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MULTIPLIERS IN SOBOLEV SPACES AND EXACT CONVERGENCE RATE ESTIMATES FOR THE FINITE-DIFFERENCE SCHEMES

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Abstract. In this paper we present some recent results concerning convergence rate estimates for finite-difference schemes approximating boundary-value problems. Special attention is given to the problem of minimal smoothness of coefficients in partial differential equations necessary for obtaining the results.

1. Introduction. Recently, increased attention is given to approximation of generalized solutions of partial differential equations with finite-difference methods.

For a problem with the solution belonging to the Sobolev space $W_p^s(\Omega)$, the convergence estimate

$$\|u-v\|_{W_p^k(\omega)} \le Ch^{s-k}\|u\|_{W_p^s(\Omega)}, \qquad s>k\,, \tag{1}$$

is said to be *compatible* with the smoothness of the solution [12]. Here $u\in W_p^s(\Omega)$ denotes the solution of the original boundary-value problem, v denotes the solution of the corresponding finite-difference scheme, h is the discretization parameter, $W_p^k(\omega)$ denotes the discrete Sobolev space, and C is a positive generic constant, independent of h and u.

Estimates of this type have been obtained for a broad class of elliptic problems (see [6, 10, 11, 13, 16]). Analogous results have also been obtained for parabolic and hyperbolic problems (see [5, 7, 8, 9]). As a rule, the Bramble–Hilbert lemma [2, 4] is used in their proofs.

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For equations with variable coefficients, the natural problem arises of establishing the minimal smoothness properties of coefficients for obtaining the same type (1) of estimate. Such coefficients belong to classes of multipliers in Sobolev spaces.

2. Multipliers in Sobolev spaces. Let Ω be a domain in $\mathbb{R}^n$. By $D(\Omega) = \dot{C}^\infty(\Omega)$ we denote the space of infinitely smooth functions with compact support in Ω, and by $D'(\Omega)$ the space of distributions. Moreover, $x = (x_1, x_2, \dots, x_n)$ denote vectors from $\mathbb{R}^n$, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ are multi-indices. Let $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Partial derivatives are denoted by

$$D_i u = \partial u/\partial x_i \quad \text{and} \quad D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} u \,.$$

Suppose V and W are two function spaces contained in $D'(\Omega)$. A function a defined on Ω is called a *pointwise multiplier*, or simply a *multiplier*, from V to W if, for every v in V, the product $a \cdot v$ belongs to W. The set of all multipliers from V to W is denoted by $M(V \to W)$. In particular, when $V = W$ we put $M(V) = M(V \to V)$.

In this section, we shall be concerned with multipliers in Sobolev spaces which belong to $M(W_p^t(\Omega) \to W_p^s(\Omega))$, $1 < p < \infty$. Naturally, we assume that $t \geq s$.

To begin, we consider multipliers in pairs of Sobolev spaces on $\mathbb{R}^n$. Motivated by the definition of multiplication of a distribution with a smooth function, for $a \in M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$ and $u \in W_{p'}^{-s}(\mathbb{R}^n)$, $1/p + 1/p' = 1$, we define the product $a \cdot u \in W_{p'}^{-t}(\mathbb{R}^n)$ by

$$\langle a \cdot u, \varphi \rangle_{W_{p'}^{-t} \times W_p^t} = \langle u, a \cdot \varphi \rangle_{W_{p'}^{-s} \times W_p^s}, \qquad \forall \varphi \in W_p^t(\mathbb{R}^n) \,.$$

This definition implies that $M(W_{p'}^{-s}(\mathbb{R}^n) \to W_{p'}^{-t}(\mathbb{R}^n)) = M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$, and therefore it suffices to explore the properties of the sets $M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$ and $M(W_p^t(\mathbb{R}^n) \to W_p^{-s}(\mathbb{R}^n))$ for $t \geq s \geq 0$.

We recall a collection of fundamental results on multipliers in Sobolev spaces (see [14]).

