^t ((N_T)_t \xi, \xi)\, ds \right| + \left| \int_0^t ((S_F)_t v_x, v_x)\, ds \right| \\ &\quad \le |||(N_T)_t|||_{t,1+\tau,0} \int_0^t (1+s)^{-(1+\tau)} \|\xi(s,\cdot)\|^2\, ds \\ &\quad\quad + |||(S_F)_t|||_{t,1+\tau,0} \int_0^t (1+s)^{-(1+\tau)} \|v_x(s,\cdot)\|^2\, ds\,, \end{aligned} \tag{3.17.e}$$

where $|||w|||_{t,1+\tau,0} = \sup\{(1+s)^{(1+\tau)}|w(s,x)| \mid (s,x) \in [0,t] \times [0,1]\}$.

Hence, integrating (3.16) from 0 to t, estimating the resulting formula by using (3.17) and (3.10) and applying Gronwall's inequality, we have

$$E_L(t) \le C\{\exp C I_1(t)\}\{E_2 + I_2(t)\}\,, \tag{3.18}$$

where

$$\begin{aligned} I_1(t) &= |||((N_T)_t, (S_F)_t)|||_{t,1+\tau}\,, \\ I_2(t) &= \|((F_L^1)_t, F_L^2, G_L)\|_{t,1+\tau,0} + \|\partial_t^L Q \cdot \theta_x\|_{t,(1+\tau)/2,0}\,, \\ E_2 &= E_L(0) + \|F_L^1(0)\|\,. \end{aligned} \tag{3.19}$$

Now, we shall estimate $E_l(t)$ for $0 \le l \le L-1$. To do this, we rewrite (3.1)–(3.3) as follows:

$$u_{tt} - (\alpha u_x - \delta\theta + A^1)_x = 0 \qquad \text{in } [0,t_0] \times \Omega\,, \tag{3.20}$$

$$\beta\theta_t - \gamma\theta_{xx} + \delta u_{tx} = -A_t^2 + B \qquad \text{in } [0,t_0] \times \Omega\,, \tag{3.21}$$

$$\alpha u_x - \delta\theta + A^1 = \theta_x = 0 \qquad \text{on } [0,t_0] \times \partial\Omega\,. \tag{3.22}$$

Here, by Taylor expansion, we have put

$$\begin{aligned} \alpha &= (\partial S/\partial F)(X_\infty, T_\infty)\,, \qquad \delta = -(\partial^2\psi/\partial T \partial F)(X_\infty, T_\infty)\,, \\ \beta &= (\partial N/\partial T)(X_\infty, T_\infty)\,, \qquad \gamma = Q(X_\infty, T_\infty) T_\infty^{-1}\,, \end{aligned} \tag{3.23}$$

$$
\begin{aligned}
&A^1 = A^1(u_x,\theta) = S(X_\infty + u_x, T_\infty + \theta) - \alpha u_x + \delta\theta\,,\\
(3.24)\qquad &A^2 = A^2(u_x,\theta) = N(X_\infty + u_x, T_\infty + \theta) - N(X_\infty, T_\infty) - \beta\theta_t - \delta u_{tx}\,,\\
&B = B(\overline{\partial}_x^1 u, \overline{\partial}_x^2\theta) = (T_\infty+\theta)^{-1}[Q(X_\infty + u_x, T_\infty+\theta)\theta_x]_x - \gamma\theta_{xx}\,.
\end{aligned}
$$

Differentiating (3.20)–(3.22) l times with respect to t, and employing the same arguments as in the proof of (3.18), we have

$$
E_l(t) \le C\{E_l(0) + \|\partial_t^l A^1(0)\| + I_3(t)\}\,, \tag{3.25}
$$

where

$$
I_3(t) = \|(A^1, A^2)\|_{t,1+\tau,L} + \|B\|_{t,1+\tau,L-1}\,. \tag{3.26}
$$

Now, we shall estimate the derivatives with respect to x. Using (3.20) and (3.21), we have

