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ON LOCAL MOTION OF A COMPRESSIBLE BAROTROPIC VISCOUS FLUID BOUNDED BY A FREE SURFACE

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Abstract. We consider the motion of a viscous compressible barotropic fluid in $\mathbb{R}^3$ bounded by a free surface which is under constant exterior pressure, both with surface tension and without it. In the first case we prove local existence of solutions in anisotropic Hilbert spaces with noninteger derivatives. In the case without surface tension the anisotropic Sobolev spaces with integration exponent $p > 3$ are used to omit the coefficients which are increasing functions of $1/T$, where T is the existence time.

1. Introduction. First we consider the motion of a viscous compressible barotropic fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$, which depends on time $t \in \mathbb{R}^1_+$. The shape of the (free) boundary S_t of Ω_t is governed by the surface tension. Let $v = v(x,t)$ be the velocity of the fluid, $\varrho = \varrho(x,t)$ the density, $f = f(x,t)$ the external force field per unit mass, $p = p(\varrho)$ the pressure, μ and ν the viscosity coefficients, σ the surface tension coefficient and p_0 the external (constant) pressure. Then the problem is described by the following system (see [4], Chs. 1, 2, 7):

$$
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) + \nabla p(\varrho) - \mu\Delta v - \nu\nabla\operatorname{div} v = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\varrho|_{t=0} = \varrho_0, \qquad v|_{t=0} = v_0 && \text{in } \Omega,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\phi_t/|\nabla\phi| && \text{on } \widetilde{S}^T,
\end{aligned}
\tag{1.1}
$$

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where $\phi(x,t)=0$ describes S_t, $\widetilde{\Omega}^T=\bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, Ω_t is the domain of the drop at time t, $\Omega_0=\Omega$ is its initial domain, $\widetilde{S}^T=\bigcup_{t\in(0,T)}S_t$, $\overline{n}$ is the unit outward vector normal to the boundary ($\overline{n}=\nabla\phi/|\nabla\phi|$), μ, ν, σ are constant coefficients. Moreover, thermodynamical considerations imply $\nu\ge 1/(3\mu)>0$, $\sigma>0$. The last condition $(1.1)_5$ means that the free boundary S_t is built up of moving fluid particles. Finally, $\mathbb{T}=\mathbb{T}(v,p)$ denotes the stress tensor of the form

$$T_{ij}=-p\delta_{ij}+\mu(\partial_{x^i}v^j+\partial_{x^j}v^i)+(\nu-\mu)\delta_{ij}\operatorname{div}v\equiv -p\delta_{ij}+D_{ij}(v)\,,\tag{1.2}$$

where $i,j=1,2,3$, $\mathbb{D}=\mathbb{D}(v)=\{D_{ij}\}$ is the deformation tensor and H is the double mean curvature of S_t, which is negative for convex domains and can be expressed in the form

$$H\overline{n}=\Delta_{S_t}(t)x\,,\qquad x=(x^1,x^2,x^3)\,,\tag{1.3}$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be determined by $x=x(s_1,s_2,t)$, $(s_1,s_2)\in U\subset\mathbb{R}^2$, where U is an open set. Then

$$\Delta_{S_t}(t)=g^{-1/2}\partial_{s_\alpha}g^{-1/2}\widehat{g}_{\alpha\beta}\partial_{s_\beta}=g^{-1/2}\partial_{s_\alpha}g^{1/2}g^{\alpha\beta}\partial_{s_\beta}\,,\qquad \alpha,\beta=1,2\,,\tag{1.4}$$

where the convention summation over repeated indices is assumed, $g=\det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}$, $g_{\alpha\beta}=x_\alpha\cdot x_\beta$, where $x_\alpha=\partial_{s_\alpha}x$, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\widehat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements for $\{g_{\alpha\beta}\}$.

Let the domain Ω be given. Then, by $(1.1)_5$, $\Omega_t=\{x\in\mathbb{R}^3: x=x(\xi,t),\ \xi\in\Omega\}$, where $x=x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t}=v(x,t),\qquad x|_{t=0}=\xi\in\Omega,\qquad \xi=(\xi^1,\xi^2,\xi^3)\,.\tag{1.5}$$

Therefore the transformation $x=x(\xi,t)$ connects the Eulerian x and Lagrangian ξ coordinates of the same fluid particle. Hence

$$x=\xi+\int_0^t u(\xi,s)\,ds\equiv X_u(\xi,t)\,,\tag{1.6}$$

where $u(\xi,t)=v(X_u(\xi,t),t)$. Moreover, the kinematic boundary condition $(1.1)_5$ implies that the boundary S_t is a material surface, so if $\xi\in S=S_0$ then $X_u(\xi,t)\in S_t$ and $S_t=\{x: x=X_u(\xi,t),\ \xi\in S\}$.

In virtue of the continuity equation $(1.1)_2$ and $(1.1)_5$ the total mass M is conserved and

$$\int_{\Omega_t}\varrho(x,t)\,dx=M\,,\tag{1.7}$$

which is a relation between ϱ and Ω_t.

The aim of Section 3 of this paper is to prove local existence of solutions for the problem (1.1). We use spaces $W_2^{l,l/2}(\Omega^T)$ with noninteger derivatives.

In Section 4 we show local existence of solutions to (1.1) with $\sigma=0$. Since the existence for the nonlinear problem was considered in [16] it is sufficient to find an estimate (see (4.3)) for the linearized problem $(1.1)_{1,3,4}$ with $\sigma = 0$ (see (4.1)) with a constant independent of T for $T < \infty$. A similar estimate for a scalar parabolic equation in the case of the Neumann boundary condition was found in [14].

Local existence of solutions in the compressible case was considered in [6, 7, 13]. In the incompressible case local existence is proved in [2, 11].

2. Notation and auxiliary results. In Section 3 of this paper we use the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(\Omega^T)$, $l \in \mathbb{R}_+$ (see [3], Ch. 18) of functions defined in $\Omega^T = \Omega \times (0,T)$. In fact $W_2^{l,l/2}$, $l \notin \mathbb{Z}$, are Besov spaces; the equivalence between $W_2^{l,l/2}$, $l \notin \mathbb{Z}$, and Besov spaces follows from considerations in [1], Ch. 7. In the case of noninteger l we introduce the following norms ($\Omega \subset \mathbb{R}^3$) for functions defined in Ω^T:

$$
\begin{aligned}
&\|u\|_{W_2^{l,0}(\Omega^T)} = \Big(\int_0^T \|u\|^2_{W_2^l(\Omega)}\,dt\Big)^{1/2},\\
&\|u\|_{W_2^{0,l/2}(\Omega^T)} = \Big(\int_\Omega \|u\|^2_{W_2^{l/2}((0,T))}\,dx\Big)^{1/2},\\
&\|u\|^2_{W_2^l(\Omega)} = \sum_{|\alpha|\le[l]} \|D_x^\alpha u\|^2_{L_2(\Omega)}\\
(2.1)\qquad &\qquad + \sum_{|\alpha|=[l]} \int_\Omega\int_\Omega \frac{|D_x^\alpha u(x,t) - D_y^\alpha u(y,t)|^2}{|x-y|^{3+2(l-[l])}}\,dx\,dy\\
&\qquad \equiv \sum_{|\alpha|\le[l]} |D_x^\alpha u|^2_{2,\Omega} + \sum_{|\alpha|=[l]} [D_x^\alpha u]^2_{l-[l],\Omega}\\
&\qquad \equiv \sum_{|\alpha|\le[l]} |D_x^\alpha u|^2_{2,\Omega} + [u]^2_{l,\Omega}\,,
\end{aligned}
$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\partial_{x_i} = \partial/\partial x_i$, $[l]$ is the integer part of l, $\varrho = l - [l] \in (0,1)$ and in the case when l is an integer the last term must be omitted,

$$
\begin{aligned}
\|u\|^2_{W_2^{l/2}((0,T))} &= \sum_{j=0}^{[l/2]} \|\partial_t^j u\|^2_{L_2((0,T))} + \int_0^T\int_0^T \frac{|\partial_t^{[l/2]} u(x,t) - \partial_\tau^{[l/2]} u(x,\tau)|^2}{|t-\tau|^{1+2(l/2-[l/2])}}\,dt\,d\tau\\
&\equiv \sum_{j=0}^{[l/2]} |\partial_t^j u|^2_{2,(0,T)} + [\partial_t^{[l/2]} u]^2_{l/2-[l/2],(0,T)}\,,
\end{aligned}
$$

where $\varrho = l/2 - [l/2] \in (0,1)$ and in the case when $l/2$ is an integer the last term must be omitted,

$$
\begin{aligned}
\|u\|^2_{W_2^{l,l/2}(\Omega^T)} &= \sum_{|\alpha|\le[l]} \|D^\alpha_{x,t}u\|^2_{L_2(\Omega^T)} \\
&\quad + \sum_{|\alpha|=[l]} \bigg(\int_0^T \int_\Omega \int_\Omega \frac{|D^\alpha_{x,t}u(x,t) - D^\alpha_{y,t}u(y,t)|^2}{|x-y|^{3+2(l-[l])}}\, dx\, dy\, dt \\
&\quad + \int_\Omega \int_0^T \int_0^T \frac{|D^\alpha_{x,t}u(x,t) - D^\alpha_{x,\tau}u(x,\tau)|^2}{|t-\tau|^{1+2(l/2-[l/2])}}\, dt\, d\tau\, dx \bigg) \\
&\equiv \sum_{|\alpha|\le[l]} |D^\alpha_{x,t}u|^2_{2,\Omega^T} \\
&\quad + \sum_{|\alpha|=[l]} ([D^\alpha_{x,t}u]^2_{l-[l],\Omega^T,x} + [D^\alpha_{x,t}u]^2_{l/2-[l/2],\Omega^T,t}) \\
&\equiv \sum_{|\alpha|\le[l]} |D^\alpha_{x,t}u|^2_{2,\Omega^T} + [u]^2_{l,\Omega^T,x} + [u]^2_{l,\Omega^T,t}\,,
\end{aligned}
$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $D^\alpha_{x,t} = \partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $|\alpha| = 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ and we use generalized (Sobolev) derivatives. Similarly using local coordinates and a partition of unity we introduce the norm in the space $W_2^{l,l/2}(S^T)$ of functions defined on $S^T = S \times (0,T)$, where $S = \partial\Omega$. We also use $W_2^l(\Omega)$ with the norm $(2.1)_3$ for functions defined in Ω. We do not distinguish norms of scalar and vector-valued functions. To simplify notation we write

$$
\begin{aligned}
\|u\|_{l,Q} &= \|u\|_{W_2^{l,l/2}(\Omega)} && \text{if } Q = \Omega^T \text{ or } Q = S^T,\ l \ge 0\,, \\
\|u\|_{l,Q} &= \|u\|_{W_2^l(Q)} && \text{if } Q = \Omega \text{ or } Q = (0,T),\ l \ge 0\,,
\end{aligned}
$$

and $W_2^{0,0}(Q) = W_2^0(Q) = L_2(Q)$. Moreover,

$$
\|u\|_{L_p(Q)} = |u|_{p,Q}, \qquad \|u\|_{l,p,\Omega^T} = \|u\|_{L_p(0,T;W_2^l(\Omega))}, \qquad 1 \le p \le \infty\,.
$$

Let us introduce a space $\Gamma_r^l(\Omega)$ with the norm

$$
\|u\|_{\Gamma_r^l(\Omega)} = \sum_{i\le[l/2]} \|\partial_t^i u\|_{l-2i,r,\Omega}\,,
$$

where $\|u\|_{l,r,Q} = \|u\|_{W_r^l(Q)}$, for Q either Ω or S. In the case when Q is either Ω^T or S^T we define $\|u\|_{l,r,Q} = \|u\|_{W_r^{l,l/2}(Q)}$.

We also define the following norms:

$$
|u|_{l,0,\infty,\Omega^T} = \sup_{t\in[0,T]} \Big(\sum_{|\alpha|\le[l]} |D^\alpha_{x,t}u|^2_{2,\Omega} + \sum_{|\alpha|=[l]} [D^\alpha_{x,t}u]_{l-[l],\Omega} \Big)^{1/2},
$$

$$
|u|_{l,0,\Omega} = \Big(\sum_{|\alpha|\le[l]} |D^\alpha_{x,t}u|^2_{2,\Omega} + \sum_{|\alpha|=[l]} [D^\alpha_{x,t}u]_{l-[l],\Omega} \Big)^{1/2}, \qquad l \in \mathbb{R}_+\,.
$$

We introduce

$$\overset{\circ}{W}{}_2^{l,l/2}(Q^T) = \{u \in W_2^{l,l/2}(Q^T) : \partial_t^i u|_{t=0} = 0,\ i \le [l/2 - 1/2]\},$$

where Q is either Ω or S.

We also need the notation

$$\begin{aligned} \|u\|_{[l]+\kappa,\Omega^T} &= \bigg(\sum_{|\alpha|=[l]} \int_0^T \frac{|D_{x,t}^\alpha u|_{2,\Omega}^2}{t^{2\kappa}}\, dt \bigg)^{1/2}, \qquad \kappa \in (0,1), \\ \|u\|_{(l),\Omega^T,\kappa} &= \|u\|_{l,\Omega^T} + \|u\|_{[l]+\kappa,\Omega^T}, \\ \|u\|_{l,\Omega^T,\kappa} &= \|u\|_{l,\Omega^T} + T^{-\kappa} \sum_{|\alpha|=[l]} |D_{x,t}^\alpha u|_{2,\Omega^T}. \end{aligned}$$

We denote by $W_{2,\kappa}^{l,l/2}(\Omega^T)$ the space with the norm $\|\ \|_{(l),\Omega^T,\kappa}$. For T finite and $\kappa \le l/2 - [l/2]$ the above norms are equivalent because in view of Lemma 2.6 below we have

$$\begin{aligned} T^{-\kappa}\langle u\rangle_{[l],\Omega^T} &\le \|u\|_{[l]+\kappa,\Omega^T} \le c[u]_{[l]+\kappa,\Omega^T,t} + cT^{-\kappa}\langle u\rangle_{[l],\Omega^T} \\ &\le cT^{l/2-[l/2]-\kappa}[u]_{l,\Omega^T,t} + cT^{-\kappa}\langle u\rangle_{[l],\Omega^T}, \end{aligned}$$

where $\kappa \le l/2 - [l/2]$, $\langle u\rangle_{[l],\Omega^T} = \sum_{|\alpha|=[l]} |D_{x,t}^\alpha u|_{2,\Omega^T}$.

Now we introduce spaces convenient for proving the existence of solutions to the linearized parabolic problem $(1.1)_{1,3,4}$ (see [8, 15]):

$$\|u\|^2_{H_\gamma^{l,l/2}(\Omega^T)} = \int_0^T e^{-2\gamma t}\|u\|^2_{l,\Omega}\, dt + \|u\|^2_{H_\gamma^{0,l/2}(\Omega^T)},$$

where $\gamma \ge 0$. Here for $l/2 \notin \mathbb{Z}$,

$$\begin{aligned} \|u\|^2_{H_\gamma^{0,l/2}(\Omega^T)} &= \gamma^l \int_0^T e^{-2\gamma t}\|u\|^2_{0,\Omega}\, dt \\ &\quad + \int_0^T e^{-2\gamma t}\, dt \int_0^\infty d\tau \frac{\|\partial_t^k u_0(\cdot, t-\tau) - \partial_t^k u_0(\cdot,t)\|^2_{0,\Omega}}{\tau^{1+2(l/2-k)}} \end{aligned}$$

and $k = [l/2]$, $u_0(x,t) = u(x,t)$ for $t > 0$, $u_0(x,t) = 0$ for $t \le 0$. For $l/2 \in \mathbb{Z}$

$$\|u\|^2_{H_\gamma^{0,l/2}(\Omega^T)} = \int_0^T e^{-2\gamma t}(\gamma^l\|u\|^2_{0,\Omega} + \|\partial_t^{l/2} u\|^2_{0,\Omega})\, dt,$$

and we assume that $\partial_t^j u|_{t=0} = 0$, $j = 0,\dots,l/2-1$, so $u_0(x,t)$ has a generalized derivative $\partial_t^{l/2} u_0$ in $L_2(\Omega \times (-\infty,T))$. Moreover, we introduce the notation $\|u\|_{l,2,\gamma,\Omega^T} = \|u\|_{H_\gamma^{l,l/2}(\Omega^T)}$.

For the purposes of Section 4 we define

$$[u]_{p,\varrho,\Omega} = \Big(\int\limits_{\Omega} \int\limits_{\Omega} dx\, dx' \frac{|u(x,t) - u(x',t)|^p}{|x - x'|^{3+p\varrho}} \Big)^{1/p}, \qquad [u]_{2,\varrho,\Omega} \equiv [u]_{\varrho,\Omega},$$

$$[u]_{l,p,\Omega} = \sum_{|\alpha|=[l]} [D_x^\alpha u]_{p,l-[l],\Omega}.$$

Next we introduce the notation used by V. A. Solonnikov (see [10]):

$$E_{n+1} = \{x_1,\dots,x_n,t\} \equiv \mathbb{R}^{n+1}, \quad E_n = \{x_1,\dots,x_n\} \equiv \mathbb{R}^n,$$
$$\widetilde{E}_n = \{(x,t) \in E_{n+1} : x_n = 0\}, \quad D_{n+1} = \{(x,t) \in E_{n+1} : t > 0\},$$
$$\widetilde{D}_{n+1} = \{(x,t) \in E_{n+1} : x_n > 0\}, \quad D_n = \{x \in E_n : x_n > 0\} \equiv \mathbb{R}^n_+,$$
$$\widetilde{D}_n = \{(x',t) \in \widetilde{E}_n : t > 0\}, \quad x' = (x_1,\dots,x_{n-1}),$$
$$R = \{(x,t) \in E_{n+1} : x_n > 0, t > 0\}, \quad E_{n-1} = \{x \in E_n : x_n = 0\} \equiv \mathbb{R}^{n-1}.$$

If G is one of the sets E_{n+1}, $\widetilde{E}_n$, D_{n+1}, $\widetilde{D}_{n+1}$, $\widetilde{D}_n$, R, we denote by $G(T)$ the set of all points in G with $t \le T$, and $G(\infty) = G$.

Moreover, we introduce the following norms and seminorms:

$$\langle u\rangle_{l,p,Q} = \sum_{|\alpha|=l} |D_{x,t}^\alpha u|_{p,Q}, \qquad l \in \mathbb{N},$$

$$[u]_{p,\varrho,\Omega^T,x} = \Big(\int\limits_0^T dt \int\limits_{\Omega} dx \int\limits_{\Omega} dx' \frac{|u(x,t) - u(x',t)|^p}{|x - x'|^{n+p\varrho}} \Big)^{1/p}, \qquad \Omega \subset \mathbb{R}^n,\ \varrho \in (0,1),$$

$$[u]_{p,\varrho,\Omega^T,t} = \Big(\int\limits_{\Omega} dx \int\limits_0^T dt \int\limits_0^T dt' \frac{|u(x,t) - u(x,t')|^p}{|t - t'|^{1+p\varrho}} \Big)^{1/p}, \qquad \varrho \in (0,1),\ p \in (1,\infty),$$

$$\begin{aligned}
[u]_{p,\varrho,\Omega^T} &= [u]_{p,\varrho,\Omega^T,x} + [u]_{p,\varrho,\Omega^T,t},\\
[u]_{l,p,\Omega^T} &= \sum_{|\alpha|=[l]} ([D_{x,t}^\alpha u]_{p,l-[l],\Omega^T,x} + [D_{x,t}^\alpha u]_{p,l/2-[l/2],\Omega^T,t})\\
&\equiv [u]_{l,p,\Omega^T,x} + [u]_{l,p,\Omega^T,t},
\end{aligned}$$

for $l \ge 1$, $l - [l] \in (0,1)$, $l/2 - [l/2] \in (0,1)$, $p \in (1,\infty)$.