LEMMA 1. *If* $a \in M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$, $t \geq s \geq 0$, *then*:

$$\begin{aligned}
a &\in M(W_p^{t-s}(\mathbb{R}^n) \to L_p(\mathbb{R}^n)) \,, \\
a &\in M(W_p^{t-\sigma}(\mathbb{R}^n) \to W_p^{s-\sigma}(\mathbb{R}^n)), \quad 0 < \sigma < s \,, \\
D^\alpha a &\in M(W_p^t(\mathbb{R}^n) \to W_p^{s-|\alpha|}(\mathbb{R}^n)), \quad |\alpha| \leq s \,, \\
D^\alpha a &\in M(W_p^{t-s+|\alpha|}(R^n) \to L_p(\mathbb{R}^n)), \quad |\alpha| \leq s \,.
\end{aligned}$$

LEMMA 2. *For* $t \geq s \geq 0$, $M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n)) \subseteq W^s_{p,\mathrm{unif}}$, *where*

$$W^s_{p,\mathrm{unif}} = \{f \mid \sup_{z\in\mathbb{R}^n} \|\eta(x-z)\cdot f(x)\|_{W_p^s} < \infty,\ \forall\eta \in D(\mathbb{R}^n),\ \eta \equiv 1 \text{ on } B_1\},$$

and B_1 *is the unit ball with center* 0. *If* $tp > n$, *then* $M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n)) = W^s_{p,\mathrm{unif}}$.

LEMMA 3. *For* $s \geq 0$, $M(W_p^s(\mathbb{R}^n)) \subseteq L_\infty(\mathbb{R}^n)$.

LEMMA 4. *Suppose* $1 < p < \infty$, *and let* s *and* t *be nonnegative integers such that* $t \geq s$. *If*

$$a = \sum_{|\alpha|\leq \mathbf{t}} D^\alpha a_\alpha$$

and $a_\alpha \in M(W_p^t(\mathbb{R}^n) \to W_p^{t-s}(\mathbb{R}^n)) \cap M(W_{p'}^s(\mathbb{R}^n) \to L_{p'}(\mathbb{R}^n))$, $1/p + 1/p' = 1$, *then* $a \in M(W_p^t(\mathbb{R}^n) \to W_p^{-s}(\mathbb{R}^n))$.

LEMMA 5. *Let* $p > 1$, $t > s > 0$, *and suppose that either* $q \in [n/t, \infty]$ *and* $tp < n$, *or* $q \in (p,\infty)$ *and* $tp = n$. *If*

$$\begin{aligned} a &\in B^s_{q,p,\mathrm{unif}} \\ &= \{f \mid \sup_{z\in\mathbb{R}^n} \|\eta(x-z)\cdot f(x)\|_{B^s_{q,p}} < \infty,\ \forall\eta \in D(\mathbb{R}^n),\ \eta \equiv 1 \text{ on } B_1\}, \end{aligned}$$

where $B^s_{q,p}$ *is the Besov space, then* $a \in M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$. *The result is also true for* $t = s$, *provided* $a \in B^s_{q,p,\mathrm{unif}} \cap L_\infty(\mathbb{R}^n)$.

LEMMA 6. *If* $a_\alpha \in M(W_p^{s-|\alpha|}(\mathbb{R}^n) \to W_p^{s-k}(\mathbb{R}^n))$, $s \geq k$, *for every multi-index* α *then the differential operator*

$$Lu = \sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u, \qquad x \in \mathbb{R}^n, \tag{2}$$

defines a continuous mapping from $W_p^s(\mathbb{R}^n)$ *to* $W_p^{s-k}(\mathbb{R}^n)$.

The analogous result holds true for $s < 0$. If $p = 2$ then the result holds true for every s. Under certain conditions we have the converse result:

LEMMA 7. *Let the operator* (2) *define a continuous mapping from* $W_p^s(\mathbb{R}^n)$ *to* $W_p^{s-k}(\mathbb{R}^n)$, *and* $p(s-k) > n$, $p > 1$. *Then* $a_\alpha \in M(W_p^{s-|\alpha|}(\mathbb{R}^n) \to W_p^{s-k}(\mathbb{R}^n))$, *for every multi-index* α.

All of these results can be transfered to Sobolev spaces in an open subset of $\mathbb{R}^n$. More precisely, if Ω is an open set in $\mathbb{R}^n$ with a Lipschitz continuous boundary and $a \in M(W_p^t(\Omega) \to W_p^s(\Omega))$, then a can be extended to a function $\widetilde{a}$, defined on the whole of $\mathbb{R}^n$, such that $\widetilde{a} \in M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$. The converse is also true: the restriction to Ω of a multiplier $a \in M(W_p^t(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n))$ is an element of $M(W_p^t(\Omega) \to W_p^s(\Omega))$.