$$
\begin{aligned}
(3.27)\quad &\|\overline{\partial}_t^{L-1-P}\partial_x^{P+2}u(t,\cdot)\|\\
&\qquad \le \alpha^{-1}\{\|\overline{\partial}_t^{L-1-P}\partial_x^P\partial_t^2 u(t,\cdot)\| + |\delta|\,\|\overline{\partial}_t^{L-1-P}\partial_x^{P+1}\theta(t,\cdot)\| + I_3(t)\}\,,\\
(3.28)\quad &\|\overline{\partial}_t^{L-1-P}\partial_x^{P+2}\theta(t,\cdot)\|\\
&\qquad \le \gamma^{-1}\{\beta\|\overline{\partial}_t^{L-1-P}\partial_t\partial_x^P\theta(t,\cdot)\| + |\delta|\,\|\overline{\partial}_t^{L-1-P}\partial_t\partial_x^{P+1}u(t,\cdot)\| + I_3(t)\}
\end{aligned}
$$

for $0 \le P \le L-1$. By (3.21) and (3.22), we also have

$$
\begin{aligned}
(3.29)\qquad \|\overline{\partial}_t^{L-1}\partial_x\theta(t,\cdot)\| &= \Big\{\sum_{l=0}^{L-1} -(\partial_t^l\theta(t,\cdot), \partial_t^l\theta_{xx}(t,\cdot))\Big\}^{1/2}\\
&\le C\{\|\overline{\partial}_t^L\theta(t,\cdot)\| + \|\overline{\partial}_t^L u_x(t,\cdot)\| + I_3(t)\}\,.
\end{aligned}
$$

Using (3.27), (3.28) and (3.29), by induction on P we have

$$
\begin{aligned}
(3.30)\quad &\|\overline{\partial}_t^{L-1-P}\partial_x^{P+2}u(t,\cdot)\| + \|\overline{\partial}_t^{L-1-P}\partial_x^{P+2}\theta(t,\cdot)\| + \|\overline{\partial}_t^{L-1-P}\partial_x^{P+1}\theta(t,\cdot)\|\\
&\qquad\qquad \le CI_4(t) \quad \text{for } 0 \le P \le L-1\,,
\end{aligned}
$$

where $I_4(t) = \|\overline{\partial}_t^L(u_t(t,\cdot), u_x(t,\cdot), \theta(t,\cdot))\| + I_3(t)$. Hence, combining (3.18), (3.25) and (3.30), we have

$$
Y_2(t) \le C\{\exp CI_1(t)\}\{E_1 + \|\overline{\partial}_t^{L-1}A^1(0)\| + \|F_L^1(0)\| + I_2(t) + I_3(t)\}\,. \tag{3.31}
$$

Now, we shall estimate the nonlinear terms. To do this, we need the following calculus lemma (cf. [3, §3] for its proof).

LEMMA 3.3. (1) *Let* $t \ge 1$ *and let* L *be an integer* ≥ 1. *Then*

$$
\|f\|_{t,K,N} \le C(K,L)\|f\|_{t,K/\alpha,0}^{\alpha}\|f\|_{t,0,L}^{1-\alpha} \quad \textit{where} \quad N \in (0,L) \textit{ and } \alpha = 1 - NL^{-1}\,.
$$

(2) *Let* $L \ge 1$ *and let* F *be a smooth function defined on* $\{u = (u_1,\dots,u_m) \in \mathbb{R}^m \mid |u| \le u_0\}$. *Assume that* $F(u) = O(|u|^k)$ *near* $u = 0$. *If* $|u(t,x)| \le u_0$ *for* $(t,x) \in [0,t_0] \times \overline{\Omega}$, *then*

$$
\|\overline{D}^L F(u(t,\cdot))\| \le C(F,L)(1 + \|\overline{D}^L u(t,\cdot)\|)^{L-k}\|\overline{D}^L u(t,\cdot)\|^k\,.
$$

(3) *Let* $L \geq 2$. *Then*

$$\|\partial_t^L(u(t,\cdot)v(t,\cdot)) - \partial_t^L u(t,\cdot)\cdot v(t,\cdot) - u(t,\cdot)\partial_t^L v(t,\cdot)\| \\ \leq C(L)\{\|\overline{D}^{L-1}u(t,\cdot)\|\,\|\overline{D}^{[L/2]+1}v(t,\cdot)\| + \|\overline{D}^{[L/2]+1}u(t,\cdot)\|\,\|\overline{D}^{L-1}v(t,\cdot)\|\}\,.$$