Finally, we define

$$\langle\!\langle u\rangle\!\rangle_{p,\varrho,E_n,x_i} = \Big(\int\limits_0^\infty |u(x_1,\dots,x_i+h,\dots,x_n) - u(x)|^p_{p,E_n} \frac{dh}{h^{1+p\varrho}} \Big)^{1/p}, \qquad \varrho \in (0,1).$$

In the case of $E_{n+1}(T)$ we have

$$\langle\!\langle u\rangle\!\rangle_{l,p,E_{n+1}(T)} = \sum_{i=1}^n \sum_{|\alpha|=[l]} \langle\!\langle D_x^\alpha u\rangle\!\rangle_{p,l-[l],E_{n+1}(T),x_i} + \langle\!\langle \partial_t^{[l/2]} u\rangle\!\rangle_{p,l/2-[l/2],E_{n+1}(T),t},$$

where

$$\langle\!\langle u\rangle\!\rangle_{p,\varrho,E_{n+1}(T),x_i} = \Big(\int\limits_0^\infty |u(x_1,\dots,x_i+h,\dots,x_n,t)-u(x,t)|^p_{p,E_{n+1}(T)}\,\frac{dh}{h^{1+p\varrho}}\Big)^{1/p},\ i=1,\dots,n\,,$$

$$\langle\!\langle u\rangle\!\rangle_{p,\varrho,E_{n+1}(T),t} = \Big(\int\limits_0^\infty |u(x,t-h)-u(x,t)|^p_{p,E_{n+1}(T)}\,\frac{dh}{h^{1+p\varrho}}\Big)^{1/p},\ \varrho\in(0,1)\,.$$

We need the following imbedding theorems and interpolation inequalities:

LEMMA 2.1 (see [3]). *For a sufficiently smooth u on Ω we have*

$$|D^\mu_x u|_{f,\Omega} \le c\|u\|_{d,\Omega}, \quad 3/2-3/f+|\mu|\le d,\ 0\le d\in\mathbb{R},\ 1<f\in\mathbb{R},\ |\mu|\in\mathbb{N}\cup\{0\}\,, \tag{2.2}$$

$$[D^\mu_x u]_{p,\varrho,\Omega} \le c\|u\|_{l,\Omega}, \quad 3/2-3/p+|\mu|+\varrho\le l,\ \varrho\in(0,1),\ 1<p\in\mathbb{R},\ |\mu|\in\mathbb{N}\cup\{0\}\,, \tag{2.3}$$

$$|D^\mu_x u|_{f,\Omega} \le \varepsilon^{1-\kappa}\langle u\rangle_{[d],\Omega} + c\varepsilon^{-\kappa}|u|_{2,\Omega}\,, \quad 0<\kappa=(3/2-3/f+|\mu|)/[d]<1,\ \varepsilon\in(0,1)\,, \tag{2.4}$$

$$|D^\mu_x u|_{f,\Omega} \le \varepsilon^{1-\kappa_1}[u]_{d,2,\Omega} + c\varepsilon^{-\kappa_1}|u|_{2,\Omega}, \quad 0<\kappa_1=(3/2-3/f+|\mu|)/d<1, \tag{2.5}$$

$$[D^\mu_x u]_{p,\varrho,\Omega} \le \varepsilon^{1-\kappa_2}[u]_{l,2,\Omega} + c\varepsilon^{-\kappa_2}|u|_{2,\Omega}\,, \quad 0<\kappa_2=(3/2-3/p+|\mu|+\varrho)/l<1\,. \tag{2.6}$$

LEMMA 2.2 (see [9, 12]). *For a sufficiently regular u we have*

$$\|\partial^i_t u\|_{2-2/p,p,\Omega} \le c(\|\partial^i_t u\|_{2,p,\Omega^T} + \|\partial^i_t u|_{t=0}\|_{2-2/p,p,\Omega})\,, \tag{2.7}$$

where the constant c does not depend on T.

LEMMA 2.3 (see [10], Sect. 12, p. 72; [12], Th. 5). *The norms $[u]_{l,p,E_{n+1}(T)}$ and $\langle\!\langle u\rangle\!\rangle_{l,p,E_{n+1}(T)}$ are equivalent.*

LEMMA 2.4 (see [10]). *Let $u=0$ for $t\le 0$. Then*

$$\langle\!\langle u\rangle\!\rangle^r_{r,\alpha,\widetilde{E}_n(T),t} \le [u]^r_{r,\alpha,\widetilde{D}_n(T),t} + \frac{2}{r\alpha}\int\limits_0^T dt\,|u(\cdot,t)|^r_{r,E_{n-1}}\frac{1}{t^{r\alpha}}\,, \tag{2.8}$$

where $\alpha\in(0,1)$.

LEMMA 2.5. *Let $f(0)=0$ and $\operatorname{supp} f\subset[0,T]$. Let $p\in(1,\infty)$, $1/p<l<1$. Then there exists a constant $A(l,p)$ such that*

$$\Big(\int\limits_0^T \frac{|f(x)|^p}{x^{pl}}\,dx\Big)^{1/p} \le A(p,l)\Big(\int\limits_0^T dx\int\limits_0^T dy\,\frac{|f(x)-f(y)|^p}{|x-y|^{1+pl}}\Big)^{1/p}, \tag{2.9}$$

where f is such that the right-hand side is finite and

$$0 < A = \alpha/[(1-(pl)^{-1/p}a^{l-1/p})a]\,,$$
$$\alpha = ((1-(1-a)^{p'(1+l)})/(p'(1+l))^{1/p'})\,, \qquad a \in (0,1)\,, 1/p + 1/p' = 1\,.$$

P r o o f. It is sufficient to prove the lemma for smooth functions vanishing near 0. We consider the identity

$$f(x) = \frac{1}{ax}\int_0^{ax} f(y)\,dy + \frac{1}{ax}\int_0^{ax}[f(x)-f(y)]\,dy\,, \qquad a < 1\,.$$

By the Hölder inequality we have

$$\begin{aligned}(2.10)\qquad |f(x)| &\le \frac{1}{ax}\Big(\int_0^{ax}|f(y)|^p\,dy\Big)^{1/p}\Big(\int_0^{ax}dy\Big)^{1/p'}\\ &\quad + \frac{1}{ax}\Big(\int_0^{ax}\frac{|f(x)-f(y)|^p}{|x-y|^{1+pl}}\,dy\Big)^{1/p}\Big(\int_0^{ax}|x-y|^{p'(1/p+l)}\,dy\Big)^{1/p'},\end{aligned}$$

where $1/p + 1/p' = 1$. We calculate the integral

$$\begin{aligned}\int_0^{ax}|x-y|^{p'(1/p+l)}\,dy &= \int_0^{ax}(x-y)^{p'(1/p+l)}\,dy\\ &= \frac{1-(1-a)^{p'(1+l)}}{p'(1+l)}x^{p'(1+l)} \equiv \alpha(p,a)^{p'}x^{p'(1+l)}\,.\end{aligned}$$

Therefore, (2.10) becomes

$$(2.11)\quad |f(x)| \le \Big(\frac{1}{ax}\int_0^{ax}|f(y)|^p\,dy\Big)^{1/p} + a^{-1}\alpha(p,a)x^l\Big(\int_0^{ax}\frac{|f(x)-f(y)|^p}{|x-y|^{1+pl}}\,dy\Big)^{1/p}.$$

In view of the Minkowski inequality, (2.11) implies

$$\begin{aligned}(2.12)\quad \Big(\int_0^T\frac{|f(x)|^p}{x^{pl}}\,dx\Big)^{1/p} &\le a^{-1/p}\Big(\int_0^T\frac{dx}{x^{1+pl}}\int_0^{ax}|f(y)|^p\,dy\Big)^{1/p}\\ &\quad + \frac{\alpha}{a}\Big(\int_0^T dx\int_0^{ax}\frac{|f(x)-f(y)|^p}{|x-y|^{1+pl}}\,dy\Big)^{1/p}\\ &= a^{-1/p}\Big(\int_0^{aT}|f(y)|^p\,dy\int_{y/a}^T\frac{dx}{x^{1+pl}}\Big)^{1/p}\\ &\quad + \frac{\alpha}{a}\Big(\int_0^T dx\int_0^{ax}\frac{|f(x)-f(y)|^p}{|x-y|^{1+pl}}\,dy\Big)^{1/p} \equiv I\,.\end{aligned}$$

Integrating the second integral in the first term yields

$$\bigg(\int\limits_{y/a}^{T}\frac{dx}{x^{1+pl}}\bigg)^{1/p}=\bigg(\frac{1}{pl}[(y/a)^{-pl}-T^{-pl}]\bigg)^{1/p}\le\bigg(\frac{1}{pl}\bigg)^{1/p}(y/a)^{-l}\,.$$

Hence we have

$$I\le(pl)^{-1/p}a^{l-1/p}\bigg(\int\limits_0^T\frac{|f(y)|^p}{y^{pl}}\,dy\bigg)^{1/p}+\frac{\alpha}{a}\bigg(\int\limits_0^T dx\int\limits_0^{ax}\frac{|f(x)-f(y)|^p}{|x-y|^{1+pl}}\,dy\bigg)^{1/p}.$$

Since $l>1/p$, assuming that

$$\bigg(\int\limits_0^T\frac{|f(x)|^p}{x^{pl}}\,dx\bigg)^{1/p}<\infty$$

(here we use $f(0)=0$) and that a is so small that $1-(pl)^{-1/p}a^{l-1/p}>0$ we obtain (2.9). This concludes the proof.

We recall Lemma 6.3 from [8].

LEMMA 2.6. *Let $\tau\in(0,1)$. Then for $u\in W_2^{0,\tau/2}(\Omega^T)$*

$$\int\limits_0^T|u|_{2,\Omega}^2\frac{dt}{t^\tau}\le c_1\int\limits_0^T dt\int\limits_0^T dt'\frac{|u(\cdot,t)-u(\cdot,t')|_{2,\Omega}^2}{|t-t'|^{1+\tau}}+c_2T^{-\tau}\int\limits_0^T|u|_{2,\Omega}^2\,dt\,,\tag{2.13}$$

where c_1, c_2 do not depend on T and u.

For $T=\infty$ the last term in (2.13) vanishes. The above result was shown in [12], Lemma 2, p. 138.

3. Local existence. To prove the local existence of solutions to (1.1) we write it in the Lagrangian coordinates introduced by (1.5) and (1.6):

$$\begin{aligned}
&\eta u_t-\mu\nabla_u^2u-\nu\nabla_u\nabla_u\cdot u+\nabla_uq=\eta g && \text{in } \Omega^T\,,\\
&\eta_t+\eta\nabla_u\cdot u=0 && \text{in } \Omega^T\,,\\
&\mathbb{T}_u(u,q)\overline{n}(\xi,t)-\sigma\Delta_{S_t}(t)X_u(\xi,t)=-p_0\overline{n}(\xi,t) && \text{on } S^T\,,\\
&u|_{t=0}=v_0(\xi) && \text{in } \Omega\,,\\
&\eta|_{t=0}=\varrho_0(\xi) && \text{in } \Omega\,,
\end{aligned}\tag{3.1}$$

where $\eta(\xi,t)=\varrho(X_u(\xi,t),t)$, $q(\xi,t)=p(X_u(\xi,t),t)$, $g(\xi,t)=f(X_u(\xi,t),t)$, $\nabla_u=\partial_x\xi^i\nabla_{\xi^i}$, $\nabla_{\xi^i}=\partial_{\xi^i}$, $\mathbb{T}_u(u,q)=-q\delta+\mathbb{D}_u(u)$, $\delta=\{\delta_{ij}\}$ and $\mathbb{D}_u(u)=\{\mu(\partial_{x^i}\xi^k\nabla_{\xi^k}u^j+\partial_{x^j}\xi^k\nabla_{\xi^k}u^i)+(\nu-\mu)\delta_{ij}\nabla_u\cdot u\}$, with $\nabla_u\cdot u=\partial_{x^i}\xi^k\nabla_{\xi^k}u^i$. Let A be the Jacobi matrix of the transformation $x=x(\xi,t)\equiv X_u(\xi,t)$ with elements $a_{ij}=\delta_{ij}+\int_0^t\partial_{\xi^j}u^i(\xi,\tau)\,d\tau$. Assuming $|\nabla_\xi u|_{\infty,\Omega^T}\le M$ we obtain

$$0<c_1(1-Mt)^3\le\det\{\partial_{\xi^j}x^i\}\le c_2(1+Mt)^3,\qquad t\le T\,,\tag{3.2}$$

where c_1, c_2 are constants and T is sufficiently small. Moreover, $\det A = \exp(\int_0^t \nabla_u \cdot u\, d\tau) = \varrho_0/\eta$.

Let S_t be determined (at least locally) by the equation $\phi(x,t) = 0$. Then S is described by $\phi(x(\xi,t),t)|_{t=0} \equiv \widetilde{\phi}(\xi) = 0$. Moreover, we have

$$\overline{n}(x(\xi,t),t) = \frac{\nabla_x \phi(x,t)}{|\nabla_x \phi(x,t)|}\bigg|_{x=x(\xi,t)}, \qquad \overline{n}_0(\xi) = \frac{\nabla_\xi \widetilde{\phi}(\xi)}{|\nabla_\xi \widetilde{\phi}(\xi)|}.$$

To prove the existence of solutions of (3.1) we consider first the following linear problem:

$$\begin{aligned} &u_t - \mu \Delta_\xi u - \nu \nabla_\xi \nabla_\xi \cdot u = f_1 && \text{in } \Omega^T, \\ &\Pi_0 \mathbb{D}_\xi(u) \overline{n}_0 = g_1 && \text{on } S^T, \\ &\overline{n}_0 \mathbb{D}_\xi(u) \overline{n}_0 - \sigma \overline{n}_0 \Delta_S(0) \int_0^t u(\tau)\, d\tau = g_2 + \sigma \int_0^t h_1(\tau)\, d\tau && \text{on } S^T, \\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{3.3}$$

where Π_0, Π are projections defined by $\Pi g = g - (g \cdot \overline{n})\overline{n}$, $\Pi_0 g = g - (g \cdot \overline{n}_0)\overline{n}_0$, and $\mathbb{D}_\xi(u) = \{\mu(\partial_{\xi^i} u^j + \partial_{\xi^j} u^i) + (\nu - \mu)\delta_{ij} \partial_{\xi^k} u^k\}$.

LEMMA 3.1. *Let* $f_1 \in W_{2,\kappa}^{l,l/2}(\Omega^T)$, $g_1, g_2 \in W_{2,\kappa}^{l+1/2,l/2+1/4}(S^T)$, $h_1 \in W_{2,\kappa}^{l-1/2,l/2-1/4}(S^T)$, $u_0 \in W_2^{l+1}(\Omega)$, $S \in W_2^{l+2}$ *and* $T < \infty$. *Let* $0 < l \notin \mathbb{Z}$ *be such that* $l/2 = n + 3/4 + \kappa$, $n \in \mathbb{N} \cup \{0\}$, $\kappa \in (0, 1/4)$. *Then there exists a solution of problem* (3.3) *such that* $u \in W_{2,\kappa}^{l+2,l/2+1}(\Omega^T)$ *and*

$$\|u\|_{l+2,\Omega^T,\kappa} \le c_1(X_1 + |u(0)|_{l+1,0,\Omega}) \le c_2(X_1 + X_2 + \|u_0\|_{l+1,\Omega}), \tag{3.4}$$

where

$$X_1 = \|f_1\|_{l,\Omega^T,\kappa} + \sum_{i=1}^2 \|g_i\|_{l+1/2,S^T,\kappa} + \|h_1\|_{l-1/2,S^T,\kappa},$$

$$X_2 = \sum_{i=0}^{[l/2-1/2]} \|\partial_t^i f_1(0)\|_{l-1-2i,\Omega},$$

$$|u(0)|_{l+1,0,\Omega} = \sum_{i=0}^{[l/2+1/2]} \|\partial_t^i u(0)\|_{l+1-2i} \le c_3(X_2 + \|u_0\|_{l+1,\Omega}),$$

and the constants c_1, c_2, c_3 *do not depend on* T *for* $T < \infty$.

Proof. Let $\varphi^i = \partial_t^i u|_{t=0} \in W_2^{l+1-2i}(\Omega)$ be calculated from $(3.3)_1$ inductively:

$$\varphi^{i+1} = \mu \nabla_\xi^2 \varphi^i + \nu \nabla_\xi \nabla_\xi \cdot \varphi^i + \partial_t^i f_1(0), \qquad i \le [l/2 - 1/2]. \tag{3.5}$$

They satisfy the following compatibility conditions:

$$
\begin{aligned}
&\Pi_0\mathbb{D}_\xi(\varphi^i)\overline{n}_0=\partial_t^i g_1(0)\,, \qquad i\le[l/2+1/4-1/2]\,,\\
&\overline{n}_0\mathbb{D}_\xi(\varphi^i)\overline{n}_0-\sigma\overline{n}_0\Delta_S(0)\varphi^{i-1}=\partial_t^i g_2(0)+\sigma\partial_t^{i-1}h_1(0)\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad 1\le i\le[l/2-1/4-1/2]\,,\\
&\overline{n}_0\mathbb{D}_\xi(\varphi^0)\overline{n}_0=g_2(0)\,, \qquad \text{for } i=0\,.
\end{aligned}
\tag{3.6}
$$

We extend the functions φ^i, $i=0,\dots,[l/2+1/2]$, to $\mathbb{R}^3$ in such a way that the extended functions $\widetilde{\varphi}^i\in W_2^{l+1-2i}(\mathbb{R}^3)$, $0\le i\le[l/2+1/2]$, satisfy

$$
\|\widetilde{\varphi}^i\|_{l+1-2i,\mathbb{R}^3}\le c\|\varphi^i\|_{l+1-2i,\Omega}\,.
$$

Now we construct a function $\widetilde{v}$ such that

$$
\partial_t^i\widetilde{v}|_{t=0}=\widetilde{\varphi}^i\,, \qquad i\le[l/2+1/2]\,. \tag{3.7}
$$

By Theorem 3 of [12] there exists a function $\widetilde{v}\in W_{2,\kappa}^{l+2,l/2+1}(\mathbb{R}^3\times\mathbb{R}_+)$, $\kappa\le l/2-[l/2]$, satisfying (3.7) and in view of Lemma 2.6 we have

$$
\begin{aligned}
\|\widetilde{v}\|_{l+2,\mathbb{R}^3\times\mathbb{R}_+,\kappa}&\le c\|\widetilde{v}\|_{l+2,\mathbb{R}^3\times\mathbb{R}_+}\le c\sum_{i=0}^{[l/2+1/2]}\|\widetilde{\varphi}^i\|_{l+1-2i,\mathbb{R}^3}\\
&\le c\sum_{i=0}^{[l/2+1/2]}\|\varphi^i\|_{l+1-2i,\Omega}=c|u(0)|_{l+1,0,\Omega}\le c(X_2+\|u_0\|_{l+1,\Omega})\,.
\end{aligned}
\tag{3.8}
$$

Let $v=\widetilde{v}|_{\Omega^T}$. Then introducing the function

$$
w=u-v\,, \tag{3.9}
$$

we see that it is a solution of the problem

$$
\begin{aligned}
&w_t-\mu\nabla_\xi^2 w-\nu\nabla_\xi\nabla_\xi\cdot w=f_1' && \text{in } \Omega^T\,,\\
&\Pi_0\mathbb{D}_\xi(w)\overline{n}_0=g_1' && \text{on } S^T\,,\\
&\overline{n}_0\mathbb{D}_\xi(w)\overline{n}_0-\sigma\overline{n}_0\Delta_S(0)\int_0^t w(\tau)\,d\tau=g_2'+\sigma\int_0^t h_1'\,d\tau && \text{on } S^T\,,\\
&w|_{t=0}=0 && \text{in } \Omega\,,
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
&f_1'=f_1-(v_t-\mu\nabla_\xi^2 v-\nu\nabla_\xi\nabla_\xi\cdot v)\in \mathring{W}_2^{l,l/2}(\Omega^T)\,,\\
&g_1'=g_1-\Pi_0\mathbb{D}_\xi(v)\overline{n}_0\in \mathring{W}_2^{l+1/2,l/2+1/4}(S^T)\,,\\
&g_2'=g_2-\overline{n}_0\mathbb{D}_\xi(v)\overline{n}_0\in \mathring{W}_2^{l+1/2,l/2+1/4}(S^T)\,,\\
&h_1'=h_1-\sigma\overline{n}_0\Delta_S(0)v\in \mathring{W}_2^{l-1/2,l/2-1/4}(S^T)\,.
\end{aligned}
\tag{3.11}
$$

Therefore, we have

$$
\begin{aligned}
(3.12)\qquad &\|f_1'\|_{l,\Omega^T}+\sum_{i=1}^2\|g_i'\|_{l+1/2,S^T}+\|h_1'\|_{l-1/2,S^T}\\
&\le c\Big(\|f_1\|_{l,\Omega^T}+\sum_{i=1}^2\|g_i\|_{(l+1/2),S^T,\kappa}+\|h_1\|_{l-1/2,S^T}+\|v\|_{l+2,\Omega^T}\Big)\\
&\quad+\big\|\mathbb{D}_\xi(v)\overline{n}_0\big\|_{[l+1/2]+\kappa,S^T}+\big\|\overline{n}_0\mathbb{D}_\xi(v)\overline{n}_0\big\|_{[l+1/2]+\kappa,S^T}\\
&\le c(X_1+\|v\|_{l+2,\Omega^T}),
\end{aligned}
$$

because

$$
(3.13)\qquad \big\|\mathbb{D}_\xi(v)\overline{n}_0\big\|_{[l+1/2]+\kappa,S^T}+\big\|\overline{n}_0\mathbb{D}_\xi(v)\overline{n}_0\big\|_{[l+1/2]+\kappa,S^T}\le c(T)\|v\|_{l+2,\Omega^T}.
$$

The restrictions imposed on l in the assumptions of the lemma imply that $l/2-[l/2]=3/4+\kappa>1/2$, $l/2-1/4-[l/2-1/4]=1/2+\kappa>1/2$, $l/2+1/4-[l/2+1/4]=\kappa<1/4$. Therefore, f_1', g_i', $i=1,2$, h_1' can be extended by zero into $t<0$ without losing regularity. Denote the extended functions by f_1'', g_1'', g_2'', h_1'', respectively. Moreover, in view of Lemma 2.5 we find that $f_1''\in H_0^{l,l/2}(\Omega^T)$, $h_1''\in H_0^{l-1/2,l/2-1/4}(S^T)$ and by Lemmas 2.3 and 2.4 that $g_1'',g_2''\in H_0^{l+1/2,l/2+1/4}(S^T)$ and

$$
\begin{aligned}
&\|f_1''\|_{l,2,0,\Omega^T}\le c\|f_1'\|_{l,\Omega^T},\\
(3.14)\qquad &\|g_i''\|_{l+1/2,2,0,S^T}\le c\|g_i'\|_{(l+1/2),S^T},\qquad i=1,2,\\
&\|h_1''\|_{l-1/2,2,0,S^T}\le c\|h_1'\|_{l-1/2,S^T},
\end{aligned}
$$

where the constant does not depend on T.