For bounded domains, $W^s_{p,\mathrm{unif}}$ and $B^s_{q,p,\mathrm{unif}}$ are replaced by standard Sobolev and Besov spaces, respectively. Employing Lemmas 2, 3, 5, imbedding theorems for Besov spaces [1, 17] and the representation of distributions from negative Sobolev spaces [18], we obtain the following results:

LEMMA 8. *Suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with a Lipschitz continuous boundary, $s > 0$ and $p > 1$. If $a \in W_q^t(\Omega)$ where*

$$q = p,\ t = s \quad \text{when } sp > n, \quad \text{and}$$
$$q \geq n/s,\ t = s + \varepsilon,\ \varepsilon > 0 \quad \text{when } sp \leq n\,,$$

then $a \in M(W_p^s(\Omega))$.

LEMMA 9. *Let Ω be a bounded open set in $\mathbb{R}^n$ with a Lipschitz continuous boundary, $s > 0$ and $p > 1$. If $a \in L_q(\Omega)$ where*

$$\begin{aligned} &q = p && \text{when } sp > n\,, \\ &q > p && \text{when } sp = n, \quad \text{and} \\ &q \geq n/s && \text{when } sp < n\,, \end{aligned}$$

then $a \in M(W_p^s(\Omega) \to L_p(\Omega))$.

LEMMA 10. *Let Ω be a bounded open set in $\mathbb{R}^n$ with a Lipschitz continuous boundary and*

$$a(x) = a_0(x) + \sum_{i=1}^{n} D_i a_i(x)\,.$$

If $a_0 \in M(W_2^t(\Omega) \to L_2(\Omega))$, and $a_i \in M(W_2^t(\Omega) \to W_2^{1-s}(\Omega)) \cap M(W_2^{t-1}(\Omega) \to L_2(\Omega))$, $i = 1, 2, \ldots, n$, where $0 < s \leq 1 \leq t < 2$, $s \neq 1/2$, then $a \in M(W_2^t(\Omega) \to W_2^{-s}(\Omega))$.

3. Boundary-value problem and its approximation. As a model problem let us consider the first boundary-value problem for a second-order linear elliptic equation with variable coefficients, in the square $\Omega = (0,1)^2$:

$$-\sum_{i,j=1}^{2} D_i(a_{ij} D_j u) + au = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega\,. \tag{3}$$

We assume that the generalized solution of the problem (3) belongs to the Sobolev space $W_2^s(\Omega)$, $1 < s \leq 3$, with the right-hand side $f(x)$ belonging to $W_2^{s-2}(\Omega)$. Consequently, the coefficients $a_{ij}(x)$ and $a(x)$ belong to the following classes of multipliers: $a_{ij} \in M(W_2^{s-1}(\Omega))$, $a \in M(W_2^s(\Omega) \to W_2^{s-2}(\Omega))$. According to Lemmas 8–10 sufficient conditions are the following:

$$a_{ij} \in W_2^{|s-1|}(\Omega)\,, \quad a \in W_2^{|s-1|-1}(\Omega)\,, \quad \text{for } |s-1| > 1\,,$$
$$a_{ij} \in W_p^{|s-1|+\delta}(\Omega)\,, \quad a = a_0 + \sum_{i=1}^{2} D_i a_i\,,$$
$$a_0 \in L_{2+\varepsilon}(\Omega)\,, \quad a_i \in W_p^{|s-1|+\delta}(\Omega)\,,$$

where $\varepsilon > 0$,

$$\begin{aligned} &\delta > 0\,, \quad p > 2/|s-1| \quad &&\text{for } 0 < |s-1| \le 1\,, \quad \text{and} \\ &\delta = 0\,, \quad p = \infty &&\text{for } s = 1\,. \end{aligned}$$

The following estimates do not depend on δ in any way, so we can put $\delta = 0$.