(4) *Let* r_j $(1 \leq j \leq m$ *and* $m \geq 2)$, K, L *and* M *be integers such that*

$$L \geq 1\,, \quad \sum_{j=1}^m r_j = L\,, \quad 0 \leq r_1 \leq r_2 \leq \ldots \leq r_m\,, \quad M \geq 1\,, \quad K + L \leq M\,.$$

Then

$$\Big\|\prod_{j=1}^m u_j\Big\|_K \leq C\prod_{j=1}^m \|u_j\|_{M-r_j}\,.$$

First of all, we show that

$$\|V\|_{t,1+\tau,[L/2]+1} \leq CY(t)\,. \tag{3.32}$$

For $t \leq 1$, (3.32) is obvious because $[L/2]+1 \leq L-1$. For $t \geq 1$, by Lemma 3.3(2),

$$\|V\|_{t,1+\tau,[L/2]+1} \leq C\|V\|^{\alpha}_{t,(1+\tau)/\alpha,0}\|V\|^{1-\alpha}_{t,0,L-1}\,,$$

where $\alpha = 1 - ([L/2]+1)(L-1)^{-1}$. Since $(1+\tau)/\alpha \leq K$ as follows from (1.12), we have (3.32).

By Sobolev's inequality and (3.32), we also have

$$|||V|||_{t,1+\tau} \leq CY(t)\,. \tag{3.33}$$

Application of (3.33) to $|||R_G|||_{t,1+\tau}$ (cf. (3.15)) immediately yields

$$I_1(t) \leq CY(t)\,. \tag{3.34}$$

Now, we estimate $I_2(t)$. For simplicity, we use the following notation for the function $Z = Z(X_\infty + u_x, T_\infty + \theta)$: $Z^0 = Z(X_\infty, T_\infty)$ and $Z^1 = Z^1(u_x,\theta) = Z - Z^0$. Note that $Z^1(u_x,\theta) = O(|(u_x,\theta)|)$. Since $\partial_t^L Q = \partial_t^L Q^1$, by (3.33) and Lemma 3.3(2),

$$\|\partial_t^L Q\cdot\theta_x\|_{t,(1+\tau)/2,0} \leq |||\theta_x|||_{t,(1+\tau)/2}\|\partial_t^L Q^1\|_{t,0,0} \leq C(1+Y(t))^{L-1}Y(t)^2\,. \tag{3.35}$$

Here and hereafter, we sometimes use the estimates $\|(u_x,\theta)\|_{t,0,L} \leq Y(t)$. By direct calculation, we have

$$(F_L^1)_t = \{\partial_t^L(S_F^1 u_{tx}) - S_F^1\partial_t^L u_{tx} - \partial_t^L S_F^1\cdot u_{tx}\} + \{\partial_t^L(S_T^1\theta_t) - S_T^1\partial_t^L\theta_t \\ - \partial_t^L S_T^1\cdot\theta_t\} + \partial_t^L S_F^1\cdot u_{tx} + \partial_t^L S_T^1\cdot\theta_t - \partial_t S_F^1\cdot\partial_t^L u_x - \partial_t S_T^1\cdot\partial_t^L\theta\,.$$

Then, applying (2) and (3) of Lemma 3.3 and using (3.32) and (3.33), we have

$$\|(F_L^1)_t\|_{t,1+\tau,0} \leq C(1+Y(t))^{L-1}Y(t)^2\,. \tag{3.36}$$

Performing the same change of the formula for F_L^2, by (2) and (3) of Lemma 3.3, (3.32) and (3.33), we also have

$$\|F_L^2\|_{t,1+\tau,0} \leq C(1+Y(t))^{L-1}Y(t)^2\,. \tag{3.37}$$

Put $T^{-1} = T_\infty^{-1} + a(\theta)$ where $a(\theta) = -\theta/(TT_\infty)$. Since