Since $T<\infty$, the norms of $H_\gamma^{l,l/2}(\Omega^T)$ and $H_0^{l,l/2}(\Omega^T)$ (and similarly for boundary norms) are equivalent. Therefore, from [8] we deduce that $f_1''\in H_\gamma^{l,l/2}(\Omega^T)$, $g_1'',g_2''\in H_\gamma^{l+1/2,l/2+1/4}(S^T)$, $h_1''\in H_\gamma^{l-1/2,l/2-1/4}(S^T)$ and there exists a constant $c(\gamma)$ such that

$$
\begin{aligned}
&\|f_1''\|_{l,2,\gamma,\Omega^T}\le c(\gamma)\|f_1''\|_{l,2,0,\Omega^T},\\
(3.15)\qquad &\|g_i''\|_{l+1/2,2,\gamma,S^T}\le c(\gamma)\|g_i''\|_{l+1/2,2,0,S^T},\qquad i=1,2,\\
&\|h_1''\|_{l-1/2,2,\gamma,S^T}\le c(\gamma)\|h_1''\|_{l-1/2,2,0,S^T}.
\end{aligned}
$$

Performing the above extension on the right-hand sides of (3.10) we obtain the following problem:

$$
\begin{aligned}
&\widetilde{w}_t-\mu\nabla_\xi^2\widetilde{w}-\nu\nabla_\xi\nabla_\xi\cdot\widetilde{w}=f_1'' &&\text{in } \Omega\times(-\infty,T),\\
&\Pi_0\mathbb{D}_\xi(\widetilde{w})\overline{n}_0=g_1'' &&\text{on } S\times(-\infty,T),\\
(3.16)\qquad &\overline{n}_0\mathbb{D}_\xi(\widetilde{w})\overline{n}_0-\sigma\overline{n}_0\Delta_S(0)\int_0^t\widetilde{w}(\tau)\,d\tau=g_2''+\sigma\int_0^t h_1''(\tau)\,d\tau\\
& &&\text{on } S\times(-\infty,T),
\end{aligned}
$$

where $\widetilde{w}$ is zero for $t \le 0$. By [15] there exists a solution of (3.16) such that $\widetilde{w} \in H_\gamma^{l+2,l/2+1}(\Omega \times (-\infty,T))$ and (3.12), (3.14), (3.15) imply

$$\|\widetilde{w}\|_{l+2,2,\gamma,\Omega\times(-\infty,T)} \le c(\gamma)(X_1 + X_2 + \|u_0\|_{l+1\Omega}).$$

For $T < \infty$ we have

$$\begin{aligned}\|\widetilde{w}\|_{l+2,2,0,\Omega\times(-\infty,T)} &\le c(\gamma,T)\|\widetilde{w}\|_{l+2,2,\gamma,\Omega\times(-\infty,T)} \\ &\le c(\gamma,T)(X_1 + X_2 + \|u_0\|_{l+1,\Omega}),\end{aligned} \tag{3.17}$$

where $c(\gamma,T)$ is an increasing function of T.

Let $w = \widetilde{w}|_{[0,T]}$. Then (3.17) yields

$$\begin{aligned}\|w\|_{l+2,\Omega^T,\kappa} &\le c_1(T)\|w\|_{l+2,\Omega^T,l/2-[l/2]} \le c_2(T)\|\widetilde{w}\|_{l+2,2,0,\Omega\times(-\infty,T)} \\ &\le c_3(\gamma,T)(X_1 + X_2 + \|u_0\|_{l+2,\Omega}),\end{aligned}$$

where c_1, c_2, c_3 are increasing functions of T.

From the above inequality, (3.8) and (3.9) we get (3.4). This concludes the proof.

Now we consider the following problem with $\eta > 0$ (we use here the considerations from [9, 16]):

$$\begin{aligned}&\eta u_t - \mu\Delta_\xi u - \nu\nabla_\xi\nabla_\xi\cdot u = F && \text{in } \Omega^T,\\ &\Pi_0\mathbb{D}_\xi(u)\overline{n}_0 = G_1 && \text{on } S^T,\\ &\overline{n}_0\mathbb{D}_\xi(u)\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u(\tau)\,d\tau = G_2 + \int_0^T H(\tau)\,d\tau && \text{on } S^T,\\ &u|_{t=0} = u_0 && \text{in } \Omega.\end{aligned} \tag{3.18}$$

Lemma 3.2. *Assume that* $F \in W_{2,\kappa}^{l,l/2}(\Omega^T)$, $G_i \in W_{2,\kappa}^{l+1/2,l/2+1/4}(S^T)$, $i = 1,2$, $H \in W_{2,\kappa}^{l-1/2,l/2-1/4}(S^T)$, $1/\eta \in L_\infty(\Omega^T)$, $\eta \in L_\infty(0,T;\Gamma_2^{l+1}(\Omega)) \cap W_2^{l+1,l/2+1/2}(\Omega^T)$, $\eta \in C^\alpha(\Omega^T)$, $\alpha \in (0,1)$ (*where* $C^\alpha(\Omega^T)$ *is the Hölder space* (see [10])), $u_0 \in W_2^{l+1}(\Omega)$, $S \in W_2^{l+2}$, $l > 3/2$ *satisfies the assumptions of Lemma* 3.1 *and* $\kappa \in (0,1/4)$. *Then there exists a solution of problem* (3.18) *such that* $u \in W_{2,\kappa}^{l+2,l/2+1}(\Omega^T)$ *and*

$$\begin{aligned}\|u\|_{l+2,\Omega^T,\kappa} &\le \varphi_1(T,|1/\eta|_{\infty,\Omega^T},|\eta|_{l+1,0,\infty,\Omega^T},\|\eta\|_{l+1,\Omega^T}) \\ &\quad\times\Big[\|F\|_{l,\Omega^T,\kappa} + \sum_{i=1}^2\|G_i\|_{l+1/2,S^T,\kappa} + \|H\|_{l-1/2,S^T,\kappa} \\ &\qquad + |u(0)|_{l+1,0,\Omega} + \|u\|_{l,\Omega^T,\kappa}\Big],\end{aligned} \tag{3.19}$$

where φ_1 *is an increasing positive function.*

Proof. We introduce a partition of unity $\{\zeta_k^{(\lambda)}(\xi,t), Q_k^{(\lambda)}\}$ (see [9, 16]), $Q_k^{(\lambda)} = \operatorname{supp}\zeta_k^{(\lambda)}$, $k = 1,\dots,N$, such that $\sum_{k=1}^N\zeta_k^{(\lambda)}(\xi,t) = 1$, $(\xi,t) \in \Omega^T$, $\lambda =$

$\max_k \operatorname{diam} Q_k^{(\lambda)}$, $\zeta_k^{(\lambda)} \ge 0$, $0 \le \mu_0 \le \sum_{k=1}^N (\zeta_k^{(\lambda)})^2 \le N_0$ and $|D_{\xi,t}^\alpha \zeta_k^{(\lambda)}(\xi,t)| \le c|\lambda|^{-|\alpha|}$. Set $u_k = u\zeta_k^{(\lambda)}$ and $h_k = h\zeta_k^{(\lambda)}$. Therefore instead of (3.18) we consider

$$
\begin{aligned}
&\eta(\xi_k,t_k)u_{kt} - \mu\Delta_\xi u_k - \nu\nabla_\xi\nabla_\xi\cdot u_k = [\eta(\xi_k,t_k)-\eta(\xi,t)]u_{kt} + \eta u\zeta_{kt}^{(\lambda)}\\
&\qquad - \mu[\Delta_\xi,\zeta_k^{(\lambda)}]u - \nu[\nabla_\xi\nabla_\xi\cdot\,,\zeta_k^{(\lambda)}]u + F_k \equiv F_k' + F_k \equiv \widetilde{F}_k\,,\\
&\Pi_0\mathbb{D}_\xi(u_k)\overline{n}_0 = \Pi_0\mathbb{D}_\xi(\zeta_k^{(\lambda)})\overline{n}_0 u + G_{1k} \equiv G_{1k}' + G_{1k} \equiv \widetilde{G}_{1k}\,,\\
&\overline{n}_0\mathbb{D}_\xi(u_k)\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u_k(\tau)\,d\tau\\
&\qquad = \overline{n}_0\mathbb{D}_\xi(\zeta_k^{(\lambda)})\overline{n}_0 u - \overline{n}_0\mathbb{D}_\xi(\zeta_k^{(\lambda)}(0))\overline{n}_0 u_0 + G_{2k}\\
&\qquad\quad + \int_0^t [-\sigma\overline{n}_0[\Delta_S(0),\zeta_k^{(\lambda)}]u + \overline{n}_0\mathbb{D}_\xi(u)\zeta_{k,\tau}^{(\lambda)}\overline{n}_0]\,d\tau\\
&\qquad\quad + \int_0^t (-G_2\zeta_{k,\tau}^{(\lambda)} + H_k(\tau))\,d\tau\\
&\qquad \equiv G_{2k}' + G_{2k} + \int_0^t (H_k'(\tau) + H_k(\tau))\,d\tau \equiv \widetilde{G}_2 + \int_0^t \widetilde{H}(\tau)\,d\tau\,,\\
&u_k|_{t=0} = u_{0k}\,,
\end{aligned}
\tag{3.20}
$$

where $(\xi_k,t_k)\in Q_k^{(\lambda)}$, $[L,u]w = L(uw) - uL(w)$, L is an operator. To obtain the boundary condition $(3.20)_3$ we differentiate $(3.18)_3$ with respect to time, multiply by $\zeta_k^{(\lambda)}$ and integrate the result with respect to time to get

$$
\begin{aligned}
&\overline{n}_0\mathbb{D}_\xi(u_k)\overline{n}_0 - \sigma\overline{n}_0\Delta_S(0)\int_0^t u_k(\tau)\,d\tau\\
&\qquad = \int_0^t [\overline{n}_0\mathbb{D}_\xi(\zeta_k^{(\lambda)})\overline{n}_0 u_\tau - \sigma\overline{n}_0[\Delta_S(0),\zeta_k^{(\lambda)}]u + \overline{n}_0\mathbb{D}_\xi(u\zeta_{k,\tau}^{(\lambda)})\overline{n}_0]\,d\tau\\
&\qquad\quad + G_{2k} - \int_0^t (G_2\zeta_{k,\tau}^{(\lambda)} + H_k)\,d\tau\,.
\end{aligned}
$$

Integrating by parts in the first term on the right-hand side we get $(3.20)_3$.

Changing variables $\tau = \eta_k^{-1}t$, $\eta_k = \eta_k(\xi_k,t_k)$, and applying Lemma 3.1 yields

$$
\begin{aligned}
&\|u_k\|_{l+2,\Omega^{T/\eta_k},\kappa}\\
&\quad \le c\Big(\|\widetilde{F}_k\|_{l,\Omega^{T/\eta_k},\kappa} + \sum_{i=1}^2 \|\widetilde{G}_{ik}\|_{l+1/2,S^{T/\eta_k},\kappa} + \|\widetilde{H}_k\|_{l-1/2,S^{T/\eta_k},\kappa} + \|u_{0k}\|_{l+1,\Omega}\Big)\,.
\end{aligned}
\tag{3.21}
$$

Going back to the variable t in (3.21) implies

$$\|u_k\|_{l+2,\Omega^T,\kappa} \le \varphi_1(1/\min\eta, \max\eta) \times \Big(\|\widetilde{F}_k\|_{l,\Omega^T,\kappa} + \sum_{i=1}^{2}\|\widetilde{G}_{ik}\|_{l+1/2,S^T,\kappa} + \|\widetilde{H}_k\|_{l-1/2,S^T,\kappa} + \|u_{0k}\|_{l+1,\Omega}\Big). \tag{3.22}$$

Now we employ the explicit forms of F'_k, G'_{ik}, H'_k, $i=1,2$, which are given by the right-hand sides of (3.20). We only consider the $W_2^{l,l/2}(\Omega^T)$ norms, because the remaining part of the $W_{2,\kappa}^{l,l/2}(\Omega^T)$ norm is easier. First we estimate the first term in the last brackets on the right-hand side of (3.22):

$$\|(\eta(\xi_k,t_k)-\eta(\xi,t))u_{kt}\|_{l,\Omega^T} \le c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}\|u_k\|_{l+2,\Omega^T} + [\eta u_{kt}]_{l,\Omega^T,x} + [\eta u_{kt}]_{l,\Omega^T,t} + \{\text{lower order terms}\}. \tag{3.23}$$

Now we estimate the second term. Let

$$D^\alpha_{\xi,t} = \sum_{2\alpha_0+|\alpha'|=|\alpha|} D_\xi^{\alpha'}\partial_t^{\alpha_0}, \qquad \alpha'=(\alpha_1,\alpha_2,\alpha_3),\ |\alpha'|=\alpha_1+\alpha_2+\alpha_3,\ |\alpha|=[l].$$

Then

$$\begin{aligned}
&[\eta u_{kt}]_{l,\Omega^T,x}\\
&\le \bigg(\int_0^t dt\int_\Omega\int_\Omega d\xi\,d\xi'\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}\frac{|D^\beta_{\xi,t}\eta D^{\alpha-\beta}_{\xi,t}u_{kt}-D^\beta_{\xi',t}\eta D^{\alpha-\beta}_{\xi',t}u_{kt}|^2}{|\xi-\xi'|^{3+2(l-[l])}}\bigg)^{1/2}\\
&\le \bigg(\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi'\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}\bigg(\frac{|D^\beta_{\xi,t}\eta-D^\beta_{\xi',t}\eta|^2|D^{\alpha-\beta}_{\xi,t}u_{kt}|^2}{|\xi-\xi'|^{3+2(l-[l])}}\\
&\qquad+\frac{|D^\beta_{\xi',t}\eta|^2|D^{\alpha-\beta}_{\xi,t}u_{kt}-D^{\alpha-\beta}_{\xi',t}u_{kt}|^2}{|\xi-\xi'|^{3+2(l-[l])}}\bigg)\bigg)^{1/2}\\
&\le \bigg(\int_0^T dt\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}\bigg([D^\beta_{\xi,t}\eta]^2_{l-[l]+\varepsilon/2,2p_\beta,\Omega}\bigg(\int_\Omega\int_\Omega d\xi\,d\xi'\frac{|D^{\alpha-\beta}_{\xi,t}u_{kt}|^{2q_\beta}}{|\xi-\xi'|^{3-\varepsilon q_\beta}}\bigg)^{1/q_\beta}\\
&\qquad+\bigg(\int_\Omega\int_\Omega d\xi\,d\xi'\frac{|D^\beta_{\xi',t}\eta|^{2q'_\beta}}{|\xi-\xi'|^{2-\varepsilon q'_\beta}}\bigg)^{1/q'_\beta}[D^{\alpha-\beta}_{\xi,t}u_{kt}]^2_{l-[l]+\varepsilon/2,2p'_\beta,\Omega}\bigg)\bigg)^{1/2}\\
&\le c\bigg(\int_0^T dt\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}([D^\beta_{\xi,t}\eta]^2_{l-[l]+\varepsilon/2,2p_\beta,\Omega}|D^{\alpha-\beta}_{\xi,t}u_{kt}|^2_{2q_\beta,\Omega}\\
&\qquad+|D^\beta_{\xi,t}\eta|^2_{2q'_\beta,\Omega}[D^{\alpha-\beta}_{\xi,t}u_{kt}]^2_{l-[l]+\varepsilon/2,2p'_\beta,\Omega})\bigg)^{1/2}\equiv I_1,
\end{aligned}$$

where $\alpha\ge\beta$, $\varepsilon>0$, $1/p_\beta+1/q_\beta=1$, $1/p'_\beta+1/q'_\beta=1$.

Using the imbedding theorems (2.2), (2.3) and the interpolation inequalities (2.5), (2.6) with $3/2-3/(2p_\beta)+|\beta|+l-[l]+\varepsilon/2\le l+1$, $3/2-3/(2q_\beta)+|\alpha|-|\beta|+2<l+2$ and $3/2-3/q'_\beta+|\beta|\le l+1$, $3/2-3/(2p'_\beta)+|\alpha|-|\beta|+l-[l]+\varepsilon/2+2<l+2$, respectively, we have

$$I_1\le c|\eta|_{l+1,0,2,\infty,\Omega^T}(\varepsilon_1\|u_k\|_{l+2,\Omega^T}+c\|u_k\|_{l,\Omega^T}).$$

Next we consider

$$\begin{aligned}
&[\eta u_{kt}]_{l,\Omega^T,t}\\
&\quad\le\Big(\int\limits_\Omega d\xi\int\limits_0^T\int\limits_0^T dt\,dt'\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}\frac{|D^\beta_{\xi,t}\eta D^{\alpha-\beta}_{\xi,t}u_{kt}-D^\beta_{\xi,t'}\eta D^{\alpha-\beta}_{\xi,t'}u_{kt'}|^2}{|t-t'|^{1+2(l/2-[l/2])}}\Big)^{1/2}\\
&\quad\le\Big(\int\limits_\Omega d\xi\int\limits_0^T\int\limits_0^T dt\,dt'\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}\Big(\frac{|D^\beta_{\xi,t}\eta-D^\beta_{\xi,t'}\eta|^2|D^{\alpha-\beta}_{\xi,t}u_{kt}|^2}{|t-t'|^{1+2(l/2-[l/2])}}\\
&\qquad\qquad+\frac{|D^\beta_{\xi,t'}\eta|^2|D^{\alpha-\beta}_{\xi,t}u_{kt}-D^{\alpha-\beta}_{\xi,t'}u_{kt'}|^2}{|t-t'|^{1+2(l/2-[l/2])}}\Big)\Big)^{1/2}\\
&\quad\le\Big(\int\limits_0^T\int\limits_0^T dt\,dt'\sum_{|\alpha|=[l]}\sum_{|\beta|=1}^{|\alpha|}\Big(\frac{|D^\beta_{\xi,t}\eta-D^\beta_{\xi,t'}\eta|^2_{2p_\beta,\Omega}|D^{\alpha-\beta}_{\xi,t}u_{kt}|^2_{2q_\beta,\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\\
&\qquad\qquad+\frac{|D^\beta_{\xi,t'}\eta|^2_{2p'_\beta,\Omega}|D^{\alpha-\beta}_{\xi,t}u_{kt}-D^{\alpha-\beta}_{\xi,t'}u_{kt'}|^2_{2q'_\beta,\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\Big)\Big)^{1/2}\equiv I_2\equiv I_3+I_4,
\end{aligned}$$

where $1/p_\beta+1/q_\beta=1$, $1/p'_\beta+1/q'_\beta=1$. First we estimate I_3. Let $|\beta|=1$. Then

$$\begin{aligned}
I_3&\le\Big(\int\limits_0^T\int\limits_0^T dt\,dt'\sum_{|\beta|=1}\frac{|D^\beta_\xi\eta(t)-D^\beta_\xi\eta(t')|^2_{2p_1,\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\sum_{|\alpha|=[l]-1}\int\limits_0^T|D^\alpha_{\xi,t}u_{kt}|^2_{2q_1,\Omega}\,dt\Big)^{1/2}\\
&\le\Big(\sup_t\sum_{|\alpha|=[l]-1}|D^\alpha_{\xi,t}u_{kt}|^2_{2q_1,\Omega}\int\limits_0^T\int\limits_0^T dt\,dt'\,\frac{\|\eta(t)-\eta(t')\|^2_{[l]+1,\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\Big)^{1/2}\equiv I_5,
\end{aligned}$$

where we used the imbedding (2.2) with

$$3/2-3/(2p_1)\le[l]. \tag{3.24}$$

To estimate the first factor in I_5 we consider two cases: $[l]=2s+1$ and $[l]=2s$, $s\in\mathbb{N}\cup\{0\}$. In the first case the highest derivative $\partial_t^{s+1}u_k$ appears in the factor which is the highest t-derivative in the $W_2^{l+2,l/2+1}(\Omega^T)$ norm but it does not appear in the $W_2^{l,l/2}(\Omega^T)$ norm where the highest t-derivative is $\partial_t^s u_k$.

