NUMERICAL ANALYSIS AND MATHEMATICAL MODELLING
BANACH CENTER PUBLICATIONS, VOLUME 29
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1994

IMPLICIT RUNGE–KUTTA METHODS FOR TRANSFERABLE DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. The numerical solution of transferable differential-algebraic equations (DAE's) by implicit Runge–Kutta methods (IRK) is studied. If the matrix of coefficients of an IRK is non-singular then the arising systems of nonlinear equations are uniquely solvable. These methods are proved to be stable if an additional contractivity condition is satisfied. For transferable DAE's with smooth solution we get convergence of order $\min(k_E, k_I + 1)$, where k_E is the classical order of the IRK and k_I is the stage order. For transferable DAE's with generalized solution convergence of order 1 is ensured, provided that $k_E \geq 1$.

1. Introduction. Numerical methods for initial value problems of transferable differential-algebraic equations (DAE's)

$$f(x'(t), x(t), t) = 0, \quad x(t_0) = x^0, \quad t \in [t_0, T], \ x : [t_0, T] \to \mathbb{R}^m \tag{1}$$

have found much interest in recent years. *Transferable* means that DAE (1) is equivalent to a system of state equations [GM86, §1.2].

Implicit Runge–Kutta methods (IRK) for (1) are given by (cf. [Pe86])

$$\begin{aligned} x_{n+1} &= x_n + h\sum_{j=1}^{s} b_j x_{n+1}^{(j)\prime}, \\ x_{n+1}^{(i)} &= x_n + h\sum_{j=1}^{s} a_{ij} x_{n+1}^{(j)\prime}, \quad f(x_{n+1}^{(i)\prime}, x_{n+1}^{(i)}, t_n + c_i h) = 0 \quad (i = 1, \ldots, s)\,, \end{aligned} \tag{2}$$

$A = ((a_{ij}))$ denotes the Runge–Kutta matrix, b_j the weights and c_i the nodes of

1991 *Mathematics Subject Classification*: 65L05.
The paper is in final form and no version of it will be published elsewhere.

the method. We assume that IRK (2) satisfies the contractivity condition

$$|R(\infty)| < 1 \tag{3}$$

with the stability function $R(\cdot)$ known from stiff ODE-theory (if the Runge–Kutta matrix A is non-singular, then $R(\infty) = 1 - \sum_{j,k} b_j a^*_{jk}$ with a^*_{jk} the elements of A^{-1}).

Assuming that matrix A is non-singular Petzold [Pe86] studies IRK with (3) applied to quasilinear DAE's of uniform index 1 and obtains convergence of order $\min(k_E, k_I + 1)$, with the classical (ODE-) order k_E and the stage order k_I defined by

$$k_I := \max\left\{k : \sum_j a_{ij} c_j^{l-1} = \frac{1}{l} c_i^l,\ \sum_j b_j c_j^{l-1} = \frac{1}{l},\ i = 1, \dots, s,\ l = 1, \dots, k\right\}. \tag{4}$$

For methods with non-singular matrix A we have $k_I \le k_E$.

In the special case of semi-explicit DAE's (1) of index 1 the methods (2) with non-singular Runge–Kutta matrix A are convergent with order $\min(k_E, k_I{+}1)$ (if (3) is satisfied or $R(\infty) = -1$) or k_I (if $R(\infty) = 1$), and diverge if $|R(\infty)| > 1$. For these semi-explicit DAE's, IRK with $c_s = 1$, $b_j = a_{sj}$ $(j = 1, \dots, s)$ are of special interest since these methods (that are denoted by IRK(DAE)) do not suffer from order reduction and are convergent with order k_E [HLR89].

Throughout the paper we assume the following properties of DAE (1) that guarantee its transferability into a system of state equations (for details see [GM86]):

- $f = f(y, x, t)$ is a continuously differentiable function $f : \mathcal{G} \to \mathbb{R}^m$ with $\mathcal{G} = \mathbb{R}^m \times \mathbb{R}^m \times [t_0, T]$. The Jacobians $f'_y(y, x, t)$ and $f'_x(y, x, t)$ are uniformly Lipschitz-continuous on $\mathcal{G}$; furthermore, $\|f'_x(y, x, t)\|$ is supposed to be uniformly bounded on $\mathcal{G}$.
- The Jacobian $f'_y(y, x, t)$ has constant rank on $\mathcal{G}$. $\ker f'_y(y, x, t)$ depends neither on y nor on x and $N(t) \equiv \ker f'_y(y, x, t)$ is a smooth subspace of $\mathbb{R}^m$, i.e., there is a continuously differentiable projector function $Q(t) : \mathbb{R}^m \to N(t)$.
- The matrix $G(y, x, t) := f'_y(y, x, t) + f'_x(y, x, t)Q(t)$ has a uniformly bounded inverse on $\mathcal{G}$.