$$\begin{aligned} G_L = &\{\partial_t^L(a(\theta)(Q\theta_x)_x) - a(\theta)\partial_t^L(Q\theta_x)_x - \partial_t^L a(\theta)\cdot(Q\theta_x)_x\} + \partial_t^L a(\theta)\cdot(Q\theta_x)_x \\ &+ T^{-1}\{\partial_t^L(Q^1\theta_{xx}) - \partial_t^L Q^1\cdot\theta_{xx} - Q^1\partial_t^L\theta_{xx}\} \\ &+ T^{-1}\{\partial_t^L(Q_x^1\theta_x) - \partial_t^L Q_x^1\cdot\theta_x - Q_x^1\partial_t^L\theta_x\}, \end{aligned}$$

and since $(Q\theta_x)_x = O(|(u_x, u_{xx}, \theta, \theta_x, \theta_{xx})|)$, $Q^1 = O(|(u_x,\theta)|)$ and $Q_x^1 = O(|(u_x, u_{xx}, \theta, \theta_x)|)$, by (2) and (3) of Lemma 3.3, (3.32) and (3.33), we have

$$\|G_L\|_{t,1+\tau,0} \le C(1+Y(t))^{L-1}Y(t)^2. \tag{3.38}$$

Combining (3.35)–(3.38), we have

$$I_2(t) \le C(1+Y(t))^{L-1}Y(t)^2. \tag{3.39}$$

Now, we shall estimate $I_3(t)$. Since $S(X_\infty, T_\infty) = 0$, $A^l(u_x,\theta)$, $l = 1,2$, are quadratic forms in (u_x,θ). Thus, we may write symbolically $A^l(u_x,\theta) = a^l(u_x,\theta)(u_x,\theta)$, where $a^l(u_x,\theta) = O(|(u_x,\theta)|)$. Applying (2) and (3) of Lemma 3.3, (3.32) and (3.33), we have

$$\begin{aligned} \|A^l\|_{t,1+\tau,L} \le\ & C\{\|a^l\|_{t,0,L-1}\|(u_x,\theta)\|_{t,1+\tau,[L/2]+1} \\ &+ \|a^l\|_{t,0,[L/2]+1}\|(u_x,\theta)\|_{t,1+\tau,L-1} + \|a^l\|_{t,0,L}|||(u_x,\theta)|||_{t,1+\tau} \\ &+ |||a^l|||_{t,1+\tau}\|(u_x,\theta)\|_{t,0,L}\} \\ \le\ & C(1+Y(t))^{L-1}Y(t)^2 \quad \text{for } l = 1,2. \end{aligned} \tag{3.40}$$

Since $B = a(\theta)(Q\theta_x)_x + T_\infty^{-1}(Q^1\theta_x)_x$, and since both $a(\theta)(Q\theta_x)_x$ and $(Q^1\theta_x)_x$ are $O(|(u_x, u_{xx}, \theta, \theta_x, \theta_{xx})|^2)$, we may write symbolically $B = b(W)W$ where $W = (u_x, u_{xx}, \theta, \theta_x, \theta_{xx})$ and $b(W) = O(|W|)$. Noting that $[(L-1)/2] \le [L/2]+1$ and employing the same arguments as in (3.40) (L should be replaced by $L-1$), by (2) and (3) of Lemma 3.3, (3.32) and (3.33), we have

$$\|B\|_{t,1+\tau,L-1} \le C(1+Y(t))^{L-2}Y(t)^2. \tag{3.41}$$

Combining (3.40) and (3.41), we have

$$I_3(t) \le C(1+Y(t))^{L-1}Y(t)^2. \tag{3.42}$$

Since we can write the estimate of the term $\|\overline{\partial}_t^{L-1}A^1(0)\|$ symbolically as follows:

$$\begin{aligned} \|\overline{\partial}_t^{L-1}A^1(0)\| \le C\Big\{&\|(u_x(0,\cdot),\theta(0,\cdot))\| \\ &+ \sum_{j=1}^{L-1}\sum_{\alpha_j}\|(u_{tx}(0,\cdot),\theta_t(0,\cdot))^{\alpha_j^1}\dots(\partial_t^j u_x(0,\cdot),\partial_t^j\theta(0,\cdot))^{\alpha_j^j}\|\Big\} \end{aligned}$$