We also assume that the following conditions hold:

$$a_{ij} = a_{ji}, \qquad \sum_{i,j=1}^{2} a_{ij} y_i y_j \le c_0 \sum_{i=1}^{2} y_i^2, \qquad x \in \Omega, \qquad c_0 = \text{const} > 0,$$
$$a(x) \ge 0 \quad \text{in the sense of distributions, i.e.}$$
$$\langle a \cdot \varphi, \varphi \rangle_{D' \times D} \ge 0, \qquad \forall \varphi \in D(\Omega)\,.$$

Let $\overline{\omega}$ be the uniform mesh in $\overline{\Omega}$ with step h, $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \cap \Gamma$, $\gamma_{ik} = \{x \in \gamma \mid x_i = k,\ 0 < x_{3-i} < 1\}$, $k = 0, 1$, and $\omega_i = \omega \cup \gamma_{i0}$. We define finite differences as usual:

$$v_{x_i} = (v^{+i} - v)/h, \qquad v_{\overline{x}_i} = (v - v^{-i})/h\,,$$

where $v^{\pm i}(x) = v(x \pm hr_i)$, and r_i is the unit vector on the x_i axis.

We also define the Steklov smoothing operators:

$$T_i^+ f(x) = \int_0^1 f(x + htr_i)\,dt = T_i^- f(x + hr_i) = T_i f(x + 0.5hr_i)\,.$$

These operators commute and transform derivatives to differences:

$$T_i^+ D_i u = u_{x_i}, \qquad T_i^- D_i u = u_{\overline{x}_i}\,.$$

We approximate the problem (3) with the following finite-difference scheme:

$$L_h v = T_1^2 T_2^2 f \quad \text{in } \omega, \qquad v = 0 \quad \text{on } \gamma \tag{4}$$

where $L_h v = -0.5 \sum_{i,j=1}^{2} [(a_{ij} v_{\overline{x}_j})_{x_i} + (a_{ij} v_{x_j})_{\overline{x}_i}] + (T_1^2 T_2^2 a) v$ and $T_i^2 = T_i^+ T_i^-$. The difference scheme (4) is a standard symmetric difference scheme (see [15]) with the right-hand side and coefficient $a(x)$ averaged. For $1 < s \le 3$ these coefficients may not be continuous, so the difference scheme with non-averaged data is not well defined.

4. Convergence of the finite-difference scheme. Let u denote the solution to the boundary value problem (3) and v the solution to the difference scheme (4). The error $z = u - v$ satisfies the conditions

$$L_h z = \sum_{i,j=1}^{2} \eta_{ij,\overline{x}_i} + \zeta \quad \text{in } \omega, \qquad z = 0 \quad \text{on } \gamma \tag{5}$$

where $\eta_{ij} = T_i^+ T_{3-i}^2 (a_{ij} D_j u) - 0.5(a_{ij} u_{x_j} + a_{ij}^{+i} u_{\overline{x}_j}^{+i})$ and $\zeta = (T_1^2 T_2^2 a) u - T_1^2 T_2^2 (au)$.

For $\theta \subseteq \overline{\omega}$ let $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{L_2(\theta)}$ and $\|\cdot\|_\theta = \|\cdot\|_{L_2(\theta)}$ denote the discrete inner product and the discrete L_2-norm on θ. We also define the discrete W_2^1-norm on ω:

$$\|v\|^2_{W_2^1(\omega)} = \|v\|^2_\omega + \|v_{x_1}\|^2_{\omega_1} + \|v_{x_2}\|^2_{\omega_2}\,.$$

Using the energy method [15] it is easy to prove the next lemma.

Lemma 11. *The finite-difference scheme* (5) *is stable in the sense of the a priori estimate*

$$\|z\|_{W_2^1(\omega)} \le C\Big(\sum_{i,j=1}^{2} \|\eta_{ij}\|_{\omega_i} + \|\zeta\|_\omega\Big)\,. \tag{6}$$

The problem of deriving the convergence rate estimate for the finite-difference scheme (4) is now reduced to estimating the right-hand side terms in (6). Estimates which follow are based on the following bilinear version of the Bramble–Hilbert lemma [3, 10]:

Lemma 12. *Let E be a bounded open set in $\mathbb{R}^n$ with a Lipschitz continuous boundary and let $\eta(u,v)$ be a bounded bilinear functional on $W_p^s(E)\times W_q^t(E)$, $1\le p,q\le\infty$, $s,t>0$, such that*

$$\eta(u,v)=0$$

if either u is a polynomial of degree $< s$ and $v\in W_q^t(E)$, or v is a polynomial of degree $< t$ and $u\in W_p^s(E)$. Then there exists a positive constant $C=C(E,p,s,q,t)$ such that

$$|\eta(u,v)| \le |u|_{W_p^s(E)}|v|_{W_q^t(E)}, \qquad \forall (u,v)\in W_p^s(E)\times W_q^t(E),$$

with the seminorms of the corresponding spaces at the right-hand side.