Under these assumptions the Runge–Kutta equations (2) are uniquely solvable for sufficiently small stepsizes h, provided that the contractivity condition (3) is satisfied and the Runge–Kutta matrix A is non-singular. For IRK(DAE) applied to transferable DAE's (1) stability and (if $k_I \ge 2$) convergence with order k_I has been shown (cf. [Ob90]). In [GM86, §2.1] IRK for transferable DAE's (1) with constant $\ker f'_y(y, x, t) \equiv N$ are studied in detail: If the contractivity condition (3) is satisfied and the matrix A is non-singular, then method (2) is stable and the convergence with order k_I (for IRK(DAE): order k_E) is proved.

In this paper we prove that the results of Petzold remain valid for transferable DAE's (1) with time-dependent $\ker f'_y(y, x, t) \equiv N(t)$. In Section 2 the existence and uniqueness of Runge–Kutta solutions and the stability of IRK (2) with (3) are stated. Using the stability estimate convergence is proved for DAE's (1) with smooth and with generalized solution (Section 3). The final Section 4 contains the somewhat technical proofs of Theorems 1 and 2.

2. Stability of implicit Runge–Kutta methods for transferable DAE's. The definition of method (2) includes systems of non-linear equations. For sufficiently small stepsizes h the existence and uniqueness of a Runge–Kutta solution is shown in the following theorem so that the IRK is well-determined. A similar theorem was proved in [Ob90] under the additional assumption that the contractivity condition (3) is satisfied.

THEOREM 1. *Suppose that the Runge–Kutta matrix A of method* (2) *is non-singular. Then equations* (2) *are uniquely solvable for sufficiently small stepsizes h.*

Proof. See Section 4.

Next we study the influence of small perturbations to the numerical solution, i.e. the stability of IRK (2) in the sense of Dahlquist. Such perturbations may be caused e.g. by round-off errors or by errors in the iterative solution of the Runge–Kutta equations. In Section 3 the exact solution of (1) is inserted into (2) to prove convergence of the method and the arising defect is once again interpreted as perturbation of (2). Let a sequence $(\widehat{x}_n)$ be defined by

$$
\begin{aligned}
&\widehat{x}_{n+1} = \widehat{x}_n + h \sum_{j=1}^{s} b_j \widehat{x}_{n+1}^{(j)\,\prime} + h\varphi_{n+1}\,, \\
(5)\qquad &\widehat{x}_{n+1}^{(i)} = \widehat{x}_n + h \sum_{j=1}^{s} a_{ij} \widehat{x}_{n+1}^{(j)\,\prime} + h\varphi_{n+1}^{(i)}, \quad f(\widehat{x}_{n+1}^{(i)\,\prime}, \widehat{x}_{n+1}^{(i)}, t_n + c_i h) = \psi_{n+1}^{(i)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \dots, s)
\end{aligned}
$$

with perturbations φ_{n+1}, $\varphi_{n+1}^{(i)}$, $\psi_{n+1}^{(i)}$ bounded by

$$
\begin{aligned}
\|P(t_{n+1})\varphi_{n+1}\| &\le \delta_{\varphi,P}^{E}, \quad & \|Q(t_{n+1})\varphi_{n+1}\| &\le \delta_{\varphi,Q}^{E}\,, \\
\|\varphi_{n+1}^{(i)}\| &\le \delta_{\varphi}^{I}, & \|\psi_{n+1}^{(i)}\| &\le \delta_{\psi}
\end{aligned}
$$

where $Q(t)$ denotes a smooth projector function onto $\ker f_y'(y,x,t)$ and $P(t) := I - Q(t)$. In Theorem 2 we give the stability estimate. For IRK(DAE) with non-singular Runge–Kutta matrix A (then (3) is satisfied with $R(\infty) = 0$) a stability estimate was given in [Ob90].