where $\alpha_j = (\alpha_j^1, \dots, \alpha_j^j)$ and α_j^k are multi-indices satisfying

$$\sum_{k=1}^{j} k|\alpha_j^k| = j \quad \text{and} \quad \sum_{k=1}^{j}|\alpha_j^k| \le j,$$

applying Lemma 3.3(4), we have

$$\|\overline{\partial}_t^{L-1}A^1(0)\| \le C(1+E_1)^{L-2}E_1\,. \tag{3.43}$$

In the same manner, we see that

$$\|F_L^1(0)\| \le C(1+E_1)^{L-2}E_1\,. \tag{3.44}$$

Since $E_1 \le 1$, combining (3.43), (3.44), (3.42), (3.39), (3.34) and (3.31), we have

$$Y_2(t) \le C\{\exp CY(t)\}\{E_1 + (1+Y(t))^{L-1}Y(t)^2\}\,. \tag{3.45}$$

Now, we estimate $Y_1(t)$. Since $Y_1(t) \le 2^{(1+\tau)}Y(t)$ for $t \le 1$, we consider the case where $t \ge 1$ below. Applying Theorem 2.1 to (3.20)–(3.22), we have

$$\|\alpha u_x - \delta\theta\|_{t,K,1} + \|(\theta_x, \theta_{xx}, \theta_{tx}, \theta_{txx})\|_{t,K,0} \le CI_5(t) \tag{3.46}$$

where $I_5(t) = E_1 + \|(\overline{\partial}_x^1 A^1, A_t^2, B)\|_{t,K+\tau+1,4K+7}$. Here, we have used the fact that $4K+8 \le L$, which follows from (1.12) and the fact that $|v(t,l)| \le C\|\overline{\partial}_x^1 v(t,\cdot)\|$ for $l = 0$ and 1.

Now, let us prove the decay property of u_{xx}, u_{tx}, u_{tt}, u_t, u_x, θ and θ_t. By the identity $u_{xx} = \alpha^{-1}(\alpha u_{xx} - \delta\theta_x) + \delta\alpha^{-1}\theta_x$, we have $\|u_{xx}\|_{t,K,0} \le CI_5(t)$. Since $(\beta + \delta^2\alpha^{-1})\theta_t = \gamma\theta_{xx} - \delta\alpha^{-1}(\alpha u_{xt} - \delta\theta_t) - A_t^2 + B$ as follows from (3.21), we have $\|\theta_t\|_{t,K,0} \le CI_5(t)$. Now, the identity $u_{tx} = \delta^{-1}(\gamma\theta_{xx} - \beta\theta_t - A_t^2 + B)$, which follows also from (3.21), implies that $\|u_{tx}\|_{t,K,0} \le CI_5(t)$. Moreover, by (3.20) we see that $\|u_{tt}\|_{t,K,0} \le CI_5(t)$. Integrating (3.1) on Ω and using (3.3) and (1.13), we have

$$\int_0^1 u_t(t,x)\,dx = \int_0^1 u_t(0,x)\,dx = \int_0^1 u_1(x)\,dx = 0 \quad \text{for } t \in [0,t_0]\,. \tag{3.47}$$

Let us recall the well-known Poincaré inequality:

$$\|v\| \le C\Big\{\|v'\| + \Big|\int_0^1 v(x)\,dx\Big|\Big\} \quad \text{for } v \in H^1\,. \tag{3.48}$$

Combining (3.47) and (3.48) with $v = u_t(t,\cdot)$, we have $\|u_t(t,\cdot)\| \le CI_5(t)$. To deal with the decay property of u_x and θ, we use the following form of Poincaré's inequality:

$$\begin{aligned} &\|v\| + \|w\| \\ &\quad \le C\Big\{\|v'\| + \|w'\| + \Big|\int_0^1 (\delta v(x) + \beta w(x))\,dx\Big| + \Big|\int_0^1 (\alpha v(x) - \delta w(x))\,dx\Big|\Big\} \end{aligned} \tag{3.49}$$