Therefore to estimate the expression

$$I_6 \equiv \sup_t |\partial_t^{s+1} u_k|_{2q_1,\Omega}$$

we use

$$\partial_t u_k = \frac{1}{\eta}(\mu \Delta_\xi u + \nu \nabla_\xi \nabla_\xi \cdot u)\zeta_k + u\zeta_{kt} + \frac{1}{\eta} F_k \,. \tag{3.25}$$

We obtain

$$I_6 \le \sum_{i=1}^{s} c_{is} \sup_t \left| \partial_t^{s-i} \frac{1}{\eta} \right|_{2q_1 r_i,\Omega} \sum_{|\alpha|=2} |\partial_t^i D_\xi^\alpha u|_{2q_1 r_i',\Omega_k} + \{\text{lower order terms}\} + \sup_t \left| \partial_t^s \left(\frac{1}{\eta} F_k \right) \right|_{2q_1,\Omega_k} \equiv I_7 \,,$$

where $1/r_i + 1/r_i' = 1$, $i = 1, \ldots, s$, and $\Omega_k = \Omega \cap \operatorname{supp} \zeta_k$.

Now in view of (2.5) and (2.7) we have

$$I_7 \le c|1/\eta|_{l+1,0,\infty,\Omega_k^T} (\varepsilon^{1-\kappa_1} \|u\|_{l+2,\Omega_k^T} + \varepsilon^{-\kappa_1} \|u\|_{l,\Omega^T} + \|F_k\|_{l,\Omega^T} + |u(0)|_{l+1,0,\Omega})$$

where

$$\begin{gathered} 3/2 - 3/(2q_1 r_i) + 2(s-i) \le l+1 \,, \\ \kappa_1 = (3/2 - 3/(2q_1 r_i') + 2i + 2)/(l+2) < 1 \,. \end{gathered} \tag{3.26}$$

In the case $[l] = 2s$ the highest t-derivative in the first factor in I_5 is $\partial_t^s u_k$. Therefore the factor can be estimated by

$$\varepsilon^{1-\kappa_2} \|u_k\|_{l+2,\Omega^T} + c\varepsilon^{-\kappa_2} \|u_k\|_{l,\Omega^T} + c|u(0)|_{l+1,0,\Omega} \,,$$

where

$$\kappa_2 = (3/2 - 3/(2q_1) + [l] + 1)/(l+2) < 1 \,. \tag{3.27}$$

Summarizing, (3.24), (3.26), (3.27) are satisfied for $l > 3/2$, where for $l \in (3/2, 2)$ we have to assume $q_1 = 1$, $p_1 = \infty$. Therefore

$$\begin{aligned} I_5 \le{} & \varphi_1'(T, |1/\eta|_{\infty,\Omega^T}, |\eta|_{l+1,0,\infty,\Omega^T}) \|\eta\|_{l+1,\Omega^T} \\ & \times (\varepsilon \|u\|_{l+2,\Omega_k^T} + c(\varepsilon) \|u\|_{l,\Omega_k^T} + \|F_k\|_{l,\Omega^T} + |u(0)|_{l+1,0,\Omega}) \,. \end{aligned}$$

Now we examine I_3 for $2 \le |\beta| \le [l]$. Then in view of (2.2) and (2.5) we have

$$I_3 \le \sum_\beta (\varepsilon^{1-\kappa_{3\beta}} |u_k|_{l+1,0,\infty,\Omega^T} + c\varepsilon^{-\kappa_{3\beta}} |u_k|_{l,0,\infty,\Omega^T}) \|\eta\|_{l+1,\Omega^T} \equiv I_6 \,,$$

where $3/2 - 3/(2p_\beta) + |\beta| \le [l] + 1$, $\kappa_{3\beta} = (3/2 - 3/(2q_\beta) + |\alpha - \beta| + 2)/(l+1) < 1$, $2 \le |\beta| \le [l]$, $|\alpha| = [l]$. Finally, from Lemma 2.2 the first factor in I_6 is estimated by $\varepsilon \|u_k\|_{l+2,\Omega^T} + c(\varepsilon) |u_k|_{l+1,0,\Omega}|_{t=0}$.

Summarizing, we have shown that

$$I_3 \le \varphi_2'(T, |1/\eta|_{\infty,\Omega^T}, |\eta|_{l+1,0,\infty,\Omega^T})(|\eta|_{l+1,0,\infty,\Omega^T} + \|\eta\|_{l+1,\Omega^T}) \times (\varepsilon\|u\|_{l+2,\Omega_k^T} + c(\varepsilon)\|u\|_{l,\Omega_k^T} + c(\varepsilon)|u_k|_{l+1,0,\Omega}|_{t=0}). \tag{3.28}$$

Now we examine I_4. From (2.4) and (2.5) we have for $2 \le |\beta| \le [l]$

$$I_4 \le |\eta|_{l+1,0,\infty,\Omega^T} \sum_\beta (\varepsilon^{1-\kappa_{4\beta}}[u_k]_{l+2,2,\Omega^T,t} + c\varepsilon^{-\kappa_{4\beta}}\|u_k\|_{l,\Omega^T}),$$

where $3/2 - 3/(2p_\beta') + |\beta| \le l+1$, $\kappa_{4\beta} = (3/2 - 3/(2q_\beta') + |\alpha-\beta| + 2)/([l]+2) < 1$ are satisfied for $l > 2$.

The case $|\beta|=1$ can be treated similarly to I_3. The terms with time derivatives of order less than $s+1$ can be estimated by the same bound as I_4 above. To estimate the expression

$$I_7 = \bigg(\int_0^T\int_0^T \sum_{|\alpha|=1} \frac{|D^\alpha_{\xi,t'}\eta|^2_{2p_1',\Omega}|\partial_t^{s+1}u_k - \partial_{t'}^{s+1}u_k|^2_{2q_1',\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\,dt\,dt'\bigg)^{1/2}$$

we use (3.25), so the above is

$$\begin{aligned}&\le c\bigg(\int_0^T\int_0^T \sum_{|\alpha|=1} \frac{|D^\alpha_{\xi,t'}\eta|^2_{2p_1',\Omega}|\partial_t^s(\frac{1}{\eta}u_{\xi\xi}\zeta_k) - \partial_{t'}^s(\frac{1}{\eta}u_{\xi\xi}\zeta_k)|^2_{2q_1',\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\,dt\,dt'\bigg)^{1/2}\\ &\quad + c\bigg(\int_0^T\int_0^T \sum_{|\alpha|=1} \frac{|D^\alpha_{\xi,t'}\eta|^2_{2p_1',\Omega}|\partial_t^s(\frac{1}{\eta}F) - \partial_{t'}^s(\frac{1}{\eta}F)|^2_{2q_1',\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\,dt\,dt'\bigg)^{1/2}\\ &\quad + \{\text{lower order terms}\}.\end{aligned}$$

We only consider the first term, because the second can be estimated similarly. By the Leibniz formula and the Hölder inequality the first term is bounded by

$$\begin{aligned}&c\bigg(\int_0^T\int_0^T dt\,dt' \sum_{|\alpha|=1}|D^\alpha_{\xi,t'}\eta|^2_{2p_1',\Omega}\sum_{i=0}^s c_{si}\bigg(\frac{|\partial_t^{s-i}\frac{1}{\eta} - \partial_{t'}^{s-i}\frac{1}{\eta}|^2_{2q_1'r_i,\Omega_k}}{|t-t'|^{1+2(l/2-[l/2])}}|\partial_t^i u_{\xi\xi}|^2_{2q_1'r_i',\Omega_k}\\ &\qquad + \bigg|\partial_t^{s-i}\frac{1}{\eta}\bigg|^2_{2q_1'\overline{r}_i,\Omega_k}\frac{|\partial_t^i u_{\xi\xi} - \partial_{t'}^i u_{\xi\xi}|^2_{2q_1'\overline{r}_i',\Omega_k}}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)\bigg)^{1/2} \equiv I_8,\end{aligned}$$

where $1/r_i + 1/r_i' = 1$, $1/\overline{r}_i + 1/\overline{r}_i' = 1$. Since $[l+2] = 2s+3$, $[l+1] = 2s+2$, from (2.2) and (2.4) we obtain

$$\begin{aligned}I_8 &\le c\bigg(\int_0^T\int_0^T dt\,dt' \sum_{|\alpha|=1}\|D^\alpha_{\xi,t}\eta\|^2_{l,\Omega}\\ &\quad\times\sum_{i=0}^s\bigg(\frac{\|\partial_t^{s-i}\frac{1}{\eta} - \partial_{t'}^{s-i}\frac{1}{\eta}\|^2_{[l]+1-2(s-i),\Omega_k}}{|t-t'|^{1+2(l/2-[l/2])}}(\varepsilon\|\partial_t^i u\|^2_{l+2-2i,\Omega_k} + c(\varepsilon)\|\partial_t^i u\|^2_{l-2i,\Omega_k})\end{aligned}$$

$$
\begin{aligned}
&+\Big\|\partial_t^{s-i}\frac{1}{\eta}\Big\|^2_{l+1-2(s-i),\Omega_k}\bigg(\varepsilon\frac{\|\partial_t^i u-\partial_{t'}^i u\|^2_{[l]+2-2i,\Omega_k}}{|t-t'|^{1+2(l/2-[l/2])}}\\
&\qquad\qquad\qquad\qquad+c(\varepsilon)\frac{\|\partial_t^i u-\partial_{t'}^i u\|^2_{[l]-2i,\Omega_k}}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)\bigg)\bigg)\equiv I_9\,,
\end{aligned}
$$

where $3/2-3/(2p_1')+1\le l+1$, $3/2-3/(2q_1'r_i)+2(s-i)\le [l]+1$, $3/2-3/(2q_1'r_i')+2i+2<l+2$, and $3/2-3/(2p_1')+1\le l+1$, $3/2-3/(2q_1'\overline{r}_i)+2(s-i)\le l+1$, $3/2-3/(2q_1'\overline{r}_i')+2i+2<[l]+2$, $i=0,\dots,s$, $[l]=2s+1$. The above restrictions are satisfied for $l>3/2$.

Summarizing, we have

$$
\begin{aligned}
I_4\le{}&\varphi_3'(T,|\eta|_{l+1,0,\infty,\Omega^T}+\|\eta\|_{l+1,\Omega^T})\\
&\times(\varepsilon\|u\|_{l+2,\Omega_k^T}+c(\varepsilon)\|u\|_{l,\Omega_k^T}+c|u(0)|_{l+1,0,\Omega_k}+\|F_k\|_{l,\Omega^T})\\
&+c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}\|u_k\|_{l+2,\Omega^T}\,.
\end{aligned}
$$

Using the fact that $[\Delta_\xi,\zeta_k^{(\lambda)}]u=\Delta_\xi\zeta_k^{(\lambda)}u+\nabla_\xi\zeta_k^{(\lambda)}u_\xi$ we consider the expression (where $\zeta_k^{(\lambda)}$ is replaced by ζ for simplicity)

$$
\|\zeta_\xi u_\xi\|_{l,\Omega^T}=[\zeta_\xi u_\xi]_{l,\Omega^T,x}+[\zeta_\xi u_\xi]_{l,\Omega^T,t}+\{\text{lower order terms}\}\,.
$$

First we consider

$$
\begin{aligned}
[\zeta_\xi u_\xi]^2_{l,\Omega^T,x}\le{}&\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi'\sum_{|\alpha|=[l]}\sum_{|\beta|\le|\alpha|}c_{\alpha\beta}\frac{|D^{\alpha-\beta}_{\xi,t}\zeta_\xi D^\beta_{\xi,t}u_\xi-D^{\alpha-\beta}_{\xi',t}\zeta_{\xi'}D^\beta_{\xi',t}u_{\xi'}|^2}{|\xi-\xi'|^{3+2(l-[l])}}\\
\le{}&\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi'\sum_{|\alpha|=[l]}\sum_{|\beta|\le|\alpha|}c_{\alpha\beta}\bigg(\frac{|D^{\alpha-\beta}_{\xi,t}\zeta_\xi-D^{\alpha-\beta}_{\xi',t}\zeta_{\xi'}|^2|D^\beta_{\xi,t}u_\xi|^2}{|\xi-\xi'|^{3+2(l-[l])}}\\
&\qquad\qquad+\frac{|D^{\alpha-\beta}_{\xi',t}\zeta_\xi|^2|D^\beta_{\xi,t}u-D^\beta_{\xi',t}u|^2}{|\xi-\xi'|^{3+2(l-[l])}}\bigg)\\
\le{}&c\int_0^T dt\Big(\sum_{|\beta|\le[l]}|D^\beta_{\xi,t}u_\xi|^2_{2,\Omega_k}+\sum_{|\beta|=[l]}[D^\beta_{\xi,t}u_\xi]^2_{l-[l],2,\Omega_k}\Big)\\
\le{}&c(\varepsilon\|u\|^2_{l+2,\Omega_k^T}+c\|u\|^2_{l,\Omega_k^T})\,,
\end{aligned}
$$

where $\varepsilon\in(0,1)$ and the last inequality follows from the interpolation inequalities (2.5) and (2.6).

Finally, we examine the expression

$$
\begin{aligned}
&[\zeta_\xi u_\xi]^2_{l,\Omega^T,t}\\
&\quad\le\int_\Omega d\xi\int_0^T\int_0^T dt\,dt'\sum_{|\alpha|=[l]}\sum_{|\beta|\le|\alpha|}c_{\alpha\beta}\frac{|D^{\alpha-\beta}_{\xi,t}\zeta_\xi D^\beta_{\xi,t}u_\xi-D^{\alpha-\beta}_{\xi,t'}\zeta_\xi D^\beta_{\xi,t'}u_\xi|^2}{|t-t'|^{1+2(l/2-[l/2])}}
\end{aligned}
$$

$$\leq \int_\Omega d\xi \int_0^T \int_0^T dt\,dt' \sum_{|\alpha|=[l]} \sum_{|\beta|\leq|\alpha|} c_{\alpha\beta}\bigg(\frac{|D_{\xi,t}^{\alpha-\beta}\zeta_\xi - D_{\xi,t'}^{\alpha-\beta}\zeta_\xi|^2|D_{\xi,t}^\beta u_\xi|^2}{|t-t'|^{1+2(l/2-[l/2])}}$$
$$+\frac{|D_{\xi,t'}^{\alpha-\beta}\zeta_\xi|^2|D_{\xi,t}^\beta u_\xi - D_{\xi,t'}^\beta u_\xi|^2}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)$$
$$\leq c\sum_{|\beta|\leq[l]}\bigg(\int_0^T dt\,|D_{\xi,t}^\beta u_\xi|_{2,\Omega_k}^2 + \int_0^T\int_0^T dt\,dt'\frac{|D_{\xi,t}^\beta u_\xi - D_{\xi,t'}^\beta u_\xi|_{2,\Omega_k}^2}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)$$
$$\leq c\sum_{|\beta|\leq[l]}\bigg(\int_0^T dt\,(\varepsilon|D_{\xi,t}^\beta u_{\xi\xi\xi}|_{0,\Omega_k}^2 + c(\varepsilon)|D_{\xi,t}^\beta u|_{0,\Omega_k}^2)$$
$$+\int_0^T\int_0^T dt\,dt'\,\frac{\varepsilon|D_{\xi,t}^\beta u_{\xi\xi\xi} - D_{\xi,t'}^\beta u_{\xi\xi\xi}|_{0,\Omega_k}^2 + c(\varepsilon)|D_{\xi,t}^\beta u - D_{\xi,t'}^\beta u|_{0,\Omega_k}^2}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)$$
$$\leq c(\varepsilon\|u\|_{l+2,\Omega_k^T}^2 + c(\varepsilon)\|u\|_{l,\Omega_k^T}^2),$$

where $\varepsilon\in(0,1)$.

Summarizing, we have

$$\begin{aligned}\|F_k'\|_{l,\Omega^T} &\leq \varphi_4'(T,|\eta|_{l+1,0,\infty,\Omega^T},\|\eta\|_{l+1,\Omega^T})\\ &\quad\times[\varepsilon\|u\|_{l+2,\Omega_k^T} + c(\varepsilon)\|u\|_{l,\Omega_k^T} + \|F_k\|_{l,\Omega^T} + |u(0)|_{l+1,0,\Omega}]\\ &\quad + c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}\|u_k\|_{l+2,\Omega^T}\,.\end{aligned} \tag{3.30}$$

After similar considerations we obtain

$$\sum_{i=1}^2\|G_{ik}'\|_{l+1/2,S^T} + \|H_k'\|_{l-1/2,S^T} \leq \varepsilon\|u\|_{l+2,\Omega_k^T} + c(\varepsilon,\|S\|_{W_2^{l+2}})\|u\|_{l,\Omega_k^T}\,. \tag{3.31}$$

By repeating similar considerations for the remaining part of the $W_{2,\kappa}^{l,l/2}(\Omega^T)$ norm, from (3.22), (3.30) and (3.31) it follows that

$$\begin{aligned}\|u_k\|_{l+2,\Omega^T,\kappa} &\leq \varphi_5'(T,|1/\eta|_{\infty,\Omega^T},|\eta|_{l+1,0,\infty,\Omega^T},\|\eta\|_{l+1,\Omega^T})\\ &\quad\times\Big[\varepsilon\|u\|_{l+2,\Omega_k^T,\kappa} + \|F_k\|_{l,\Omega^T,\kappa} + \sum_{i=1}^2\|G_{ik}\|_{l+1/2,S^T,\kappa} + \|H_k\|_{l-1/2,S^T,\kappa}\\ &\quad + c(\varepsilon)\|u\|_{l,\Omega_k^T,\kappa} + |u_{0k}(0)|_{l+1,0,\Omega}\Big] + c\lambda^\alpha|\eta|_{C^\alpha(\Omega^T)}\|u_k\|_{l+2,\Omega^T,\kappa}\,.\end{aligned} \tag{3.32}$$

Assuming that λ is so small that

$$\frac{c\lambda^\alpha}{\min_{\Omega^T}\eta}|\eta|_{C^\alpha(\Omega^T)} < 1\,, \tag{3.33}$$

summing (3.32) over all $k\in\{1,\dots,N\}$, using the fact that the norms $\sum_{k=1}^N\|w\|_{l+3,\Omega_k^{(\lambda)}}$ and $\|w\|_{l+3,\Omega^T}$ are equivalent (see for example [10], Ch. 5), then assuming that ε is sufficiently small we get (3.19). This concludes the proof.

Finally, we examine the problem

$$
\begin{aligned}
&\eta u_t - \mu \Delta_w u - \nu \nabla_w \nabla_w \cdot u = F && \text{in } \Omega^T, \\
&\Pi_0 \Pi \mathbb{D}_w(u)\overline{n} = G_1 && \text{on } S^T, \\
&\overline{n}_0 \mathbb{D}_w(u)\overline{n} - \sigma \overline{n}_0 \Delta_{S_t}(t) \int_0^t u \, d\tau = G_2 + \sigma \int_0^t H \, d\tau && \text{on } S^T, \\
&u|_{t=0} = u_0 && \text{in } \Omega,
\end{aligned} \tag{3.34}
$$

where w is treated as a given vector, $\overline{n} = \overline{n}(X_w(\xi, t))$ and $\Delta_{S_t}(t)$ also depends on w.