THEOREM 2. *An implicit Runge–Kutta method* (2) *applied to a transferable DAE* (1) *is stable if the matrix A is non-singular and the contractivity condition* (3) *is satisfied. For sufficiently small stepsizes h and for $t_0 + nh \le T$ we have the* *estimate*

$$\|\widehat{x}_n - x_n\| \le C_0(\|\widehat{x}_0 - x_0\| + \delta^E_{\varphi,P} + h\delta^E_{\varphi,Q} + h\delta^I_{\varphi} + \delta_{\psi}) \tag{6}$$

with a constant C_0 being independent of h, n and of the perturbations.

Proof. See Section 4.

3. Convergence of implicit Runge–Kutta methods for transferable DAE's. Provided that the solution $x(t)$ of DAE (1) is sufficiently smooth, expansion of $x(t)$ in a Taylor series gives the estimates

$$x(t_n + h) = x(t_n) + h\sum_{j=1}^{s} b_j x'(t_n + c_j h) + \mathcal{O}(h^{k_E+1}) \quad \text{and}$$

$$x(t_n + c_i h) = x(t_n) + h\sum_{j=1}^{s} a_{ij} x'(t_n + c_j h) + \mathcal{O}(h^{k_I+1}) \quad (i = 1, \ldots, s)$$

(note that an IRK of order k_E satisfies the condition $\mathbf{B}(k_E)$: $\sum_j b_j c_j^{l-1} = 1/l$, $l = 1, \ldots, k_E$ [But87]). Inserting the analytical solution $x(t)$ in the Runge–Kutta equations (2) we therefore get a defect of order $\delta^E_{\varphi,P}$, $\delta^E_{\varphi,Q} = \mathcal{O}(h^{k_E})$, $\delta^I_{\varphi} = \mathcal{O}(h^{k_I})$, $\delta_{\psi} = 0$ (set $\widehat{x}_n = x(t_n)$, $\widehat{x}^{(i)}_{n+1} = x(t_n + c_i h)$, $\widehat{x}^{(i)\,\prime}_{n+1} = x'(t_n + c_i h)$ in (5)). Thus stability estimate (6) gives the proof of

THEOREM 3. *Suppose that IRK* (2) *has a non-singular Runge–Kutta matrix A and satisfies the contractivity condition* (3). *Applying* (2) *to a transferable DAE* (1) *with smooth solution the method is convergent with order* $\min(k_E, k_I + 1)$ $(x_0 := x(t_0))$. ■

We thus have convergence for Radau IA, Radau IIA and Lobatto IIIC methods since these methods satisfy the assumptions of Theorem 3 (see [But87]). The order of convergence (for transferable DAE's) is s (Radau IA and Lobatto IIIC methods) or $\min(2s-1, s+1)$ (Radau IIA methods). Theorem 3 does *not* apply to Gauß–Legendre methods (because of $|R(\infty)| = 1$) and to Lobatto IIIA and Lobatto IIIB methods (since A is singular). Furthermore, for IRK(DAE) we get order $\min(k_E, k_I + 1)$ only. Until now we have no idea how to show order k_E as in the special case of $\ker f'_y(y, x, t) \equiv N$ (and as suggested by the numerical tests discussed in [Pe86]).

In practical applications DAE's appear that have non-differentiable solution components. For such DAE's Griepentrog and März [GM86, §1.2] introduce the concept of generalized solutions. A continuous function $x(t)$ is called a *generalized solution* of DAE (1) if the projection $P(t)x(t)$ is continuously differentiable and

$$f((P(t)x(t))' - P'(t)x(t), x(t), t) = 0 \tag{7}$$

is satisfied (because of $x'(t) = P(t)x'(t) + Q(t)x'(t) = (P(t)x(t))' - P'(t)x(t) +$

$Q(t)x'(t)$ and $Q(t)x'(t) \in \ker f'_y(y,x,t)$, a smooth solution of (1) also satisfies (7)). We have

THEOREM 4. *Under the assumptions of Theorem* 3, *IRK* (2) (*of classical order* $k_E \geq 1$) *applied to a transferable DAE* (1) *with generalized solution* $x(t)$ *is convergent with order* 1, *provided that* $x(t)$ *and the derivatives* $(P(t)x(t))'$ *and* $Q'(t)$ *are Lipschitz-continuous.*