for $v, w \in H^1$. In fact, if we put $p = \delta v + \beta w$ and $q = \alpha v - \delta w$, noting that $v = (\alpha\beta + \delta^2)^{-1}(\delta p + \beta q)$ and $w = (\alpha\beta + \delta^2)^{-1}(\alpha p - \delta q)$ and applying (3.48) to p and q, we have (3.49) immediately. Applying (3.49) to $v = u_x(t,\cdot)$ and $w = \theta(t,\cdot)$, we have

$$\|u_x(t,\cdot)\| + \|\theta(t,\cdot)\| \le C\Big\{\|u_{xx}(t,\cdot)\| + \|\theta_x(t,\cdot)\| \tag{3.50}$$

$$+\Big| \int_0^1 (\delta u_x(t,x) + \beta\theta(t,x))\,dx\Big| + \Big| \int_0^1 (\alpha u_x(t,\cdot) - \delta\theta(t,x))\,dx\Big|\Big\}\,.$$

By (1.5) and (1.6.a), we see that

$$\varepsilon(X_\infty, T_\infty) = \int_0^1 \varepsilon(X_\infty + u_x(t,x), T_\infty + \theta(t,x))\,dx + \frac{1}{2}\int_0^1 u_t(t,x)^2\,dx\,. \tag{3.51}$$

By Taylor expansion, (3.23), (1.6.b), (1.7) and (A.2), we have

$$\varepsilon(X_\infty + u_x, T_\infty + \theta) = \varepsilon(X_\infty, T_\infty) + T_\infty(\delta u_x + \beta\theta) + \varepsilon'(u_x,\theta)\,,$$

where $\varepsilon'(u_x,\theta) = O(|(u_x,\theta)|^2)$. Since $|V(t,x)| \le \sigma$, by (3.51) we have

$$\Big| \int_0^1 (\delta u_x(t,x) + \beta\theta(t,x))\,dx\Big| \le C\sigma(\|u_x(t,\cdot)\| + \|\theta(t,\cdot)\|) + (\sigma/2T_\infty)\|u_t(t,\cdot)\|\,. \tag{3.52}$$

Combining (3.50) and (3.52) and choosing $\sigma > 0$ small enough, we infer that $\|u_x\|_{t,K,0}, \|\theta\|_{t,K,0} \le CI_5(t)$. Hence

$$Y_1(t) \le CI_5(t) \tag{3.53}$$

provided that $\sigma > 0$ is small enough and $|V(t,x)| \le \sigma$ for all $(t,x) \in [0,t_0]\times[0,1]$.

Finally, we estimate $I_5(t)$. First, we show that

$$\|V\|_{t,(K+\tau+1)/2,4K+7} \le CY(t)\,. \tag{3.54}$$

Since $4K+7 \le L-1$, (3.54) is valid for $t \le 1$. By Lemma 3.3(1), we have

$$\|V\|_{t,(K+\tau+1)/2,4K+7} \le C\|V\|^{\alpha}_{t,(K+\tau+1)/2\alpha,0}\|V\|^{1-\alpha}_{t,0,L-1}$$

where $\alpha = 1-(4K+7)(L-1)^{-1}$. Since $(K+\tau+1)/(2\alpha) \le K$ as follows from (1.12), we have (3.54). Recall that $A^l = a^l(u_x,\theta)(u_x,\theta)$ and $B = b(W)W$ where $W = (u_x, u_{xx}, \theta, \theta_x, \theta_{xx})$, $a^l(u_x,\theta) = O(|(u_x,\theta)|)$ and $b(W) = O(|W|)$. Applying Lemma 3.3(2) to A^1, A^2 and B, noting that $4K+7 \le L-1$ and using (3.54), we have

$$\begin{aligned}\|(\overline{\partial}_x^1 A^1, A_t^2, B)\|_{t,K+\tau+1,4K+7} &\le C\{\|(a^1,a^2)\|_{t,(K+\tau+1)/2,4K+8}\|(u_x,\theta)\|_{t,(K+\tau+1)/2,4K+8}\\ &\quad + \|b\|_{t,(K+\tau+1)/2,4K+7}\|W\|_{t,(K+\tau+1)/2,4K+7}\}\\ &\le C(1+Y(t))^{L-1}Y(t)^2\,.\end{aligned}$$

Thus, we have

$$Y_1(t) \le C\{E_1 + (1+Y(t))^{L-1}Y(t)^2\}\,. \tag{3.55}$$

Since $\exp CY(t) \le 1$, combining (3.54) and (3.55) completes the proof of Theorem 3.2.