To prove the existence of solutions to (3.34) we write it in the form

$$
\begin{aligned}
&\eta u_t - \mu \Delta_\xi u - \nu \nabla_\xi \nabla_\xi \cdot u \\
&\qquad = -\mu(\Delta_\xi - \Delta_w)u - \nu(\nabla_\xi \nabla_\xi \cdot - \nabla_w \nabla_w \cdot)u + F \equiv \widetilde{F} + F, \\
&\Pi_0 \mathbb{D}_\xi(u)\overline{n}_0 = \Pi_0 \mathbb{D}_\xi(u)\overline{n}_0 - \Pi_0 \Pi \mathbb{D}_w(u)\overline{n} + G_1 \equiv \widetilde{G} + G_1, \\
&\overline{n}_0 \mathbb{D}_\xi(u)\overline{n}_0 - \sigma \overline{n}_0 \Delta_S(0) \int_0^t u(\tau)\, d\tau \\
&\qquad = \overline{n}_0 \mathbb{D}_\xi(u)(\overline{n}_0 - \overline{n}) + \overline{n}_0(\mathbb{D}_\xi(u) - \mathbb{D}_w(u))\overline{n} \\
&\qquad\quad - \sigma \overline{n}_0(\Delta_S(0) - \Delta_{S_t}(t)) \int_0^t u(\tau)\, d\tau + G_2 + \sigma \int_0^t H(\tau)\, d\tau \\
&\qquad \equiv \widetilde{G}_2 + G_2 + \sigma \int_0^t (\widetilde{H}(\tau) + H(\tau))\, d\tau, \\
&u|_{t=0} = u_0.
\end{aligned} \tag{3.35}
$$

LEMMA 3.3. *Let the assumptions of Lemma 3.2 be satisfied. Let* $w \in W_2^{l+2,l/2+1}(\Omega^T)$ *and let* $\delta_1 \in (0,1)$. *Let* l *and* κ *be the same as in Lemma 3.1. Let*

$$
\begin{aligned}
&T^a(\|w\|_{l+2,\Omega^T} + |w|_{l+1,0,\infty,\Omega^T}) \\
&\times \varphi_3(T, T^a(\|w\|_{l+2,\Omega^T} + |w|_{l+1,0,\infty,\Omega^T})|1/\eta|_{\infty,\Omega^T}, \|\eta\|_{l+1,\Omega^T}, |\eta|_{l+1,0,\infty,\Omega^T}) \le \delta_1,
\end{aligned} \tag{3.36}
$$

where $\varphi_3 = \varphi_1\varphi_2$, φ_1 *and* φ_2 *are determined by* (3.19) *and* (3.42) *below, respectively*, $a > 0$ *and* δ_1 *is sufficiently small. Then for sufficiently small* T *there exists* *a solution of problem* (3.34) *such that* $u \in W^{l+2,l/2+1}_{2,\kappa}(\Omega^T)$ *and*

$$\|u\|_{l+2,\Omega^T,\kappa} \le \varphi_4(T, |1/\eta|_{\infty,\Omega^T}, \|\eta\|_{l+1,\Omega^T}, |\eta|_{l+1,0,\infty,\Omega^T}) \times \Big[\|F\|_{l,\Omega^T,\kappa} + \sum_{i=1}^2 \|G_i\|_{l+1/2,S^T,\kappa} + \|H\|_{l-1/2,S^T,\kappa} + |u(0)|_{l+1,0,\Omega} + \|u\|_{l,\Omega^T,\kappa}\Big], \tag{3.37}$$

where φ_4 *is a positive increasing function.*

Proof. We have to examine the right-hand sides of (3.35). In view of their strong nonlinearity with respect to w we only consider their qualitative behaviour. We restrict our considerations to terms without the coefficient $T^{-\kappa}$. Examining them we have

$$\begin{aligned} &\|\widetilde{F}\|_{l,\Omega^T} \le c\Big(\Big\|f_1 \int_0^t w_{\xi\xi}\,d\tau\, u_\xi\Big\|_{l,\Omega^T} + \Big\|f_2 \int_0^t w_\xi\,d\tau\, u_{\xi\xi}\Big\|_{l,\Omega^T}\Big),\\ &\|\widetilde{G}_1\|_{l+1/2,S^T} + \|\widetilde{G}_2\|_{l+1/2,S^T} \le c\Big\|f_3 \int_0^t w_\xi\,d\tau\, u_\xi\phi_\xi\Big\|_{l+1/2,S^T},\\ &\|\widetilde{H}\|_{l-1/2,S^T} \le c\Big\|f_4\Big(\int_0^t w_{\xi\xi}\,d\tau\, u_\xi + \int_0^t w_\xi\,d\tau\, u_{\xi\xi}\Big)\phi_\xi\Big\|_{l-1/2,S^T}, \end{aligned} \tag{3.38}$$

where $f_i = f_i(\delta + \int_0^t w_\xi\,d\tau)$, $i = 1,\dots,4$, are smooth and δ is the Kronecker delta. Comparing the above expressions we see that it is sufficient to estimate only the terms on the right-hand side of the first inequality. We have to bound them by an expression with the factor T^a, $a > 0$. We shall restrict our considerations to the first norm on the right-hand side in $(3.38)_1$. It is sufficient to consider the main terms in the norm. Therefore, we consider the expression

$$\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=[l]} \Big(\Big[D^{\alpha_1}_{\xi,t} f D^{\alpha_2}_{\xi,t} \int_0^t w_{\xi\xi}\,d\tau\, D^{\alpha_3}_{\xi,t} u_\xi\Big]^2_{l-[l],\Omega^T,x} + \Big[D^{\alpha_1}_{\xi,t} f D^{\alpha_2}_{\xi,t} \int_0^t w_{\xi\xi}\,d\tau\, D^{\alpha_3}_{\xi,t} u_\xi\Big]^2_{l/2-[l/2],\Omega^T,t}\Big) \equiv I_1 + I_2,$$

where f_1 was replaced by f for simplicity. We have the Leibniz formula (see for instance [5, 17])

$$D^\alpha_{\xi,t} f = \sum c_{\alpha\beta_1\dots\beta_s}(\partial^{|\alpha|+1-s} f)\Big(D^1_{\xi,t}\int_0^t w_\xi\,d\tau\Big)^{\beta_1} \dots \Big(D^s_{\xi,t}\int_0^t w_\xi\,d\tau\Big)^{\beta_s}, \tag{3.39}$$

where $s, \beta_1, \dots, \beta_s \in \mathbb{N} \cup \{0\}$, $\alpha, \alpha_1, \alpha_2, \alpha_3$ are multiindices, $\beta_1 + \dots + \beta_s = |\alpha| + 1 - s$, $s \le |\alpha|$, $\beta_1 + 2\beta_2 + \dots + s\beta_s = |\alpha|$. Below we introduced the shortened notation $D^i = D^i_{\xi,t} = \sum_{|\alpha|=i} D^\alpha_{\xi,t}$. Below D'^i is either $D^i_{\xi',t}$ or $D^i_{\xi,t'}$. Then we have

$$
\begin{aligned}
I_1 \le c \sum_{i=1}^{s} \int_0^T dt \int_\Omega \int_\Omega d\xi\, d\xi' \Big| D'^1 \int_0^t w_{\xi'}\, d\tau \Big|^{2\beta_1} \dots \\
\dots \Big| \Big(D^i \int_0^t w_\xi\, d\tau \Big)^{\beta_i} - \Big(D'^i \int_0^t w_\xi\, d\tau \Big)^{\beta_i} \Big|^2 \dots \\
\dots \Big| D^s \int_0^t w_\xi\, d\tau \Big|^{2\beta_s} \Big| D^{|\alpha_2|} \int_0^t w_{\xi\xi}\, d\tau \Big|^2 |D^{|\alpha_3|} u_\xi|^2 / |\xi - \xi'|^{3+2(l-[l])} \\
+ c \int_0^T dt \int_\Omega \int_\Omega d\xi\, d\xi' \Big| D'^1 \int_0^t w_{\xi'}\, d\tau \Big|^{2\beta_1} \dots \Big| D'^s \int_0^t w_{\xi'}\, d\tau \Big|^{2\beta_s} \\
\times \Big| D^{|\alpha_2|} \int_0^t w_{\xi\xi}\, d\tau - D'^{|\alpha_2|} \int_0^t w_{\xi'\xi'}\, d\tau \Big|^2 |D^{|\alpha_3|} u_\xi|^2 / |\xi - \xi'|^{3+2(l-[l])} \\
+ c \int_0^T dt \int_\Omega \int_\Omega d\xi\, d\xi' \Big| D^1 \int_0^t w_\xi\, d\tau \Big|^{2\beta_1} \dots \\
\dots \Big| D^s \int_0^t w_\xi\, d\tau \Big|^{2\beta_s} \Big| D^{\alpha_2} \int_0^t w_{\xi\xi}\, d\tau \Big|^2 \\
\times |D^{|\alpha_3|} u_\xi - D'^{|\alpha_3|} u_{\xi'}|^2 / |\xi - \xi'|^{3+2(l-[l])} \equiv I_{11} + I_{12} + I_{13}\,.
\end{aligned}
$$

We only consider I_{12}. The other integrals can be estimated in the same way. Moreover, we shall restrict our examinations to two cases. The first is when the time derivatives only appear in the last factor. The second is when at least one time derivative appears in each factor.

In the first case, applying the Hölder inequality we have

$$
\begin{aligned}
I_{12} \le c \int_0^T dt \bigg(\int_\Omega \int_\Omega d\xi\, d\xi' \, \frac{|\int_0^T D_\xi^2 w\, d\tau|^{2\beta_1 p_1}}{|\xi - \xi'|^{3-\varepsilon}} \bigg)^{1/p_1} \dots \\
\dots \bigg(\int_\Omega \int_\Omega d\xi\, d\xi' \, \frac{|\int_0^T D_\xi^{s+1} w\, d\tau|^{2\beta_s p_s}}{|\xi - \xi'|^{3-\varepsilon}} \bigg)^{1/p_s} \\
\times \bigg(\int_\Omega \int_\Omega d\xi\, d\xi' \, \frac{|\int_0^T (D_\xi^{|\alpha_2|+2} w - D_{\xi'}^{|\alpha_2|+2} w)\, d\tau|^{2p_{s+1}}}{|\xi - \xi'|^{3+2p_{s+1}(l-[l])+\varepsilon(p_{s+1}-1)}} \bigg)^{1/p_{s+1}} \\
\times \bigg(\int_\Omega \int_\Omega d\xi\, d\xi' \, \frac{|D_\xi^{|\alpha_3|+1} u|^{2p_{s+2}}}{|\xi - \xi'|^{3-\varepsilon}} \bigg)^{1/p_{s+2}} \equiv J_1\,,
\end{aligned}
$$

where $\varepsilon > 0$ and $1/p_1 + \ldots + 1/p_{s+2} = 1$, $\beta_1 + \ldots + \beta_s = |\alpha_1| + 1 - s$, $s \le |\alpha_1|$,

$\beta_1 + 2\beta_2 + \ldots + s\beta_s = |\alpha_1|$. As $\varepsilon > 0$ we obtain

$$
\begin{aligned}
J_1 \le cT^{2(\Sigma\beta_i+1)-(1/p_1+\ldots+1/p_{s+1})}\Big(\int_0^T |D_\xi^2 w|_{2\beta_1p_1,\Omega}^{2\beta_1p_1}\,d\tau\Big)^{1/p_1}\ldots \\
\ldots\Big(\int_0^T |D_\xi^{s+1} w|_{2\beta_sp_s,\Omega}^{2\beta_sp_s}\,d\tau\Big)^{1/p_s} \\
\times\Big(\int_0^T [D_\xi^{|\alpha_2|+2} w]_{2p_{s+1},l-[l]+(\varepsilon/2)(1-1/p_{s+1}),\Omega,x}^{2p_{s+1}}\Big)^{1/p_{s+1}}\ldots \\
\ldots\Big(\int_0^T |D_\xi^{|\alpha_3|+1} u|_{2p_{s+2},\Omega}^2\,dt\Big)^{1/2} \equiv J_2\,.
\end{aligned}
$$

We now use the imbedding theorems (2.2), (2.3), for which we need the conditions

$$
\begin{aligned}
&3/2-3/(2\beta_1p_1)+2\le l+1,\ldots,3/2-3/(2\beta_sp_s)+s+1\le l+1\,,\\
&3/2-3/(2p_{s+1})+|\alpha_2|+2+l-[l]+\varepsilon(1/2-1/(2p_{s+1}))\le l+1\,,\\
&3/2-3/(2p_{s+2})+|\alpha_3|+1\le l+2\,,
\end{aligned}
\tag{3.40}
$$

which are satisfied if $\varepsilon(1/2-1/(2p_{s+1}))\le(l-3/2)\sum\beta_i+l-3/2$, which holds for $l\ge 3/2$. Thus, we get

$$
\begin{aligned}
J_2 &\le cT^{2(\Sigma\beta_i+1)-(1/p_1+\ldots+1/p_{s+1})}\Big(\int_0^T \|w\|_{l+1,\Omega}^{2\beta_1p_1}\,d\tau\Big)^{1/p_1}\ldots \\
&\quad\ldots\Big(\int_0^T \|w\|_{l+1,\Omega}^{2\beta_sp_s}\,d\tau\Big)^{1/p_s}\Big(\int_0^T \|w\|_{l+1,\Omega}^{2p_{s+1}}\,d\tau\Big)^{1/p_{s+1}}\int_0^T \|u\|_{l+2,\Omega}^2\,d\tau \\
&\le cT^{2\Sigma\beta_i+2}\sup_t\|w\|_{l+1,\Omega}^{2\Sigma\beta_i+2}\|u\|_{l+2,\Omega^T}^2\,.
\end{aligned}
$$

Now we consider the second case. Then I_{12} takes the form

$$
\begin{aligned}
I_{12} &= c\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi' \\
&\quad\times\bigg(|w_\xi|^{2\beta_1}\ldots|D_{\xi,t}^{s-1}w|^{2\beta_s}\frac{|D_{\xi,t}^{|\alpha_2|}w-D_{\xi',t}^{|\alpha_2|}w|^2}{|\xi-\xi'|^{3+2(l-[l])}}|D_{\xi,t}^{|\alpha_3|+1}u|^2\bigg) \\
&\le c\int_0^T dt\,(|w_\xi|_{2\beta_1p_1,\Omega}^{2\beta_1}\ldots|D_{\xi,t}^{s-1}w|_{2\beta_sp_s,\Omega}^{2\beta_s} \\
&\quad\times[D_{\xi,t}^{|\alpha_2|}w]_{l-[l]+\varepsilon(1/2-1/p_{s+1}),2p_{s+1},\Omega}^2|D_{\xi,t}^{|\alpha_3|+1}u|_{2p_{s+1},\Omega}^2)\equiv J_3\,.
\end{aligned}
$$

We use the imbedding theorems (2.2), (2.3), for which we need the conditions

$$
\begin{aligned}
&3/2-3/(2\beta_2p_2)+1\le l+1,\dots,3/2-3/(2\beta_sp_s)+s-1\le l+1\,,\\
&3/2-3/(2p_{s+1})+|\alpha_2|+l-[l]+\varepsilon(1/2-1/p_{s+1})\le l+1\,,\\
&3/2-3/(2p_{s+2})+|\alpha_3|+1\le l+1\,,
\end{aligned}
\tag{3.41}
$$

which hold if the following inequality is valid:

$$\varepsilon(1/2-1/(2p_{s+1}))\le(l+1/2)\sum\beta_i+l-3/2\,,$$

and it holds for $l\ge 3/2$. Therefore, we have

$$J_3\le cT|w|^{2(\Sigma\beta_i+1)}_{l+1,0,\Omega}|u|^2_{l+1,0,\Omega}\,.$$

Continuing as above we obtain

$$I_1\le cT^a\varphi(|w|_{l+1,0,\infty,\Omega^T},\|w\|_{l+2,\Omega^T},T)(\|u\|^2_{l+2,\Omega^T}+|u|^2_{l+1,0,\infty,\Omega^T})\,,$$

for some $a>0$.

Now we examine I_2. By the Leibniz formula we have

$$
\begin{aligned}
I_2\le c\sum_{i=1}^s&\int\limits_\Omega d\xi\int\limits_0^T\int\limits_0^T dt\,dt'\,\Big|D'^1\int\limits_0^{t'}w_\xi\,d\tau\Big|^{2\beta_1}\dots\\
&\dots\Big|\Big(D^i\int\limits_0^t w_\xi\,d\tau\Big)^{2\beta_i}-\Big(D'^i\int\limits_0^{t'}w_\xi\,d\tau\Big)^{2\beta_i}\Big|\dots\\
&\dots\Big|D^s\int\limits_0^t w_\xi\,d\tau\Big|^{2\beta_s}\Big|D^{|\alpha_2|}\int\limits_0^t w_{\xi\xi}\,d\tau\Big|^2|D^{|\alpha_3|}u_\xi|^2|t-t'|^{-(1+2(l/2-[l/2]))}\\
&+c\int\limits_\Omega d\xi\int\limits_0^T\int\limits_0^T dt\,dt'\Big|D'^1\int\limits_0^{t'}w_\xi\,d\tau\Big|^{2\beta_1}\dots\Big|D'^s\int\limits_0^{t'}w_\xi\,d\tau\Big|^{2\beta_s}\\
&\times\Big|D^{|\alpha_2|}\int\limits_0^t w_{\xi\xi}\,d\tau-D'^{|\alpha_2|}\int\limits_0^{t'}w_{\xi\xi}\,d\tau\Big|^2|D^{|\alpha_3|}u_\xi|^2|t-t'|^{-(1+2(l/2-[l/2]))}\\
&+c\int\limits_\Omega d\xi\int\limits_0^T\int\limits_0^T dt\,dt'\Big|D'^1\int\limits_0^{t'}w_\xi\,d\tau\Big|^{2\beta_1}\dots\\
&\dots\Big|D'^s\int\limits_0^{t'}w_\xi\,d\tau\Big|^{2\beta_s}\Big|D^{|\alpha_2|}\int\limits_0^t w_{\xi\xi}\,d\tau\Big|^2\\
&\times|D^{|\alpha_3|}u_\xi-D'^{|\alpha_3|}u_\xi|^2|t-t'|^{-(1+2(l/2-[l/2]))}\equiv I_{21}+I_{22}+I_{23}\,.
\end{aligned}
$$

We only consider I_{22}. The other integrals can be estimated in a similar way. Moreover, we restrict our considerations to the same two cases as in the estimate of I_{12}.

In the first case, by the Hölder inequality we obtain

$$I_{22} \le c \int_0^T \int_0^T dt\, dt' \Big| \int_0^T w_{\xi\xi}\, d\tau \Big|^{2\beta_1}_{2\beta_1 p_1,\Omega} \dots \Big| \int_0^{t'} D^{s+1}_\xi d\tau \Big|^{2\beta_s}_{2\beta_s p_s,\Omega}$$
$$\times \Big| \int_0^t D^{|\alpha_2|+2}_\xi w - \int_0^{t'} D^{|\alpha_2|+2}_\xi w \Big|^2_{2p_{s+1},\Omega} |D^{|\alpha_3|}_\xi u_\xi|^2_{2p_{s+2},\Omega} |t-t'|^{-(1+2(l/2-[l/2]))} \equiv J_4 ,$$

where $1/p_1 + \ldots + 1/p_{s+2} = 1$. In view of the Minkowski inequality,

$$J_4 \le c\Big(\int_0^T |w_{\xi\xi}|_{2\beta_1 p_1,\Omega}\, d\tau \Big)^{2\beta_1} \dots \Big(\int_0^T |D^{s+1}_\xi w|_{2\beta_s p_s,\Omega}\, d\tau \Big)^{2\beta_s}$$
$$\times \int_0^T \int_0^T dt\, dt'\, |D^{|\alpha_2|+2}_\xi w(\widetilde t)|^2_{2p_{s+1},\Omega} |D^{|\alpha_3|}_\xi u_\xi|^2_{2p_{s+2},\Omega} |t-t'|^{1-2(l/2-[l/2])} \equiv J_5 ,$$

where $\widetilde t \in [t,t']$ and we have used the relation

$$\int_0^t D^{|\alpha_2|+2}_\xi w\, d\tau - \int_0^{t'} D^{|\alpha_2|+2}_\xi w\, d\tau = D^{|\alpha_2|+2}_\xi w(\widetilde t)(t-t') .$$

We employ the imbeddings (2.2) with the restrictions $3/2 - 3/(2\beta_1 p_1) + 2 \le l+1, \ldots, 3/2 - 3/(2\beta_s p_s) + s + 1 \le l+1$, $3/2 - 3/(2p_{s+1}) + |\alpha_2| + 2 \le l+2$, $3/2 - 3/(2p_{s+2}) + |\alpha_3| + 1 \le l+1$, in the case $|\alpha_2| \le [l]$. In the case $|\alpha_3| \le [l]$ the last two inequalities should be replaced by $3/2 - 3/(2p_{s+1}) + |\alpha_2| + 2 \le l+1$, $3/2 - 3/(2p_{s+2}) + |\alpha_3| + 1 \le l+2$. The above inequalities are satisfied if $0 \le (l-3/2)\sum \beta_i + l - 3/2 + l - [l]$, which holds for $l \ge 3/2$. Since $|\alpha_1| + |\alpha_2| + |\alpha_3| = [l]$ we have

$$J_5 \le cT^{2\Sigma\beta_i + 2(1-(l-[l]))} |w|^{2\Sigma\beta_i}_{l+1,0,\Omega} \chi ,$$

where

$$\chi = \begin{cases} \|w\|_{l+2,\Omega^T} |u|_{l+1,0,\Omega} & \text{for } |\alpha_2| \le [l], \\ |w|_{l+1,0,\Omega} \|u\|_{l+2,\Omega^T} & \text{for } |\alpha_3| \le [l]. \end{cases}$$

Finally, we consider the second case. Then

$$I_{22} \le c \int_\Omega d\xi \int_0^T \int_0^T dt\, dt'\, |w_\xi|^{2\beta_1} \dots |D^{s-2}_{\xi,t} w_\xi|^{2\beta_s}$$
$$\times |D^{|\alpha_2|}_{\xi,t} w - D^{|\alpha_2|}_{\xi,t'} w|^2 |D^{|\alpha_3|}_{\xi,t} u_\xi|^2 |t-t'|^{-(1+2(l/2-[l/2]))} \equiv J_6 .$$

In view of the Hölder inequality,

$$J_6 \le c \int_0^T \int_0^T dt\, dt'\, |w_\xi|^{2\beta_1}_{2\beta_1 p_1,\Omega} \dots |D^{s-2}_{\xi,t} w_\xi|^{2\beta_s}_{2\beta_s p_s,\Omega} |D^{|\alpha_2|}_{\xi,t} w - D^{|\alpha_2|}_{\xi,t'} w|^2_{2p_{s+1},\Omega}$$
$$\times |D^{|\alpha_2|}_{\xi,t} u|^2_{2p_{s+2},\Omega} |t-t'|^{-(1+2(l/2-[l/2]))} \equiv J_7 .$$

As $|\alpha_1|+|\alpha_2|+|\alpha_3|=[l]$ we have $|\alpha_2|\le[l]$ so we can always use the formula $|D^{|\alpha_2|}_{\xi,t}w-D^{|\alpha_2|}_{\xi,t'}w|=|D^{|\alpha_2|+2}_{\xi,t}w(\widetilde{t})(t-t')|$, where $\widetilde{t}\in[t,t']$. Using the imbedding theorems (2.2) with the restrictions $3/2-3/(2\beta_2p_2)+1\le l+1,\dots$, $3/2-3/(2\beta_sp_s)+s-1\le l+1$, $3/2-3/(2p_{s+1})+|\alpha_2|+2\le l+1$, $3/2-3/(2p_{s+2})+|\alpha_3|+1\le l+1$, which hold if $0\le(l+1/2)\sum\beta_i+l-[l]+l-5/2$ so they are satisfied for $l\ge3/2$, because $\sum\beta_i\ge1$. Therefore,

$$
\begin{aligned}
J_7&\le c|w|^{2\Sigma\beta_i+2}_{l+1,0,\Omega}|u|^2_{l+1,0,\Omega}\int_0^T\int_0^T dt\,dt'\,|t-t'|^{1-2(l/2-[l/2])}\\
&\le cT^{3-2(l/2-[l/2])}|w|^{2\Sigma\beta_i+2}_{l+1,0,\Omega}|u|^2_{l+1,0,\Omega}\,.
\end{aligned}
$$

Summarizing, we obtain

$$
\begin{aligned}
(3.42)\qquad &\|\widetilde{F}\|_{l,\Omega^T}+\sum_{i=1}^2\|\widetilde{G}_i\|_{l+1/2,S^T}+\|\widetilde{H}\|_{l-1/2,S^T}\\
&\le T^a(\|w\|_{l+2,\Omega^T}+|w|_{l+1,0,\infty,\Omega^T})\varphi_2(T,T^a(\|w\|_{l+2,\Omega^T}+|w|_{l+1,0,\infty,\Omega^T}))\\
&\quad\times(\|u\|_{l+2,\Omega^T}+|u|_{l+1,0,\infty,\Omega^T})\,.
\end{aligned}
$$

Similar considerations can be applied in the case of the seminorm $\|\ \|_{[l]+\kappa,\Omega^T}$. Using Lemma 3.2 and the estimates (3.19), (3.42) we obtain (3.37) for sufficiently small δ_1. To prove the existence of solutions we put u_m in the right-hand sides of (3.35) and u_{m+1} in the left-hand sides. Then for sufficiently small δ_1 we have convergence of the sequence assuming that u_0 is constructed similarly to v in the proof of Lemma 3.1. This concludes the proof.