Proof. To use the stability estimate (6) we set $\widehat{x}_n = x(t_n)$, $\widehat{x}^{(i)}_{n+1} = x(t_n + c_i h)$ as before and choose now $\widehat{x}^{(i)\,\prime}_{n+1} = (Px)'(t_n + c_i h) - (P'x)(t_n + c_i h)$. Then $\delta_\psi = 0$, $\delta^E_{\varphi,Q} = \mathcal{O}(1)$ and $\delta^I_\varphi = \mathcal{O}(1)$ follow immediately. Because of $k_E \geq 1$ the order condition $\sum_j b_j = 1$ is satisfied and thus

$$\begin{aligned} P(t_{n+1})\varphi_{n+1} &= \frac{P(t_{n+1})(x(t_{n+1}) - x(t_n))}{h} \\ &\quad - \sum_j b_j P(t_{n+1})((Px)'(t_n + c_j h) - (P'x)(t_n + c_j h)) \\ &= P(t_{n+1}) \sum_j b_j \bigg(\bigg(\frac{(Px)(t_{n+1}) - (Px)(t_n)}{h} - (Px)'(t_n + c_j h) \bigg) \\ &\quad - \bigg(\frac{P(t_{n+1}) - P(t_n)}{h} x(t_n) - (P'x)(t_n + c_j h) \bigg) \bigg) . \end{aligned}$$

Applying the mean-value theorem to $(Px)(\cdot)$ and $P(\cdot)$ we get $\delta^E_{\varphi,P} = \mathcal{O}(h)$ and order 1 of convergence (see (6)). ■

4. Technical details. In this section the proofs of Theorems 1 and 2 are given.

Proof of Theorem 1. We define $Q_i = Q(t_n + c_i h)$ and $P_i = P(t_n + c_i h)$ and get $Q_i P_i = P_i Q_i = 0$, $Q_i + P_i = I$, $Q_i^2 = Q_i$ and $P_i^2 = P_i$. With the notations

$$u_i = P_i x^{(i)}_{n+1}, \quad w_i = Q_i x^{(i)}_{n+1} + P_i x^{(i)\,\prime}_{n+1}, \quad v_i = Q_i x^{(i)\,\prime}_{n+1}$$

IRK (2) reads

$$x_{n+1} = u_i + Q_i w_i, \quad x^{(i)\,\prime}_{n+1} = P_i w_i + v_i \tag{8}$$

and

$$f(w_i + v_i - Q_i w_i, u_i + Q_i w_i, t_n + c_i h) = 0\,, \tag{9}$$

$$u_i + Q_i w_i = x_n + h \sum_j a_{ij}(P_j w_j + v_j)\,; \tag{10}$$

Q_i is a projector onto $\ker f'_y(y, x, t_n + c_i h)$, so that

$$f(w_i, u_i + Q_i w_i, t_n + c_i h) = 0 \tag{11}$$

because of (9) (cf. [GM86, §1.2, Theorem 1]).

Equation (10) implies

$$hv_j = \sum_k a^*_{jk}(u_k + Q_k w_k - x_n) - hP_j w_j \tag{12}$$

with a^*_{jk} the elements of the matrix A^{-1}. We multiply (12) by Q_j, apply $Q_j v_j = v_j$, $Q_j P_j w_j = 0$ and $Q_j(u_k + Q_k w_k - x_n) = Q_k w_k - Q_k x_n + (Q_j - Q_k)(u_k + Q_k w_k - x_n)$ and get

$$h\sum_j a_{ij} v_j = Q_i w_i - Q_i x_n + \sum_{j,k} a_{ij} a^*_{jk}(Q_j - Q_k)(u_k + Q_k w_k - x_n)\,, \tag{13}$$

$$\begin{aligned} u_i - \sum_{j,k} a_{ij} a^*_{jk}(Q_j - Q_k)(u_k + Q_k w_k) - h\sum_j a_{ij} P_j w_j & \\ = P_i x_n - \sum_{j,k} a_{ij} a^*_{jk}(Q_j - Q_k) x_n\,. & \end{aligned} \tag{14}$$

Equations (14) and (11) form a system of $2ms$ equations with $2s$ unknown vectors $u_1, \ldots, u_s, w_1, \ldots, w_s$. The Jacobian of this system has the form

$$J = \begin{pmatrix} I + \mathcal{O}(h) & \mathcal{O}(h) \\ \mathcal{O}(1) & G \end{pmatrix} \tag{15}$$

with I the identity matrix of order ms and

$$G = \operatorname*{diag}_i(G(w_i, u_i + Q_i w_i, t_n + c_i h)), \quad G(y,x,t) = f'_y(y,x,t) + f'_x(y,x,t)Q(t)$$