4. Proof of Theorem 1.1. To prove Theorem 1.1, first of all we shall quote the local existence theorem for the following problem:

$$X_{tt} = S(X_x, T)_x \quad \text{in } [t_1, t_1 + t_2] \times \Omega\,, \tag{4.1}$$

$$TN(X_x, T)_t = (Q(X_x, T)T_x)_x \quad \text{in } [t_1, t_1 + t_2] \times \Omega\,, \tag{4.2}$$

$$S(X_x, T) = T_x = 0 \quad \text{on } [t_1, t_1 + t_2] \times \partial\Omega\,, \tag{4.3}$$

$$X(0, x) = v_0(x)\,, \quad X_t(0, x) = v_1(x)\,, \quad T(0, x) = \xi_0(x) \quad \text{in } \Omega\,. \tag{4.4}$$

To state the regularity of initial data and the compatibility condition, for a moment we assume the existence of solutions X and T to (4.1)–(4.4) satisfying the conditions

$$\begin{gathered} X \in \bigcap_{j=0}^{L+2} C^j([t_1, t_1 + t_2], H^{L+2-j})\,, \\ T \in C^{L+1}([t_1, t_1 + t_2], L^2) \cap \bigcap_{j=0}^{L} C^j([t_1, t_1 + t_2], H^{L+2-j})\,, \end{gathered} \tag{4.5}$$

$$(X_x(t, x), T(t, x)) \in G(B) \quad \text{for } (t, x) \in [t_1, t_1 + t_2] \times [0, 1]\,. \tag{4.6}$$

Put

$$v_{j+2}(x) = \partial_t^{j+2} X(t_1, x) \quad \text{and} \quad \xi_{j+1}(x) = \partial_t^{j+1} T(t_1, x) \quad \text{for } 0 \le j \le L\,. \tag{4.7}$$

As stated in §1, v_{j+1} and ξ_{j+1} are determined successively from v_0, v_1 and ξ_0 by differentiating (4.1) and (4.2) j times with respect to t at $t = t_1$. Next, differentiating (4.3) with respect to t at $t = t_1$, we have the conditions at $t = t_1$ on $\partial\Omega$ for v_0, v_1 and ξ_0 through $v_2, v_3, \ldots, v_{L+1}$ and $\xi_1, \ldots, \xi_L$, namely,

$$\partial_t^j S(X_x, T)|_{t=t_1} = \xi_{jx} = 0 \quad \text{for } x \in \partial\Omega \text{ and } j = 0, 1, \ldots, L\,. \tag{4.8}$$

We shall say that v_0, v_1 and ξ_0 satisfy the *compatibility condition of order* L *at* $t = t_1$.

THEOREM 4.1 (cf. Shibata [5]). *Assume that*

$$v_j \in H^{L+2-j} \quad (0 \le j \le L + 1)\,, \qquad \xi_j \in H^{L+2-j} \quad (0 \le j \le L)\,, \tag{4.9}$$

$$(v_0'(x), \xi_0(x)) \in G(B) \quad \textit{for } x \in [0, 1]\,, \tag{4.10}$$

and that v_0, v_1 *and* ξ_0 *satisfy the compatibility condition of order* $L - 2$ *at* $t = t_1$. *Let* $B' > 0$ *be a constant such that*

$$\sum_{j=0}^{2} \|v_j\|_{3-j} + \sum_{j=0}^{1} \|\xi_j\|_{3-j} \le B'\,. \tag{4.11}$$

Then there exists a t_2 *depending on* B' *but independent of* t_1 *such that the problem* (4.1)–(4.4) *admits a unique solution* X, T *satisfying* (4.5) *and* (4.6).