Remark 3.4. To estimate the term $\int_0^t\widetilde{H}(\tau)\,d\tau$ we assume that S is described by $\xi=\xi(s^1,s^2)$, and S_t is determined by $x=x(\xi(s^1,s^2),t)$, so

$$
\begin{aligned}
g_{\alpha\beta}(t)=x^i_{\xi^j}\xi^j_\alpha x^i_{\xi^l}\xi^l_\beta=g_{\alpha\beta}(0)+\int_0^t w^i_{\xi^j}\,d\tau\,\xi^j_\alpha\xi^i_\beta+\int_0^t w^i_{\xi^l}\,d\tau\,\xi^i_\alpha\xi^l_\beta\\
+\int_0^t w^i_{\xi^j}\,d\tau\cdot\int_0^t w^i_{\xi^l}\,d\tau\,\xi^j_\alpha\xi^l_\beta\,,
\end{aligned}
$$

where $g_{\alpha\beta}(0)=\xi^i_\alpha\xi^i_\beta$ and $\xi^i_\alpha=\partial_{s^\alpha}\xi^i$ and the summation convention is used $(i,j,l=1,2,3,\ \alpha,\beta=1,2)$.

Moreover, considering problem (3.31) we see that G_2 must contain the term $\sigma\overline{n}_0\Delta_{S_t}(t)\xi=\sigma\overline{n}_0(\Delta_{S_t}(t)-\Delta_S(0))\xi+\sigma H(0)$, where $H(0)=n_0^kg^{\alpha\beta}(0)\xi^k_{\alpha\beta}$ is the double mean curvature of S. Taking into account the difference $g_{\alpha\beta}(t)-g_{\alpha\beta}(0)$ we have

$$
(3.43)\qquad \|\overline{n}_0(\Delta(t)-\Delta(0))\xi\|_{l-1/2,S^T}\le T^a\|w\|_{l+2,\Omega^T}\varphi_5(T,T^a\|w\|_{l+2,\Omega^T})\,,
$$

where φ_5 is an increasing positive function.

A solution of $(3.1)_{2,5}$ has the form

$$\eta(\xi,t)=\varrho_0(\xi)\exp\Big[-\int_0^t\nabla_u\cdot u(\xi,\tau)\,d\tau\Big]. \tag{3.44}$$

LEMMA 3.5. *Assume that* $\varrho_0, 1/\varrho_0\in W_2^{l+1}(\Omega)\cap C^\alpha(\Omega)$, $u\in W^{l+2,l/2+1}(\Omega^T)\cap C(0,T;\Gamma_2^{l+1}(\Omega))$, l *is as in Lemma* 3.1 *and* $l>3/2$, $\alpha\in(0,1)$. *Let* $\kappa\in(0,1/4)$. *Then the solution* (3.44) *of* $(3.1)_{2,5}$ *satisfies* $\eta, 1/\eta\in W_{2,\kappa}^{l+1}(\Omega^T)\cap C(0,T;\Gamma_2^{l+1}(\Omega))\cap C^{\alpha,\alpha/2}(\Omega^T)$ *and*

$$\begin{aligned}\|\eta\|_{l+1,\Omega^T,\kappa}&+\|1/\eta\|_{l+1,\Omega^T,\kappa}\\&\le T^a(\|\varrho_0\|_{l+1,\Omega}+\|1/\varrho_0\|_{l+1,\Omega})\varphi_6(T,T^b(\|u\|_{l+2,\Omega^T}+|u|_{l+1,0,\infty,\Omega^T})),\end{aligned} \tag{3.45}$$

$$|\eta|_{l+1,0,\infty,\Omega^T}\le\|\varrho_0\|_{l+1,\Omega}\varphi_7(T,T^b(\|u\|_{l+2,\Omega^T}+|u|_{l+1,0,\infty,\Omega^T})), \tag{3.46}$$

$$|\eta|_{C^{\alpha,\alpha/2}(\Omega^T)}\le|\varrho_0|_{C^\alpha(\Omega)}\varphi_8(T^b\|u\|_{l+2,\Omega^T}), \tag{3.47}$$

where a,b *are positive numbers*, $\varphi_6,\varphi_7,\varphi_8$ *are increasing positive functions*, *and* $C^{\alpha,\alpha/2}(\Omega^T)$, $C^\alpha(\Omega)$ *are the Hölder spaces* (see [10]).

Proof. First we prove (3.45). Using the explicit dependence $\xi_x=f(\delta+\int_0^t u_\xi\,d\tau)$ we have

$$\|\eta\|_{l+1,\Omega^T}+\|1/\eta\|_{l+1,\Omega^T}\le c\Big\|r_0(\xi)f\Big(\delta+\int_0^t u_\xi\,d\tau\Big)\Big\|_{l+1,\Omega^T}\equiv K_1\,,$$

where $r_0(\xi)=\varrho_0(\xi)+1/\varrho_0(\xi)$. To estimate K_1 it is sufficient to consider the highest derivatives. Therefore, we consider

$$\begin{aligned}\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}&([D_{\xi,t}^{\alpha-\beta}r_0D_{\xi,t}^\beta f]_{l+1-[l+1],\Omega^T,x}\\&+[D_{\xi,t}^{\alpha-\beta}r_0D_{\xi,t}^\beta f]_{(l+1)/2-[(l+1)/2],\Omega^T,t})\equiv K_2+K_3\,.\end{aligned}$$

By the Leibniz formula (see the notation in the proof of Lemma 3.3) we have

$$\begin{aligned}K_2&\le P_1\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}\Big[D_{\xi,t}^{\alpha-\beta}r_0\Big(D_{\xi,t}^1\int_0^t u_\xi\,d\tau\Big)^{\beta_1}\dots\\&\quad\dots\Big(D_{\xi,t}^s\int_0^t u_\xi\,d\tau\Big)^{\beta_s}\Big]_{l-[l],\Omega^T,x}\\&\le P_1\Big(\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi'\\&\quad\times\Big[\frac{|D_{\xi,t}^{\alpha-\beta}r_0-D_{\xi',t}^{\alpha-\beta}r_0|^2}{|\xi-\xi'|^{s+2(l-[l])}}\Big|D_{\xi',t}^1\int_0^t u_{\xi'}\,d\tau\Big|^{2\beta_1}\dots\end{aligned}$$

$$\ldots \Big| D^s_{\xi',t} \int\limits_0^t u_{\xi'}\,d\tau \Big|^{2\beta s} + \sum_{i=1}^s |D^{\alpha-\beta}_{\xi,t} r_0|^2 \ldots$$
$$\ldots \frac{|(D^i_{\xi,t}\int_0^t u_\xi\,d\tau)^{2\beta_i} - (D^i_{\xi',t}\int_0^t u_\xi\,d\tau)^{2\beta_i}|}{|\xi-\xi'|^{3+2(l-[l])}} \ldots \Big| D^s_{\xi',t} \int\limits_0^t u_\xi\,d\tau \Big|^{2\beta_s} \bigg]\bigg)$$
$$\equiv K_4 + K_5\,,$$

where $\beta_1+\ldots+\beta_s = |\beta|+1-s$, $\beta_1+2\beta_2+\ldots+s\beta_s = |\beta|$, $s \le [l+1]$, $\beta_1,\ldots,\beta_s$, $s, i \in \mathbb{N}\cup\{0\}$, α,β are multiindices and $P_1 = P_1(\int_0^t |u_\xi|_{\infty,\Omega}\,d\tau)$ is an increasing positive function. We only examine K_5. Let $i < [l+1] \equiv k$. Then using

$$\Big(D^i_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big)^{2\beta_i} - \Big(D^i_{\xi',t}\int\limits_0^t u_\xi\,d\tau\Big)^{2\beta_i} = 2\beta_i\Big(D^i_{\xi,t}\int\limits_0^t u_{\xi\xi}(\widetilde{\xi})\,d\tau\Big)^{2\beta_i-1}(\xi-\xi')$$

for some $\widetilde{\xi} \in [\xi,\xi']$, and the Hölder inequality yields

$$K_5 \le P_1\Big(\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}\sum_{i=0}^k \int\limits_0^T dt\,|D^{\alpha-\beta}_\xi r_0|^2_{2p_0,\Omega}\Big|D^1_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big|^{2\beta_1}_{2\beta_1p_1,\Omega}\ldots$$
$$\ldots\Big|D^i_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big|^{2\beta_i-1}_{(2\beta_i-1)p_i,\Omega}\Big|D^i_{\xi,t}\int\limits_0^t u_{\xi\xi}\,d\tau\Big|^2_{2p_i',\Omega}\ldots$$
$$\ldots\Big|D^k_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big|^{2\beta_k}_{2\beta_kp_k,\Omega}\Big)^{1/2} \equiv K_6,$$

where $1/p_0+1/p_1+\ldots+1/p_s+1/p_i' = 1$, $i=0,\ldots,k$.

First we consider the case when the t-derivatives do not appear in the derivative D. We use the Minkowski inequality and the imbedding (2.2) with the restrictions $3/2-3/(2p_0)+|\alpha|-|\beta| \le l+1$, $3/2-3/(2\beta_jp_j)+j+1 \le l+2$, $j\ne i$, $3/2-3/((2\beta_i-1)p_i)+i+1 \le l+2, 3/2-3/(2p')+i+2 \le l+2$, which hold if $0 \le (l-1/2)\sum\beta_\sigma + l+1-[l+1]+(1/2)(3/2+l-i)$. The relation is satisfied for $l>1/2$ and $\sum\beta_i \ge 1$.

Hence, we obtain

$$K_6 \le T^a\varphi_1'(T, T^b\|u\|_{l+2,\Omega^T})\|r_0\|_{l+1,\Omega}\,,$$

where $a>0$, $b>0$ and φ_1' is an increasing positive function.

Consider the case when at least one t-derivative appears in each derivative $D_{\xi,t}$. Then we use the imbedding (2.2) with $3/2-3/(2p_0)+|\alpha|-|\beta| \le l+1$, $3/2-3/(2\beta_jp_j)+j-1 \le l+1$, $3/2-3/((2\beta_i-1)p_i)+i-1 \le l+1$, $3/2-3/(2p_i')+i \le l+1$, $j\ne i, j\ge 2$, which hold for $3/4 \le (l+1/2)\sum\beta_\sigma + l+1-[l+1]+(l-i)/2$, which is satisfied for $\sum\beta_\sigma \ge 1$ and $l \ge 1/2$. Therefore, we obtain the estimate

$$K_6 \le P_1T\|r_0\|^2_{l+1,\Omega}\|u\|^{2\Sigma\beta_\sigma}_{l+1,0,\infty,\Omega^T}\,.$$

In the case $i=[l+1]\equiv k$ we have $\beta_k=1$, $\beta_j=0$, $j<k$ and

$$K_5=P_1|r_0|^2_{\infty,\Omega}\int\limits_0^T dt\int\limits_\Omega\int\limits_\Omega d\xi\,d\xi'\frac{|D^k_{\xi,t}\int_0^t u_\xi\,d\tau-D^k_{\xi',t}\int_0^t u_{\xi'}\,d\tau|^2}{|\xi-\xi'|^{3+2(l+1-[l+1])}}\,.$$

In the case when no t-derivative appears in $D_{\xi,t}$ we have

$$\begin{aligned}K_5&\le P_1|r_0|^2_{\infty,\Omega}T^2[D^{k+1}_\xi u]^2_{l+2-[l+2],2,\Omega^T,x}\\&\le P_1T^2\|r_0\|^2_{l+1,\Omega}\|u\|^2_{l+2,\Omega^T}\,.\end{aligned}$$

If at least one t-derivative appears in $D_{\xi,t}$ we have

$$\begin{aligned}K_5&=P_1|r_0|^2_{\infty,\Omega}\int\limits_0^T dt\int\limits_\Omega\int\limits_\Omega d\xi\,d\xi'\frac{|D^{k-2}_{\xi,t}u_\xi-D^{k-2}_{\xi',t}u_{\xi'}|^2}{|\xi-\xi'|^{3+2(l+1-[l+1])}}\\&\le cP_1\|r_0\|^2_{l+1,\Omega}\int\limits_0^T dt\,|D^k_{\xi,t}u|^2_{2,\Omega}\le cT\|r_0\|^2_{l+1,\Omega}\sup_t|u|^2_{l+1,\Omega}\,.\end{aligned}$$

Now we estimate K_3. By the Leibniz formula,

$$\begin{aligned}K_3&\le P_2\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}\Big[D^{\alpha-\beta}_{\xi,t}r_0\Big(D^1_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big)^{\beta_1}\dots\\&\qquad\dots\Big(D^s_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big)^{\beta_s}\Big]_{(l+1)/2-[(l+1)/2],\Omega^T,t}\\&\le P_2\bigg(\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}\int\limits_\Omega d\xi\int\limits_0^T\int\limits_0^T dt\,dt'\bigg[\frac{|D^{\alpha-\beta}_{\xi,t}r_0-D^{\alpha-\beta}_{\xi,t'}r_0|^2}{|t-t'|^{1+2((l+1)/2-[(l+1)/2])}}\\&\qquad\times\Big|D^1_{\xi,t'}\int\limits_0^{t'}u_\xi\,d\tau\Big|^{2\beta_1}\dots\Big|D^s_{\xi,t'}\int\limits_0^{t'}u_\xi\,d\tau\Big|^{2\beta_s}\\&\qquad+\sum_{i=0}^s|D^{\alpha-\beta}_{\xi,t}r_0|^2\Big|D^1_{\xi,t}\int\limits_0^t u_\xi\,d\tau\Big|^{2\beta_1}\dots\\&\qquad\dots\frac{|(D^i_{\xi,t}\int_0^t u_\xi\,d\tau)^{2\beta_i}-(D^i_{\xi,t'}\int_0^{t'}u_\xi\,d\tau)^{2\beta_i}|^2}{|t-t'|^{1+2((l+1)/2-[(l+1)/2])}}\dots\\&\qquad\dots\Big|D^s_{\xi,t'}\int\limits_0^{t'}u_\xi\,d\tau\Big|^{2\beta_s}\bigg]^{1/2}\bigg)\equiv K_7+K_8\,,\end{aligned}$$

where $P_2=P_2(\int_0^t|u_\xi|_{\infty,\Omega}\,d\tau)$ and the numbers β_i are the same as before. We only examine K_8. Consider the case $s<[l+1]\equiv k$. First we assume that no

t-derivative appears in $D_{\xi,t}$. Since

$$\Big(\int_0^t D_\xi^{i+1}u\,d\tau\Big)^{2\beta_i}-\Big(\int_0^{t'} D_\xi^{i+1}u\,d\tau\Big)^{2\beta_i} = 2\beta_i\Big(\int_0^{\tilde t} D_\xi^{i+1}u_\xi\,d\tau\Big)^{2\beta_i-1}D_\xi^{i+1}u(\widetilde t)(t-t')\,,$$

for some $\widetilde t\in(t,t')$, we obtain

$$\begin{aligned}K_8 \le P_2\Big(&\sum_{|\alpha|=[l+1]}\sum_{|\beta|\le|\alpha|}\int_0^T\int_0^T dt\,dt'\,|D_\xi^{\alpha-\beta}r_0|^2_{2p_0,\Omega}\Big|D_\xi^1\int_0^t u_\xi\,d\tau\Big|^{2\beta_1}_{2\beta_1p_1,\Omega}\dots\\ &\dots\Big|\int_0^t D_\xi^{i+1}u\,d\tau\Big|^{2\beta_i-1}_{(2\beta_i-1)p_i,\Omega}|D_\xi^{i+1}u(\widetilde t)|^2_{2p_i',\Omega}\dots\Big|\int_0^{t'} D_\xi^{s+1}u\,d\tau\Big|^{2\beta_s}_{2\beta_sp_s,\Omega}\\ &\times|t-t'|^{1-2((l+1)/2-[(l+1)/2])}\Big)^{1/2}\equiv K_9\,,\end{aligned}$$

where $1/p_0+1/p_1+\ldots+1/p_s+1/p_i'=1$, $i\in\{0,\ldots,s\}$. We use the imbedding (2.2) with the restrictions $3/2-3/(2p_0)+|\alpha|-|\beta|\le l+1$, $3/2-3/(2\beta_jp_j)+j+1\le l+2$, $j\ne i$, $3/2-3/((2\beta_i-1)p_i)+i+1\le l+2$, $3/2-3/(2p_i')+i+1\le l+2$, which are satisfied because the relation $1/4\le l+1-[l+1]+(l-i)/2+(l-1/2)\sum\beta_\sigma$ holds for $l>3/4$ and $\sum\beta_\sigma\ge 1$. Then from the Minkowski and Hölder inequalities it follows that

$$K_9\le\|r_0\|_{l+1,\Omega}T^a\varphi_2'(T,T^b\|u\|_{l+2,\Omega^T})\qquad\text{for some } a>0\,,\ b>0\,.$$

Considering the case when in each derivative $D_{\xi,t}$ at least one t-derivative appears we obtain

$$K_8\le\|r_0\|_{l+1,\Omega}T^a\varphi_3'(T,T^b|u|_{l+1,0,\infty,\Omega^T})\qquad\text{for some } a>0\,,\ b>0\,,$$

where we have used the Hölder inequality and the imbedding (2.2).

Similar considerations apply in the remaining cases. In this way (3.45) is proved.

To prove (3.46) the same considerations must be used but since the L_∞ norm with respect to t is taken the factor T^a does not appear.

To prove that $\eta\in C(0,T;\Gamma_2^{l+1}(\Omega))$ we have to show that $[D^\alpha_{\xi,t}\eta]_{l+1,\Omega,x}$ is continuous with respect to t. This follows from the form (3.44) of η. This concludes the proof.

Now we prove the main result of this section.