(note that $Q_j - Q_k = \mathcal{O}(h)$). Because of the transferability of DAE (1), the matrix G is regular and has a bounded inverse, thus for sufficiently small stepsizes h the Jacobian J is regular as well and has a bounded inverse. Due to Hadamard's theorem (see e.g. [ORh70]) (14) and (11) have a unique solution $u_1, \ldots, u_s, w_1, \ldots, w_s$. Subsequently, x_{n+1} is computed using (2) and (8). ∎

Proof of Theorem 2. First we will give estimates for $\|P_+(\widehat{x}_{n+1} - x_{n+1})\|$ and $\|Q_+(\widehat{x}_{n+1} - x_{n+1})\|$ in terms of $\|P_0(\widehat{x}_n - x_n)\|$, $\|Q_0(\widehat{x}_n - x_n)\|$, $\delta^E_{\varphi,P}$, $\delta^E_{\varphi,Q}$, δ^I_φ and δ_ψ (where $P_0 = P(t_n)$, $Q_0 = Q(t_n)$, $P_+ = P(t_{n+1})$, $Q_+ = Q(t_{n+1})$). Similarly to (13) we obtain

$$\begin{aligned} \widehat{x}_n - x_n + h\sum_j b_j(\widehat{v}_j - v_j) &= P_+(\widehat{x}_n - x_n) + \Big(1 - \sum_{j,k} b_j a^*_{jk}\Big) Q_+(\widehat{x}_n - x_n) \\ &\quad + Q_+ \sum_{j,k} b_j a^*_{jk} Q_k(\widehat{w}_k - w_k) - hQ_+ \sum_{j,k} b_j a^*_{jk} \varphi^{(k)}_{n+1} + R \end{aligned}$$

with

$$R = \sum_{j,k} b_j a^*_{jk}((Q_j - Q_k)(\widehat{u}_k - u_k) + (Q_j - Q_+)(Q_k(\widehat{w}_k - w_k) - (\widehat{x}_n - x_n) - h\varphi^{(k)}_{n+1}))\,.$$

The projections of $\widehat{x}_{n+1} - x_{n+1}$ are given by

$$\begin{aligned}(16)\qquad P_+(\widehat{x}_{n+1} - x_{n+1}) &= P_0(\widehat{x}_n - x_n) + (P_+ - P_0)(\widehat{x}_n - x_n)\\ &+ h\sum_j b_j P_+ P_j(\widehat{w}_j - w_j) + hP_+\varphi_{n+1} + P_+R\,,\end{aligned}$$

$$\begin{aligned}(17)\qquad Q_+(\widehat{x}_{n+1} - x_{n+1}) &= \Big(1 - \sum_{j,k} b_j a^*_{jk}\Big) Q_0(\widehat{x}_n - x_n)\\ &+ \Big(1 - \sum_{j,k} b_j a^*_{jk}\Big)(Q_+ - Q_0)(\widehat{x}_n - x_n)\\ &+ Q_+\sum_{j,k} b_j a^*_{jk} Q_k(\widehat{w}_k - w_k)\\ &+ h\sum_j b_j Q_+ P_j(\widehat{w}_j - w_j) + hQ_+\varphi_{n+1}\\ &- hQ_+\sum_{j,k} b_j a^*_{jk}\varphi^{(k)}_{n+1} + Q_+R\,.\end{aligned}$$

The vectors u_i, w_i and $\widehat{u}_i, \widehat{w}_i$ are solutions of the systems of equations (14), (11) and

$$\begin{aligned}(18)\qquad &\widehat{u}_i - \sum_{j,k} a_{ij}a^*_{jk}(Q_j - Q_k)(\widehat{u}_k + Q_k\widehat{w}_k) - h\sum_j a_{ij}P_j\widehat{w}_j\\ &= P_i\widehat{x}_n - \sum_{j,k} a_{ij}a^*_{jk}(Q_j - Q_k)\widehat{x}_n + hP_i\varphi^{(i)}_{n+1} - h\sum_{j,k} a_{ij}a^*_{jk}(Q_j - Q_k)\varphi^{(k)}_{n+1}\,,\end{aligned}$$

$$(19)\qquad f(\widehat{w}_i, \widehat{u}_i + Q_i\widehat{w}_i, t_n + c_ih) = \psi^{(i)}_{n+1}$$

respectively. (System (18), (19) has a unique solution $\widehat{u}_1, \dots, \widehat{u}_s, \widehat{w}_1, \dots, \widehat{w}_s$ (cf. the proof of Theorem 1).) For sufficiently small stepsizes h the Jacobian J of these systems (cf. (15)) has a uniformly bounded inverse, so that the implicit function theorem gives

$$\begin{aligned}(20)\qquad &\|\widehat{u}_i - u_i\| + \|\widehat{w}_i - w_i\|\\ &\le \mathcal{O}(1)\|P_0(\widehat{x}_n - x_n)\| + \mathcal{O}(h)\|Q_0(\widehat{x}_n - x_n)\| + \mathcal{O}(h)\delta^I_\varphi + \mathcal{O}(1)\delta_\psi\,.\end{aligned}$$