Now, by using Theorems 3.1 and 4.1, we prove Theorem 1.1. Let $\bar{t}$ be the supremum of the numbers t_0 such that solutions X and T in $[0, t_0]$ exist. Suppose that $\bar{t} < \infty$. In view of Theorem 4.1, we know that $\bar{t} > 0$. Let t_0 be any number in $(0, \bar{t})$. Let X, T be the solution in $[0, t_0]$. Below we use the same notation as in §3. Consider the continuation of X and T by using Theorem 4.1. To do this, let us give the initial data for the problem (4.1)–(4.4) with $t_1 = t_0$ by $v_0(x) = X(t_0, x)$, $v_1(x) = \partial_t X(t_0, x)$ and $\xi_0(x) = T(t_0, x)$. Since X and T satisfy (3.1) and (3.2), differentiating (3.1) and (3.2) with respect to t at $t = t_0$, we see that $\partial_t^{j+2} X(t_0, x) = v_{j+2}(x)$ and $\partial_t^{j+1} T(t_0, x) = \xi_{j+1}(x)$ for $0 \le j \le L$, where v_{j+2} and ξ_{j+1} are the same as the functions defined in (4.7). Then, differentiating (3.3) with respect to t at $t = t_0$, we also see that v_0, v_1 and ξ_0 satisfy the compatibility condition of order L at $t = t_0$. Obviously, it follows from (3.5) that $(v_0'(x), \xi_0(x)) \in G(B)$ for all $x \in [0, 1]$. Also, it follows from (3.6) that $v_j \in H^{L+2-j}$ $(0 \le j \le L+1)$ and $\xi_j \in H^{L+2-j}$ $(0 \le j \le L)$. By Theorem 3.1 we see that $Y(t) \le 1$ for $t \in [0, t_0]$ provided that $E \le \delta$ for some $\delta > 0$. Note that the choice of δ is independent of t_0. Since

$$\|X(t_0,\cdot)\| \le \|X(0,\cdot)\| + \int_0^{t_0} \|X_s(s,\cdot)\|\, ds \le (1/2)^{1/2} + \|u_0\| + t_0 Y(t_0)\,,$$
$$\|X_x(t_0,\cdot)\|_2 = \|X_\infty + u_x(t_0,\cdot)\|_2 \le |X_\infty| + Y(t_0)\,,$$
$$\|T(t_0,\cdot)\|_3 = \|T_\infty + \theta(t_0,\cdot)\|_3 \le |T_\infty| + Y(t_0)\,,$$

where we have used the fact that $X(0, x) = x + u_0(x)$, and since $v_1(x) = u_t(t_0, x)$, $v_2(x) = u_{tt}(t_0, x)$, $\xi_1(x) = \theta_t(t_0, x)$, $t_0 < \bar{t}$ and $Y(t_0) \le 1$, we have

$$\sum_{j=0}^{2} \|v_j\|_{3-j} + \sum_{j=0}^{1} \|\xi_j\|_{3-j} \le \|X(t_0,\cdot)\| + \|X_x(t_0,\cdot)\|_2 + \|u_t(t_0,\cdot)\|_2$$
$$+ \|u_{tt}(t_0,\cdot)\|_1 + \|T(t_0,\cdot)\|_3 + \|\theta_t(t_0,\cdot)\|_2 \le B'$$

where $B' = (1/2)^{1/2} + \|u_0\| + \bar{t} + |X_\infty| + |T_\infty| + 6$. By Theorem 4.1, we see that there exists a $t_2 > 0$ independent of t_0 such that the problem (4.1)–(4.4) admits a solution X', T' satisfying (4.5) and (4.6). Moreover, it follows from (4.7) that $\partial_t^{j+2} X'(t_0, x) = \partial_t^{j+2} X(t_0, x)$ and $\partial_t^{j+1} T'(t_0, x) = \partial_t^{j+1} T(t_0, x)$ for $0 \le j \le L$. If we put $Z''(t, x) = Z(t, x)$ for $0 \le t \le t_0$ and $= Z'(t, x)$ for $t_0 \le t \le t_0 + t_2$ where $Z = X$, T, we easily see that X'', T'' is a solution in $[0, t_0 + t_2]$. Since t_2 is independent of t_0, if we choose t_0 in such a way that $t_0 = \bar{t} - t_2/2$, we have $t_0 + t_2 = \bar{t} + t_2/2 > \bar{t}$, which contradicts the maximality of $\bar{t}$. Thus, $\bar{t} = \infty$, which completes the proof of Theorem 1.1.

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