First, we decompose η_{ij} in the following way:

$$\eta_{ij} = \eta_{ij1}+\eta_{ij2}+\eta_{ij3}+\eta_{ij4}, \qquad \text{where}$$

$$\eta_{ij1} = T_i^+T_{3-i}^2(a_{ij}D_ju) - (T_i^+T_{3-i}^2a_{ij})\cdot(T_i^+T_{3-i}^2D_ju)\,,$$

$$\eta_{ij2} = [T_i^+T_{3-i}^2a_{ij} - 0.5(a_{ij}+a_{ij}^{+i})]\cdot(T_i^+T_{3-i}^2D_ju)\,,$$

$$\eta_{ij3} = 0.5(a_{ij}+a_{ij}^{+i})\cdot[T_i^+T_{3-i}^2D_ju - 0.5(u_{x_j}+u_{\overline{x}_j}^{+i})], \qquad \text{and}$$

$$\eta_{ij4} = -0.25(a_{ij}-a_{ij}^{+i})\cdot(u_{x_j}-u_{\overline{x}_j}^{+i})\,.$$

For $1<s\le 2$ we set $\zeta=\zeta_0+\zeta_1+\zeta_2$, where

$$\zeta_0 = (T_1^2T_2^2a_0)u - T_1^2T_2^2(a_0u), \qquad \text{and}$$

$$\zeta_i = (T_1^2T_2^2D_ia_i)u - T_1^2T_2^2(D_ia_i\cdot u), \qquad i=1,2\,.$$

For $2<s\le 3$ we set $\zeta=\zeta_3+\zeta_4$, where

$$\zeta_3 = (T_1^2T_2^2a)\cdot(u - T_1^2T_2^2u), \qquad \text{and}$$

$$\zeta_4 = (T_1^2T_2^2a)\cdot(T_1^2T_2^2u) - T_1^2T_2^2(a\cdot u)\,.$$

Let us introduce the elementary rectangles $e = e(x) = \{y \mid |y_j - x_j| \le h,\ j = 1,2\}$ and $e_i = e_i(x) = \{y \mid x_i \le y_i \le x_i + h,\ |y_{3-i} - x_{3-i}| \le h\}$, $i = 1,2$.

The value η_{ij1} at the node $x \in \omega_i$ is a bounded bilinear functional on $W_q^\lambda(e_i) \times W^\mu_{2q/(q-2)}(e_i)$ where $\lambda \ge 0$, $\mu \ge 1$ and $q > 2$. Moreover, $\eta_{ij1} = 0$ if either a_{ij} is a constant or u is a first-degree polynomial. Using Lemma 12 and a procedure proposed in [11], developed in [10], we obtain

$$|\eta_{ij1}| \le C(h)|a_{ij}|_{W_q^\lambda(e_i)}|u|_{W^\mu_{2q/(q-2)}(e_i)}, \quad 0 \le \lambda \le 1,\ 1 \le \mu \le 2\,,$$

where $C(h) = Ch^{\lambda+\mu-2}$. Summation with the use of the Hölder inequality yields

$$\|\eta_{ij1}\|_{\omega_i} \le Ch^{\lambda+\mu-1}|a_{ij}|_{W_q^\lambda(\Omega)}|u|_{W^\mu_{2q/(q-2)}(\Omega)}, \quad 0 \le \lambda \le 1,\ 1 \le \mu \le 2\,. \tag{7}$$

Set $\lambda = s-1$, $\mu = 1$ and $q = p$. By the imbedding theorem [17], $W_2^s \subseteq W^1_{2p/(p-2)}$ for $1 < s \le 2$. Therefore, from (7) we obtain

$$\|\eta_{ij1}\|_{\omega_i} \le Ch^{s-1}\|a_{ij}\|_{W_p^{s-1}(\Omega)}\|u\|_{W_2^s(\Omega)}, \quad 1 < s \le 2\,. \tag{8}$$

Similar estimates hold for η_{ij2}, η_{ij4}, ζ_1 and ζ_2.