THEOREM 3.6. *Let* $v_0\in W_2^{l+1}(\Omega)$, $\varrho_0\in W_2^{l+1}(\Omega)$, $f\in C^{l+1}(\mathbb{R}^3\times(0,T))$, $S\in W_2^{l+5/2}$, *l is as in Lemma* 3.1, $l\ge 3/2$ *and* $\kappa\in(0,1/4)$. *Let G be the function from* (3.55) (see the proof) *and suppose that* $A>G(0,0,\alpha,\beta,\gamma)$, *where* α,β,γ *are defined by* (3.53). *Let* $|v(0)|_{l+1,0,\Omega}\le A$. *Let* δ_1 *be sufficiently small.*

Let T_* be so small that

$$T_*^a A\varphi_3(T_*, T_*^a A, A, A) \le \delta_1 \quad \text{(see (3.36))},$$
$$0 < c_1(1 - AT_*)^3 \le \det\{\partial x/\partial\xi\} \le c_2(1 + AT_*)^3,$$

where $x(\xi,t) = \xi + \int_0^t \widetilde{v}_0(\xi,\tau)\,d\tau$, $t \le T_*$, $G(T_*, T_*^a A, \alpha, \beta, \gamma) \le A$, $a > 0$ and $\widetilde{v}_0(\xi,t)$ is defined below. Then there exists T_{**}, $0 < T_{**} \le T_*$ (see (3.62), (3.64)), *such that for $T \le T_{**}$ there exists a unique solution to problem* (3.1) *such that $u \in W_{2,\kappa}^{l+2,l/2+1}(\Omega^T)$, $\eta \in W_{2,\kappa}^{l+1,l/2+1/2}(\Omega^T) \cap C([0,T]; \Gamma_2^{l+1}(\Omega))$ and*

$$\|u\|_{l+2,\Omega^T,\kappa} \le A,$$
$$(3.48) \qquad \|\eta\|_{l+1,\Omega^T,\kappa} + \|\eta\|_{l+1,\infty,\Omega^T} + \|1/\eta\|_{l+1,\Omega^T,\kappa} \le (\|\varrho_0\|_{l+1,\Omega} + \|1/\varrho_0\|_{l+1,\Omega})\varphi_6(T, T^a A),$$

where φ_6 is an increasing positive function.

Proof. We prove the existence of solutions to problem (3.1) by the following method of successive approximations:

$$(3.49) \qquad \begin{aligned} &\eta_m\partial_t u_{m+1} - \mu\nabla_{u_m}^2 u_{m+1} - \nu\nabla_{u_m}\nabla_{u_m}\cdot u_{m+1} \\ &\qquad = -\nabla_{u_m} q(\eta_m) + \eta_m g && \text{in } \Omega^T, \\ &\Pi_0 \Pi_{u_m}\mathbb{D}_{u_m}(u_{m+1})\overline{n}(u_m) = 0 && \text{on } S^T, \\ &\overline{n}\mathbb{D}_{u_m}(u_{m+1})\overline{n}(u_m) - \sigma\overline{n}_0\Delta_m(t)\int_0^t u_{m+1}(\tau)\,d\tau \\ &\qquad = \overline{n}_0\cdot\overline{n}(u_m)(q(\eta_m) - p_0) + \sigma\overline{n}_0\Delta_m(t)\xi && \text{on } S^T, \\ &u_{m+1}|_{t=0} = v_0 && \text{in } \Omega, \end{aligned}$$

and

$$(3.50) \qquad \begin{aligned} &\partial_t\eta_m + \eta_m\nabla_{u_m}\cdot u_m = 0 && \text{in } \Omega^T, \\ &\eta_m|_{t=0} = \varrho_0(\xi) && \text{in } \Omega, \end{aligned}$$

where $m = 0, 1, \ldots$, $u_0 = \widetilde{v}_0$ and Π_{u_m}, Δ_m denote that they depend on u_m. Now we define $\widetilde{v}_0$. Let us introduce the functions $\varphi^i = \partial_t^i u|_{t=0}$, $i \le [l/2 + 1/2]$, which are calculated inductively from (3.1). The functions φ^i satisfy the following compatibility conditions:

$$\partial_t^i(\mathbb{T}_u(u,q)\overline{n}(\xi,t) - \sigma\Delta_{S_t}(t)X_u(\xi,t) + p_0\overline{n}(\xi,t))|_{t=0} = 0,$$

where $\partial_t^i u|_{t=0}$, $\partial_t^i\eta|_{t=0}$ have to be calculated inductively from $(3.1)_{1,2}$ and $(3.1)_{4,5}$. Next we extend φ^i to functions $\widetilde{\varphi}^i$ on $\mathbb{R}^3$, and define $\widetilde{v}$ to be the solution of the Cauchy problem

$$(\partial_t - \Delta)^{[l/2-1/2]}\widetilde{v} = 0, \quad \partial_t^i\widetilde{v}|_{t=0} = \widetilde{\varphi}^i, \quad i \le [l/2 - 1/2].$$

Finally, $\widetilde{v}_0 = \widetilde{v}|_\Omega$.

Assume that (3.36) is satisfied (with $w = u_m$, $\eta = \eta_m$) with sufficiently small δ_1 and use Remark 3.4. Then by Lemma 3.3 there exists a unique solution to problem (3.49) such that $u_{m+1} \in W_{2,\kappa}^{l+2,l/2+1}(\Omega^T)$, where $T = T(\delta_1)$ is also small and

$$
\begin{aligned}
(3.51)\qquad & \|u_{m+1}\|_{l+2,\Omega^T,\kappa} + |u_{m+1}|_{l+1,0,\infty,\Omega^t} \\
&\le \varphi_1(|\eta_m|_{l+1,0,\infty,\Omega^T}, |1/\eta_m|_{l+1,0,\infty,\Omega^T}) \Big[\|\nabla_\xi q(\eta_m)\|_{l,\Omega^T,\kappa} \\
&\quad + |\eta_m|_{l+1,0,\infty,\Omega^T} \|g\|_{l,\Omega^T,\kappa} + \int_0^T \|u_{m+1}\|_{l+2,\Omega^t,\kappa}\, dt \\
&\quad + T^a \|u_m\|_{l+2,\Omega^T,\kappa} \varphi_5(T, T^a(\|u_m\|_{l+2,\Omega^T} + |u_m|_{l+1,0,\infty,\Omega^T}), \|1/\eta_m\|_{l+1,\Omega^T}) \\
&\qquad + |u_{m+1}(0)|_{l+1,0,\Omega} \Big],
\end{aligned}
$$

where we have used $\|u\|_{l,\Omega^T}^2 \le c \int_0^T \|u\|_{l+1,\Omega^t}^2\, dt + |u(0)|_{l,0,\Omega}^2$, which follows from $u(t) = \int_0^t \partial_\tau u(\tau)\, d\tau + u(0)$. By Lemma 3.5 we have

$$
(3.52)\qquad
\begin{aligned}
& \|\eta_m\|_{l+1,\Omega^T,\kappa} + \|1/\eta_m\|_{l+1,\Omega^T,\kappa} \\
&\qquad \le (\|\varrho_0\|_{l+1,\Omega} + \|1/\varrho_0\|_{l+1,\Omega}) T^a \varphi_6(T, T^a \|u_m\|_{l+2,\Omega^T}), \\
& |\eta_m|_{l+1,0,\infty,\Omega^t} + |1/\eta_m|_{l+1,0,\infty,\Omega^t} \\
&\qquad \le (\|\varrho_0\|_{l+1,\Omega} + \|1/\varrho_0\|_{l+1,\Omega}) \varphi_7(T, T^a \|u_m\|_{l+2,\Omega^T}),
\end{aligned}
$$

where T must be sufficiently small (see (3.2)) and $t \le T$.

Introduce

$$
(3.53)\qquad
\begin{aligned}
& y_m(t) = \|u_m\|_{l+2,\Omega^t} + |u_m|_{l+1,0,\infty,\Omega^t}, \\
& \alpha = \|\varrho_0\|_{l+2,\Omega}, \quad \beta = \|f\|_{C^{l+1}(\mathbb{R}^3 \times (0,T))}, \quad \gamma = |u(0)|_{l+1,0,\Omega}, \quad t \le T.
\end{aligned}
$$

Then (3.51) and (3.52) imply

$$
(3.54)\qquad y_{m+1}(t) \le \varphi_8(t, t^a y_m(t), \alpha) \int_0^t y_{m+1}(\tau)\, d\tau + \varphi_9(t, t^a y_m(t), \alpha, \beta, \gamma),
$$

where φ_8 and φ_9 are positive increasing functions. Finally, from (3.54) we obtain

$$
\begin{aligned}
(3.55)\qquad y_{m+1}(t) &\le \varphi_9(t, t^a y_m(t), \alpha, \beta, \gamma) \exp[t \varphi_8(t, t^a y_m(t), \alpha)] \\
&\equiv G(t, t^a y_m(t), \alpha, \beta, \gamma).
\end{aligned}
$$

The function G is such that $G(0, 0, \alpha, \beta, \gamma) \equiv G_0(\alpha, \beta, \gamma) > 0$. Let $A > 0$ be such that $|u(0)|_{l+1,0,\Omega} \le A$ and $G_0(\alpha, \beta, \gamma) < A$. Then for $y_m \le A$ there exists a time $T_* \le T$ such that for $t \le T_*$ we have

$$
(3.56)\qquad y_{m+1} \le G(t, t^a A, \alpha, \beta, \gamma) \le A.
$$

Thus we have proved that

$$
(3.57)\qquad y_m(t) \equiv \|u_m\|_{l+2,\Omega^t} + |u|_{l+1,0,\infty,\Omega^t} \le A, \qquad m = 0, 1, \ldots,\ t \le T_*.
$$

Now we prove the convergence of the sequence $\{u_m,\eta_m\}$. Consider the following system of problems for the differences $U_m=u_m-u_{m-1}$, $H_m=\eta_m-\eta_{m-1}$:

$$
\begin{aligned}
&\eta_m\partial_tU_{m+1}-\mu\nabla^2_{u_m}U_{m+1}-\nu\nabla_{u_m}\nabla_{u_m}\cdot U_{m+1}\\
&\qquad=-H_m\partial_tu_m-\mu(\nabla^2_{u_m}-\nabla^2_{u_{m-1}})u_m\\
&\qquad\quad-\nu(\nabla_{u_m}\nabla_{u_m}\cdot-\nabla_{u_{m-1}}\nabla_{u_{m-1}}\cdot)u_m\\
&\qquad\quad+\nabla_{u_m}q(\eta_m)-\nabla_{u_{m-1}}q(\eta_{m-1})+H_mg\\
&\qquad\equiv F_1+F_2,\\
&\varPi_0\varPi_{u_m}\mathbb{D}_{u_m}(U_{m+1})\overline{n}(u_m)\\
&\qquad=\varPi_0[\varPi_{u_m}\mathbb{D}_{u_m}(u_m)\overline{n}(u_m)-\varPi_{u_{m-1}}\mathbb{D}_{u_{m-1}}(u_m)\overline{n}(u_{m-1})]\equiv G_1\,,\\
&\overline{n}_0\mathbb{D}_{u_m}(U_{m+1})\overline{n}(u_m)-\sigma\overline{n}_0\varDelta_m(t)\int\limits_0^tU_{m+1}(\tau)\,d\tau\\
&\qquad=\overline{n}_0[\mathbb{D}_{u_m}(u_m)\overline{n}(u_m)-\mathbb{D}_{u_{m-1}}(u_m)\overline{n}(u_{m-1})]\\
&\qquad\quad-\sigma\overline{n}_0(\varDelta_m(t)-\varDelta_{m-1}(t))\int\limits_0^tu_m(\tau)\,d\tau\\
&\qquad\quad+\overline{n}_0\cdot[\overline{n}(u_m)q(\eta_m)-\overline{n}(u_{m-1})q(\eta_{m-1})]-p_0\overline{n}_0\cdot(\overline{n}(u_m)\\
&\qquad\quad-\overline{n}(u_{m-1}))+\sigma\overline{n}_0(\varDelta_m(t)-\varDelta_{m-1}(t))\xi\equiv G_2+G_2'\,,\\
&U_{m+1}|_{t=0}=0\,,
\end{aligned}\tag{3.58}
$$

where

$$
F_2=-H_m\partial_tu_m+H_mg+\nabla_{u_m}q(\widetilde{\eta}_m)H_m\,,\qquad G_2'=\overline{n}(u_m)(q(\eta_m)-q(\eta_{m-1}))\,,
$$

and F_1,G_2 are determined by the remaining terms on the right-hand sides of $(3.58)_{1,3}$, respectively.

To estimate the right-hand sides of (3.58) we only consider their qualitative forms:

$$
\begin{aligned}
&F_1=f_1\int\limits_0^tU_{m\xi}\,d\tau\,u_{m\xi\xi}+f_2\int\limits_0^tU_{m\xi\xi}\,d\tau\,u_{m\xi}+f_3\int\limits_0^tU_{m\xi}\,d\tau\,f_1'\eta_{m-1,\xi}\,,\\
&G_1=f_4\int\limits_0^tU_{m\xi}\,d\tau\,,\\
&G_2=f_5\int\limits_0^tU_{m\xi}\,d\tau\,(1+u_{m\xi})+f_2'\int\limits_0^tU_{m\xi}\,d\tau\,,\qquad G_2'=f_6f_3'H_m\,,
\end{aligned}
$$

where $f_i=f_i(\delta+\int_0^tu_{m\xi}\,d\tau,\delta+\int_0^tu_{m-1,\xi}\,d\tau)$, $i=1,\dots,6$, $f_j'=f_j'(\eta_m)$, $j=1,2,3$, are C^∞ functions of their arguments.

Now we examine

$$\|H_m\partial_t u_m\|_{l,\Omega^T} = [H_m\partial_t u_m]_{l,2,\Omega^T,x} + [H_m\partial_t u_m]_{l/2,2,\Omega^T,t} + \{\text{lower order terms}\}.$$

We need only estimate the highest order terms. We omit the subscript m for simplicity. Then

$$\begin{aligned}
[Hu_t]_{l,2,\Omega^T,x} &= \sum_{|\alpha|=[l]}\sum_{|\beta|=0}^{|\alpha|} c_{\alpha\beta}[D^\beta_{\xi,t}HD^{\alpha-\beta}_{\xi,t}u_t]_{2,l-[l],\Omega^T,x}\\
&\le c\sum_{|\alpha|=[l]}\bigg(\bigg(\sum_{|\beta|=0}^{|\alpha|}\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi'\,\frac{|D^\beta_{\xi,t}H - D^\beta_{\xi',t}H|^2}{|\xi-\xi'|^{3+2(l-[l])}}|D^{\alpha-\beta}_{\xi,t}u_t|^2\bigg)^{1/2}\\
&\quad+\bigg(\sum_{|\beta|=0}^{|\alpha|}\int_0^T dt\int_\Omega\int_\Omega d\xi\,d\xi'\,|D^\beta_{\xi,t}H|^2\frac{|D^{\alpha-\beta}_{\xi,t}u_t - D^{\alpha-\beta}_{\xi',t}u_t|^2}{|\xi-\xi'|^{3+2(l-[l])}}\bigg)^{1/2}\bigg)\\
&\le c\sum_{|\alpha|=[l]}\bigg(\sum_{|\beta|=0}^{|\alpha|}\bigg(\int_0^T dt\,[D^\beta_{\xi,t}H]^2_{2p_\beta,l-[l]+\varepsilon/2,\Omega,x}|D^{\alpha-\beta}_{\xi,t}u_t|^2_{2q_\beta,\Omega}\bigg)^{1/2}\\
&\quad+c\sum_{|\beta|=0}^{|\alpha|}\bigg(\int_0^T dt\,[D^{\alpha-\beta}_{\xi,t}u_t]^2_{2p'_\beta,l-[l]+\varepsilon/2,\Omega,x}|D^\beta_{\xi,t}H|^2_{2q'_\beta,\Omega}\bigg)^{1/2}\bigg)\equiv L_1\,,
\end{aligned}$$

where $1/p_\beta + 1/q_\beta = 1$, $1/p'_\beta + 1/q'_\beta = 1$ and $\varepsilon > 0$ was used. By the imbedding theorems (2.2) and (2.3) with $3/2-3/(2p_\beta)+|\beta|+l-[l]+\varepsilon/2 \le l+1$, $3/2-3/(2q_\beta)+|\alpha|-|\beta| \le l$ and $3/2-3/(2p'_\beta)+|\alpha|-|\beta|+l-[l]+\varepsilon/2 \le l$, $3/2-3/(2q'_\beta)+|\beta| \le l+1$, which are satisfied if $1/2+\varepsilon/2 \le l$, we obtain

$$L_1 \le c|H|_{l+1,0,\infty,\Omega^T}\|u\|_{l+2,\Omega^T}\,.$$

Consider now the expression

$$\begin{aligned}
&[Hu_t]_{l/2,2,\Omega^T,t}\\
&\quad=\sum_{|\alpha|=[l]}\bigg(\sum_{|\beta|\le|\alpha|}c_{\alpha\beta}\int_0^T\int_0^T dt\,dt'\int_\Omega d\xi\,\frac{|D^\beta_{\xi,t}H - D^\beta_{\xi,t'}H|^2}{|t-t'|^{1+2(l/2-[l/2])}}|D^{\alpha-\beta}_{\xi,t}u_t|^2\\
&\qquad+|D^\beta_{\xi,t'}H|^2\frac{|D^{\alpha-\beta}_{\xi,t}u_t - D^{\alpha-\beta}_{\xi,t'}u_t|^2}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)^{1/2}\equiv L_2\,.
\end{aligned}$$

In view of the Hölder inequality we have

$$\begin{aligned}
L_2 &\le c\sum_{|\alpha|=[l]}\sum_{|\beta|\le|\alpha|}\bigg[\bigg(\int_0^T\int_0^T dt\,dt'\,\frac{|D^\beta_{\xi,t}H - D^\beta_{\xi,t'}H|^2_{2p_\beta,\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}|D^{\alpha-\beta}_{\xi,t}u_t|^2_{2q_\beta,\Omega}\bigg)^{1/2}\\
&\quad+\bigg(\int_0^T\int_0^T dt\,dt'\,|D^\beta_{\xi,t'}H|^2_{2q'_\beta,\Omega}\frac{|D^{\alpha-\beta}_{\xi,t}u_t - D^{\alpha-\beta}_{\xi,t'}u_{t'}|^2_{2p'_\beta,\Omega}}{|t-t'|^{1+2(l/2-[l/2])}}\bigg)^{1/2}\bigg]\equiv L_3\,.
\end{aligned}$$

Employing the imbedding (2.2) with $3/2-3/(2p_\beta)+|\beta|\le[l]+1$, $3/2-3/(2q_\beta)+|\alpha|-|\beta|+2\le l+1$, which hold for $l\ge 3/2$, and $3/2-3/(2q'_\beta)+|\beta|\le l+1$, $3/2-3/(2p'_\beta)+|\alpha|-|\beta|+2\le[l]+2$, the latter being satisfied for $l\ge 1/2$, we obtain

$$L_3\le c\|H\|_{l+1,\Omega^T}|u|_{l+1,0,\infty,\Omega^T}+c|H|_{l+1,0,\infty,\Omega^T}\|u\|_{l+2,\Omega^T}\,.$$

Summarizing, we have shown

$$\begin{aligned}(3.59)\qquad &\|H_m u_{mt}\|_{l,\Omega^T}\\ &\quad\le c(\|H_m\|_{l+1,\Omega^T}+|H_m|_{l+1,0,\infty,\Omega^T})(\|u_m\|_{l+2,\Omega^T}+|u_m|_{l+1,0,\infty,\Omega^T})\,.\end{aligned}$$

Hence after similar considerations we obtain

$$\begin{aligned}(3.60)\qquad \|F_2\|_{l,\Omega^T}+\|G'_2\|_{l+1/2,S^T}&\le c(\|H_m\|_{l+1,\Omega^T}+|H_m|_{l+1,0,\infty,\Omega^T})\\ &\quad\times(\|u_m\|_{l+2,\Omega^T}+|u_m|_{l+1,0,\infty,\Omega^T})\,.\end{aligned}$$

Considering the expressions F_1,G_1,G_2 we see that in each of their terms there appears at least one factor which is an integral with respect to time. Therefore, repeating the proof of Lemma 3.4 yields

$$\begin{aligned}(3.61)\qquad &\|F_1\|_{l,\Omega^T}+\|G_1\|_{l+1/2,S^T}+\|G_2\|_{l+1/2,\Omega^T}\\ &\quad\le\varphi_{10}(A)T^a(\|U_m\|_{l+2,\Omega^T}+|U_m|_{l+1,0,\infty,\Omega^T})\,.\end{aligned}$$

Applying Lemma 3.3 and Remark 3.4 to problem (3.58) gives

$$\begin{aligned}(3.62)\qquad \|U_{m+1}\|_{l+2,\Omega^t,\kappa}&\le\varphi_{11}(t,A)t^a(\|U_m\|_{l+2,\Omega^t,\kappa}+|U_m|_{l+1,0,\infty,\Omega^t})\\ &\quad+\varphi_{12}(t,A)(\|H_m\|_{l+1,\Omega^t,\kappa}+|H_m|_{l+1,0,\infty,\Omega^t})\,.\end{aligned}$$

Next we consider the problem

$$(3.63)\qquad \begin{aligned}&\partial_t H_m+H_m\operatorname{div}_{u_m}u_m=-\eta_{m-1}(\operatorname{div}_{u_m}u_m-\operatorname{div}_{u_{m-1}}u_{m-1})\,,\\ &H_m|_{t=0}=0\,.\end{aligned}$$

Integrating (3.63) with respect to time yields

$$\begin{aligned}H_m(\xi,t)=&-\exp\Big[-\int_0^t\operatorname{div}_{u_m}u_m\,dt'\Big]\\ &\times\int_0^t\Big(\eta_{m-1}(\operatorname{div}_{u_m}u_m-\operatorname{div}_{u_{m-1}}u_{m-1})\exp\int_0^{t'}\operatorname{div}_{u_m}u_m\,dt''\Big)dt'\,,\end{aligned}$$

so we get

$$(3.64)\qquad \|H_m\|_{l+1,\Omega^t,\kappa}+|H_m|_{l+1,0,\infty,\Omega^t}\le\varphi_{13}(t,A)t^a\|U_m\|_{l+2,\Omega^t,\kappa}\,.$$

Thus, by (3.62) and (3.64), for $t\le T_{**}$, where T_{**} is sufficiently small, the sequence $\{u_m,\eta_m\}$ converges to a limit

$$\begin{aligned}\{u,\eta\}\in W^{l+2,l/2+1}_{2,\kappa}(\Omega^t)&\cap L_\infty(0,T;\Gamma^{l+1}_0(\Omega))\\ &\times W^{l+1,l/2+1/2}_{2,\kappa}(\Omega^t)\cap C([0,t];\Gamma^{l+1}_2(\Omega))\,,\end{aligned}$$

$t \le \min\{T_*, T_{**}\}$, which is a solution to (3.1). Uniqueness can be proved in the standard way. This concludes the proof.