(Note that $P_i(\widehat{x}_n - x_n) = P_0(\widehat{x}_n - x_n) + (P_i - P_0)(\widehat{x}_n - x_n)$, $Q_j - Q_k = \mathcal{O}(h)$ and $\widehat{x}_n - x_n = P_0(\widehat{x}_n - x_n) + Q_0(\widehat{x}_n - x_n)$.) We insert (20) in (16) and (17) and arrive at

$$(21)\qquad \begin{aligned}\|P_+(\widehat{x}_{n+1} - x_{n+1})\| &\le (1 + Ch)\|P_0(\widehat{x}_n - x_n)\|\\ &\qquad + Ch\|Q_0(\widehat{x}_n - x_n)\| + hC\varDelta\\ \|Q_+(\widehat{x}_{n+1} - x_{n+1})\| &\le C\|P_0(\widehat{x}_n - x_n)\|\\ &\qquad + (\alpha + Ch)\|Q_0(\widehat{x}_n - x_n)\| + C\varDelta\end{aligned}$$

with $\varDelta = \delta^E_{\varphi,P} + h\delta^E_{\varphi,Q} + h\delta^I_\varphi + \delta_\psi$, a constant $C \ge 0$ (independent of h, n and $\varDelta$) and $\alpha = 1 - \sum_{j,k} b_j a^*_{jk}$, i.e., α is the value of the stability function at infinity: $\alpha = R(\infty)$.

Because of the contractivity condition (3), Lemma 1 (see below) is applicable here with $y_n = \|P(t_n)(\widehat{x}_n - x_n)\|$, $z_n = \|Q(t_n)(\widehat{x}_n - x_n)\|$, $n \ge 0$, and gives the stability estimate (2). ∎

LEMMA 1. *Let sequences* (y_n) *and* (z_n) *of non-negative numbers be given that satisfy*

$$y_{n+1} \le (1 + hL)y_n + hMz_n + h\delta, \quad z_{n+1} \le Ny_n + (\alpha + h\gamma)z_n + \delta$$

with non-negative constants L, M, N, α, γ, δ *and* $\alpha < 1$.

Then there is an $h^* > 0$ *so that for all* $h \in (0, h^*]$

$$y_n + z_n \le C^*(1 + hL^*)^n(y_0 + (\alpha^n + h)z_0 + (1 + nh)\delta)$$

with constants C^* *and* L^* *independent of* h *and* δ.

Proof. The proof is similar to that of Lemma 2 in [DHZ87] (cf. also [Arn90]). ∎

References

[Arn90] M. Arnold, *Numerische Behandlung von semi-expliziten Algebrodifferentialgleichungen vom Index* 1 *mit linear-impliziten Verfahren*, PhD thesis, Martin-Luther-Universität Halle, Sektion Mathematik, 1990.

[But87] J. C. Butcher, *The Numerical Analysis of Ordinary Differential Equations*: *Runge–Kutta and General Linear Methods*, Wiley, Chichester 1987.

[DHZ87] P. Deuflhard, E. Hairer and J. Zugck, *One-step and extrapolation methods for differential-algebraic systems*, Numer. Math. 51 (1987), 501–516.

[GM86] E. Griepentrog and R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, Teubner-Texte Math. 88, Leipzig 1986.

[HLR89] E. Hairer, Ch. Lubich and M. Roche, *The Numerical Solution of Differential-Algebraic Systems by Runge–Kutta Methods*, Lecture Notes in Math. 1409, Springer, Berlin 1989.

[Ob90] H. Oberdörfer, *Zur numerischen Behandlung von Algebrodifferentialgleichungen mit Runge–Kutta-Verfahren*, PhD thesis, Ernst-Moritz-Arndt-Universität Greifswald, Sektion Mathematik, 1990.

[ORh70] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York 1970.

[Pe86] L. R. Petzold, *Order results for implicit Runge–Kutta methods applied to differential/algebraic systems*, SIAM J. Numer. Anal. 23 (1986), 837–852.