Let now $q > 2$ be a constant. The following imbeddings hold: $W_2^{\lambda+\mu-1} \subseteq W_q^\lambda$ for $\mu > 2 - 2/q$, and $W_2^{\lambda+\mu} \subseteq W^\mu_{2q/(q-2)}$ for $\lambda > 2/q$. Setting $\lambda + \mu = s$ we obtain from (7),

$$\|\eta_{ij1}\|_{\omega_i} \le Ch^{s-1}\|a_{ij}\|_{W_2^{s-1}(\Omega)}\|u\|_{W_2^s(\Omega)}, \quad 2 < s \le 3\,. \tag{9}$$

In the same manner we can estimate η_{ij4}.

For $s > 2$, $\eta_{ij2}(x)$ is a bounded bilinear functional on $W_2^{s-1}(e_i) \times W^1_\infty(e_i)$ which vanishes if either a_{ij} is a first-degree polynomial or u is a constant. Using the same lemma and the imbedding $W_2^s \subseteq W^1_\infty$ we obtain for η_{ij2} an estimate of the form (9).

Similarly, $\eta_{ij3}(x)$ is a bounded bilinear functional on $L_\infty(e_i) \times W_2^s(e_i)$, $s > 1$, which vanishes if u is a second-degree polynomial. In the same way, using the imbeddings $W_p^{s-1} \subseteq L_\infty$ (for $1 < s \le 2$) and $W_2^{s-1} \subseteq L_\infty$ (for $s > 2$) we again obtain estimates of the forms (8) and (9).

Let $2 < q < 2/(3-s)$. For $2 < s \le 3$, $\zeta_3(x)$ is a bounded bilinear functional on $L_q(e) \times W^{s-1}_{2q/(q-2)}(e)$. Moreover, $\zeta_3 = 0$ if u is a first-degree polynomial. Using the Bramble–Hilbert lemma and the imbeddings $W_2^{s-2} \subseteq L_q$ and $W_2^s \subseteq W^{s-1}_{2q/(q-2)}$ we obtain the estimate

$$\|\zeta_3\|_\omega \le Ch^{s-1}\|a\|_{W_2^{s-2}(\Omega)}\|u\|_{W_2^s(\Omega)}, \quad 2 < s \le 3\,. \tag{10}$$

For $2 < s \le 3$, $\zeta_4(x)$ is a bounded bilinear functional on $W_2^{s-2}(e) \times W^1_\infty(e)$. Using the same methodology and the imbedding $W_2^s \subseteq W^1_\infty$, we obtain for ζ_4 an estimate of the form (10).

Finally, let $2 < q < \min\{2+\varepsilon,\ 2/(2-s)\}$. For $1 < s \le 2$, $\zeta_0(x)$ is a bounded bilinear functional on $L_q(e) \times W^{s-1}_{2q/(q-2)}(e)$ which vanishes if u is a constant. Using the imbeddings $L_{2+\varepsilon} \subseteq L_q$ and $W_2^s \subseteq W_{2q/(q-2)}^{s-1}$, we obtain the estimate

$$\|\zeta_0\|_\omega \le Ch^{s-1}\|a_0\|_{L_{2+\varepsilon}(\Omega)}\|u\|_{W_2^s(\Omega)}, \quad 1 < s \le 2\,. \tag{11}$$

Combining (6) with (8)–(11) we obtain the final result:

THEOREM. *The finite-difference scheme* (4) *converges and the following estimates hold*:

$$\|u-v\|_{W_2^1(\omega)} \le Ch^{s-1}(\max_{i,j}\|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|a\|_{W_2^{s-2}(\Omega)})\|u\|_{W_2^s(\Omega)}\,, \quad \textit{for } 2 < s \le 3\,, \tag{12}$$

and

$$\|u-v\|_{W_2^1(\omega)} \le Ch^{s-1}(\max_{i,j}\|a_{ij}\|_{W_p^{s-1}(\Omega)} + \max_i\|a_i\|_{W_p^{s-1}(\Omega)} + \|a_0\|_{L_{2+\varepsilon}(\Omega)})\|u\|_{W_2^s(\Omega)}, \quad \textit{for } 1 < s \le 2\,. \tag{13}$$

The obtained convergence-rate estimates (12) and (13) are compatible with the smoothness of data. An analogous estimate in L_2-norm is obtained in [6]. Non-stationary problems were considered in [7, 8].

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