Having shown the local existence of solutions to (3.1) we find a more appropriate estimate which will be useful in the proof of global existence. Recall that $R_t = (\frac{3}{4\pi}|\Omega_t|)^{1/3}$, $t \ge 0$. In view of Definition 1.1 from [18] of an equilibrium state we shall look for motions of (1.1) which are close to the equilibrium state. Assuming that the initial motion is sufficiently close to the equilibrium state we introduce the quantity $q_\sigma = q - p_0 - q_0$, where $q_0 = 2\sigma/R_0$. The quantity describes the deviation of the pressure from the sum of the external pressure p_0 and the pressure (q_0) of the surface tension in the case when the drop is a ball. Therefore we consider

$$\begin{aligned}
&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u = \nabla_u q_\sigma + \eta g\,,\\
&\Pi_0\Pi\mathbb{D}_u(u)\overline{n} = 0\,,\\
&\overline{n}_0\mathbb{D}_u(u)\overline{n} - \sigma\Delta_{S_t}(t)\int_0^t u(\tau)\,d\tau\\
&\qquad = \overline{n}_0\cdot\overline{n}q_\sigma + \sigma\overline{n}_0(\Delta_{S_t}(t) - \Delta_S(0))\xi + \sigma(H(\xi,0) + 2/R_0)\,,\\
&u|_{t=0} = v_0
\end{aligned}\tag{3.65}$$

and

$$\begin{aligned}
&q_{\sigma t} = -q_\sigma\Psi(\eta)\operatorname{div}_u u - (p_0+q_0)\operatorname{div}_u u\,,\\
&q_\sigma|_{t=0} = p(\varrho_0) - p_0 - q_0\,,
\end{aligned}\tag{3.66}$$

where $\Psi(\eta) = p_\eta(\eta)\eta/p(\eta)$, $p_\eta = \partial_\eta p$.

By Theorem 3.6 we have the existence of solutions to (3.65) and (3.66). Moreover, we obtain

Remark 3.7. Let u,η be a solution of problem (1.1). Then from (3.65) and (3.66) for sufficiently small T we obtain the estimate

$$\begin{aligned}
&\|u\|_{l+2,\Omega^T,\kappa} + \|q_\sigma\|_{l+1,\Omega^T,\kappa} + |q_\sigma|_{l+1,0,\infty,\Omega^T}\\
&\quad\le \varphi_{14}(T, \|v_0\|_{l+1,\Omega}, \|\varrho_0\|_{l+1,\Omega}, \|f\|_{C^{l+1}(\mathbb{R}^3\times(0,T))}, \|S\|_{W_2^{l+5/2}})\\
&\qquad\times [\|f\|_{C^{l+1}(\mathbb{R}^3\times(0,T))} + \|v_0\|_{l+1,\Omega}\\
&\qquad+ \|p(\varrho_0) - p_0 - q_0\|_{l+1,\Omega} + \|H(\xi,0) + 2/R_0\|_{l+1/2,S}]\,,
\end{aligned}\tag{3.67}$$

where $l > 3/2$.

Proof. Applying Lemma 3.3 to (3.65) yields

$$\begin{aligned}
\|u\|_{l+2,\Omega^T,\kappa} &\le c(T, A, \alpha, \beta, \gamma, \|S\|_{W_2^{l+5/2}})\\
&\quad\times [\|q_\sigma\|_{l+1,\Omega^T,\kappa} + \|g\|_{l,\Omega^T,\kappa} + \|H(\xi,0) + 2/R_0\|_{l+1/2,S}]\,.
\end{aligned}\tag{3.68}$$

Integrating (3.66) implies

$$q_\sigma(\xi,t) = -\exp\Big[-\int_0^t \Psi(\eta)\operatorname{div}_u u\,dt'\Big] \times\Big[\int_0^t \Big[(p_0+q_0)\operatorname{div}_u u \exp\int_0^{t'}\Psi(\eta)\operatorname{div}_u u\,dt''\Big]dt' + p(\varrho_0)-p_0-q_0\Big]. \tag{3.69}$$

From (3.69) we have

$$\|q_\sigma\|_{l+1,\Omega^T,\kappa} + |q_\sigma|_{l+1,0,\infty,\Omega^T} \le c(T,A,\alpha,\beta,\gamma,\|S\|_{W_2^{l+5/2}}) \times(T^a\|u\|_{l+2,\Omega^T,\kappa} + \|p(\varrho_0)-p_0-q_0\|_{l+1,\Omega}), \quad a>0. \tag{3.70}$$

From (3.68) and (3.70) for sufficiently small T we get (3.67). This concludes the proof.

4. Existence for (1.1) with $\sigma=0$. In this section we show the existence and an estimate for solutions of the linear problem

$$\begin{aligned} &Lu \equiv u_t - \mu\Delta u - \nu\nabla\operatorname{div} u = f && \text{in } \Omega^T,\\ &Bu \equiv \mathbb{D}(u)\overline{n} = g && \text{on } S^T,\\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain, $\overline{n}$ and $\mathbb{D}(u)$ are defined in Section 1.

To prove the existence of solutions of problem (1.1) with $\sigma=0$ we have to show the existence and find a suitable estimate for solutions of problem (4.1). Having found this we use [16] to prove the existence of solutions of the nonlinear problem (1.1) with $\sigma=0$.

We consider problem (4.1) under the following conditions:

$$f\in W_r^{2l,l}(\Omega^T), \quad g\in W_r^{2l+1-1/r,l+1/2-1/(2r)}(S^T), \quad u_0\in W_r^{2l+2-2/r}(\Omega), \tag{4.2}$$

where $l\in\mathbb{N}\cup\{0\}$, $1<r\in\mathbb{R}$ (for the definition of these spaces see [10]).

Considering problem (4.1) with the data (4.2) we have to impose the following compatibility conditions:

$$D(\partial_t^i u(0))\overline{n} = \partial_t^i g(0) \quad \text{on } S, \qquad i<l+1/2-3/(2r), \tag{4.3}$$

where $\partial_t^i u(0) = A\partial_t^{i-1}u(0) + \partial_t^{i-1}f(0)$ are calculated inductively and $A=\mu\Delta+\nu\nabla\operatorname{div}$, $u(0)=u_0$.

First we define functions φ_i, $i\le[l+1-1/r]$, by

$$\varphi_i = \partial_t^i u|_{t=0} \quad \text{in } \Omega\subset\mathbb{R}^n. \tag{4.4}$$

Using the Hestenes–Whitney method (see [3], Ch. 3) we have

Lemma 4.1. *Assume that $\varphi_i\in W_r^{2l+2-2i-2/r}(\Omega)$, $i\le[l+1-1/r]$. Then there exists an extension $\widetilde{\varphi}_i$ of φ_i to $\mathbb{R}^n$ such that $\widetilde{\varphi}_i\in W_r^{2l+2-2i-2/r}(\mathbb{R}^n)$ and*

$$|\widetilde{\varphi}_i|_{2l+2-2i-2/r,r,\mathbb{R}^n} \le c|\varphi_i|_{2l+2-2i-2/r,r,\Omega}, \qquad \text{where } c=c(\Omega). \tag{4.5}$$

LEMMA 4.2. *Let* $\varphi_i \in W_r^{2l+2-2i-2/r}(\Omega)$, $i \le [l+1-1/r]$. *Let* $T < \infty$. *Then there exists* $v \in W_r^{2l+2,l+1}(\Omega^T)$ *such that*

$$\partial_t^i v|_{t=0} = \varphi_i \quad \text{in } \Omega\,, \tag{4.6}$$

$$\|v\|_{2l+2,r,\Omega^T} \le c\sum_{i=0}^{l} \|\varphi_i\|_{2l+2-2i-2/r,r,\Omega}\,, \tag{4.7}$$

where the constant c *does not depend on* T.

Proof. In view of Lemma 4.1 there exist functions $\widetilde{\varphi}_i \in W_r^{2l+2-2i-2/r}(\mathbb{R}^n)$ and the estimates (4.5) hold.

Now we define $\widetilde{v}$ to be the solution of the Cauchy problem

$$\begin{aligned} &(\partial_t - \Delta)^{l+1}\widetilde{v} = 0 && \text{in } \mathbb{R}^n \times \mathbb{R}^1_+\,,\\ &\partial_t^i \widetilde{v}|_{t=0} = \widetilde{\varphi}_i && \text{in } \mathbb{R}^n\,,\ i = 0,\dots,[l+1-1/r]\,. \end{aligned} \tag{4.8}$$

By potential techniques (see [10], Sect. 12) we have

$$\begin{aligned} \sum_{|\alpha|=2l+2} |D^\alpha_{x,t}\widetilde{v}|_{r,D_{n+1}} \\ \le c_1 \sum_{i=0}^{l} \sum_{|\alpha_i|=[2l+2-2i-2/r]} [D_x^{\alpha_i}\widetilde{\varphi}_i]_{r,2l+2-2i-2/r-[2l+2-2i-2/r],E_n}\,, \end{aligned} \tag{4.9}$$

where c_1 is an absolute constant.

To estimate the lower derivatives we use the fact that $T < \infty$ and the formula (see [10], Sect. 18)

$$\begin{aligned} g(x,t) = \sum_{i=0}^{k} \frac{t^i}{i!}\partial_\tau^i g(x,\tau)|_{\tau=0} \\ + \frac{1}{(k-1)!}\int_0^t (t-\tau)^{k-1}[\partial_\tau^k g(x,\tau) - \partial_\tau^k g(x,\tau)|_{\tau=0}]\,d\tau\,. \end{aligned} \tag{4.10}$$

Using the initial conditions $(4.8)_2$ and (4.10) for $g = \widetilde{v}$ we obtain

$$\begin{aligned} \sum_{i<2l+2}\sum_{|\alpha|=i} |D^\alpha_{x,t}\widetilde{v}|_{r,D_{n+1}(T)} \\ \le c_2(T)\Big[\sum_{|\alpha|=2l+2} |D^\alpha_{x,t}\widetilde{v}|_{r,D_{n+1}(T)} + \sum_{i=0}^{l}\|\widetilde{\varphi}_i\|_{2l+2-2i-2/r,r,E_n}\Big], \end{aligned} \tag{4.11}$$

where $c_2(T)$ is an increasing function of T. Therefore, from (4.9) and (4.11) we obtain

$$\|\widetilde{v}\|_{2l+2,r,D_{n+1}(T)} \le c_3 \sum_{i=0}^{l}\|\widetilde{\varphi}_i\|_{2l+2-2i-2/r,r,E_n}\,, \tag{4.12}$$

where c_3 does not depend on T for $T < \infty$.

Introducing $v = \widetilde{v}|_\Omega$ and using Lemma 4.1 we obtain (4.7). This concludes the proof.

Now we introduce the function

$$w = u - v \tag{4.13}$$

which is a solution of the problem

$$\begin{aligned} &Lw = f - Lv \equiv f_1 && \text{in } \Omega^T\,,\\ &\mathbb{D}(w)\overline{n} = g - \mathbb{D}(v)\overline{n} \equiv g_1 && \text{on } S^T\,,\\ &w|_{t=0} = 0 && \text{in } \Omega\,, \end{aligned} \tag{4.14}$$

where in view of the compatibility conditions (4.3) we have

$$\begin{aligned} &\partial_t^i f_1|_{t=0} = 0\,, && i = 0,1,\dots,l-1\,,\\ &\partial_t^i g_1|_{t=0} = 0\,, && i \le [l + 1/2 - 1/(2r)]\,,\\ &\partial_t^i w|_{t=0} = 0\,, && i = 0,1,\dots,l\,. \end{aligned} \tag{4.15}$$

Therefore, $f_1 \in \overset{\circ}{W}{}_2^{2l,l}(\Omega^T)$, $g_1 \in \overset{\circ}{W}{}_r^{2l+1-2/r,l+1/2-1/(2r)}(S^T)$. It is sufficient to consider problem (4.14) in $R^T = \mathbb{R}_+ \times [0,T]$ only because using a partition of unity and appriopriate norms (see [10], Sects. 20, 21) we obtain the existence and an appriopriate estimate of solutions for problem (4.14) in a bounded domain Ω. To examine (4.14) in R^T we have to perform the following extensions. First we extend f_1, g_1, w by zero for $t<0$. Denote the extensions by f_2, g_2, w_1. Thus (4.14) is replaced by the problem

$$\begin{aligned} &Lw_1 = f_2 && \text{in } \widetilde{D}_{n+1}(T) = \mathbb{R}_+^n \times (-\infty, T]\,,\\ &\mathbb{D}(w_1)\overline{n} = g_2 && \text{on } \widetilde{E}_n(T) = \mathbb{R}^{n-1} \times (-\infty, T]\,. \end{aligned} \tag{4.16}$$

Next we extend f_2 by the method of Hestenes–Whitney with respect to the hyperplane $t = T$, so denoting the extended function by f_3 we have $f_3 \in W_r^{2l,l}(\widetilde{D}_{n+1})$ and

$$\langle f_3\rangle_{2l,r,\widetilde{D}_{n+1}} \le c\langle f_2\rangle_{2l,r,\widetilde{D}_{n+1}(T)}\,, \tag{4.17}$$

with a constant independent of T.

We extend f_3 by the Hestenes–Whitney method with respect to the hyperplane $x_n = 0$. Denote the extended function by f_4. Then $f_4 \in W_r^{2l,l}(E_{n+1})$ and

$$\langle f_4\rangle_{2l,r,E_{n+1}} \le c\langle f_3\rangle_{2l,r,\widetilde{D}_{n+1}}\,. \tag{4.18}$$

Now we consider the problem

$$Lw_2 = f_4 \quad \text{in } E_{n+1}. \tag{4.19}$$

Potential techniques imply (see [10], Sect. 12)

$$\langle w_2\rangle_{2l+2,r,R(T)} \le \langle w_2\rangle_{2l+2,r,E_{n+1}} \le c\langle f_4\rangle_{2l,r,E_{n+1}} \le c\|f_1\|_{2l,r,R^T}\,. \tag{4.20}$$

Extend g_2 to $\widetilde{E}_n$ and denote the extension by g_3. Then the function

$$w_3 = w_1 - w_2 \tag{4.21}$$

is a solution of the problem

$$\begin{aligned} &Lw_3 = 0 && \text{in } \widetilde{D}_{n+1}\,, \\ &\mathbb{D}(w_3)\overline{n} = g_3 - \mathbb{D}(w_1)\overline{n}|_{x=0} \equiv g_4 && \text{in } \widetilde{E}_n\,. \end{aligned} \tag{4.22}$$

By potential techniques (see [10], Sect. 12) we have

$$\begin{aligned} \langle w_3\rangle_{2l+2,r,R(T)} &\le \langle w_3\rangle_{2l+2,r,\widetilde{D}_{n+1}} \le c[g_4]_{2l+1-1/r,r,\widetilde{E}_n} \\ &\le c([g_3]_{2l+1-1/r,r,\widetilde{E}_n} + \|w_1\|_{2l+2,r,\widetilde{D}_{n+1}}) \\ &\le c([g_2]_{2l+1-2/r,r,\widetilde{E}_n(T)} + \|f_2\|_{2l,r,R^T})\,, \end{aligned} \tag{4.23}$$

where the last inequality follows from the Hestenes–Whitney extension.

By Lemma 2.3 we have

$$[g_2]_{2l+1-1/r,r,\widetilde{E}_n(T)} \le c\langle\!\langle g_2\rangle\!\rangle_{2l+1-1/r,r,\widetilde{E}_n(T)}\,. \tag{4.24}$$

We also have

$$\sum_{i=1}^{n-1}\langle\!\langle \partial_{x_i}^{\nu} g_2\rangle\!\rangle_{r,2l+1-1/r-\nu,\widetilde{E}_n(T),x_i} = \sum_{i=1}^{n-1}\langle\!\langle \partial_{x_i}^{\nu} g_2\rangle\!\rangle_{r,2l+1-1/r-\nu,\widetilde{D}_n(T),x_i}\,, \tag{4.25}$$

where $\nu = [2l+1-1/r]$. Next by Lemma 2.4 we obtain

$$\begin{aligned} \langle\!\langle \partial_t^{\mu} g_2\rangle\!\rangle^r_{r,l+1/2-1/(2r)-\mu,\widetilde{E}_n(T),t} &\le [\partial_t^{\mu} g_2]^r_{r,l+1/2-1/(2r)-\mu,\widetilde{D}_n(T),t} \\ &\quad + 2\int_0^T |\partial_t^{\mu} g_2|^r_{r,E_{n-1}} \frac{dt}{t^{r(l+1/2-1/(2r)-\mu)}}\,, \end{aligned} \tag{4.26}$$

where $\mu = [l+1/2-1/(2r)]$.

In view of Lemma 2.5 the second term on the right-hand side of (4.26) is estimated for $r > 3$ by the first term with a constant independent of T, so (4.26) becomes

$$\langle\!\langle \partial_t^{\mu} g_2\rangle\!\rangle_{r,l+1/2-1/(2r)-\mu,\widetilde{E}_n(T),t} \le c[\partial_t^{\mu} g_2]_{r,l+1/2-1/(2r)-\mu,\widetilde{D}_n(T),t}\,, \tag{4.27}$$

where $\mu = [l+1/2-1/(2r)]$.

Summarizing, we have

$$\langle w_3\rangle_{2l+2,r,R(T)} \le c([g_2]_{2l+1-1/r,r,\widetilde{D}_n(T)} + \|f_2\|_{2l,r,R(T)})\,. \tag{4.28}$$

Therefore, using (4.10), (4.20) and (4.28), employing an appriopriate partition of unity and corresponding norms (see [10], Sects. 20, 21) we obtain

Lemma 4.3. *Assume that*

$$S \in W_r^{2l+2-1/r}, \quad f \in W_r^{2l,l}(\Omega^T),$$
$$g \in W_r^{2l+1-1/r,l+1/2-1/(2r)}(S^T), \quad u_0 \in W_r^{2l+2-2/r}(\Omega),$$

$r > 3$, $T < \infty$. *Then there exists a solution of* (4.1) *such that* $u \in W_r^{2l+2,l+1}(\Omega^T)$ *and*

$$\Vert u\Vert_{2l+2,r,\Omega^T} \le c(\Vert f\Vert_{2l,r,\Omega^T} + \Vert g\Vert_{2l+1-1/r,r,S^T} + |u(0)|_{2l+2-2/r,0,r,\Omega}), \tag{4.29}$$

where c does not depend on T and

$$|u(0)|_{2l+2-2/r,r,\Omega} = \sum_{i\le[l+1-1/r]} \langle \partial_t^i u|_{t=0}\rangle_{[2l+2-2/r-2i],r,\Omega}. \tag{4.30}$$

Lemma 4.3 and [16] yield

Theorem 4.4. *Let* $v_0 \in \Gamma_r^{2l+2-2/r}(\Omega)$, $f \in C^{2l,l}(\mathbb{R}^3 \times (0,T))$, $\varrho_0, 1/\varrho_0 \in W_r^{2l+1}(\Omega)$, $S \in W_r^{2l+2-1/r}$, $2l+2 > 3/r$, $r > 3$, $l - 1/2 - 3/(2r) \notin \mathbb{Z}$. *Then for a sufficiently small T there exists a solution of problem* (1.1) *with* $\sigma = 0$ *such that*

$$v \in W_r^{2l+2,l+1}(\Omega^T) \cap L_\infty(0,T;\Gamma_r^{2l+2-2/r}(\Omega)),$$
$$\varrho, 1/\varrho \in W_r^{2l+1,l+1/2}(\Omega^T) \cap C(0,T;\Gamma_r^{2l+1}(\Omega)).$$

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