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REAL ANALYTIC MAXIMUM MODULUS MANIFOLDS IN STRICTLY PSEUDOCONVEX BOUNDARIES

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1. Introduction. Let D be a domain in $\mathbb{C}^n$. We shall denote by $A^k(D)$ the algebra of the holomorphic functions in D which have a C^k extension to $\overline{D}$ and by $\mathcal{O}(\overline{D})$ the algebra of the holomorphic functions in a neighborhood of $\overline{D}$. A subset E of ∂D is locally a maximum modulus set for $A^k(D)$ ((LMA^k) for short) if for every $p \in \partial D$, there exist a neighborhood U of p and $f \in A^k(D \cap U)$ such that $|f| = 1$ on $E \cap U$ and $|f| < 1$ on $\overline{D} \cap U \backslash E$. Similarly, E is locally a peak set for $A^k(D)$ ((LPA^k) for short) if for every $p \in E$, there exist a neighborhood U of p and $f \in A^k(D \cap U)$ such that $f = 1$ on $E \cap U$ and $|f| < 1$ on $\overline{D} \cap U \backslash E$. We have the same definitions for the algebra $\mathcal{O}(\overline{D})$ of the holomorphic functions in a neighborhood of $\overline{D}$ ((LMH) and (LPH) for short).

The characterization of the subsets of the boundary of a bounded strictly pseudoconvex domain with C^∞ boundary which are (LPA^∞) is well known: these are sets which are locally contained in totally real complex-tangential submanifolds of dimension $n-1$ ([HS] and [CC2]). In fact, these sets are also global peak sets for $A^\infty(D)$ [FH].

For instance, few things are known about the sets which are (LMA^k) and are not (LPA^k). The situation is clear only for the real analytic submanifolds M of dimension n in the boundary of strictly pseudoconvex domains with real analytic boundary: M is (LMH) if and only if M is totally real and admits a real analytic foliation by complex-tangential submanifolds of codimension 1 [DS]. In general, a subset E of the boundary of a strictly pseudoconvex domain with C^∞ boundary which is (LMA^∞) is locally contained in totally real submanifolds of dimension n which admit a foliation of dimension 1 which is complex-tangential at the points of E [I2]. But a set which is (LMH) is not in general a global maximum modulus set

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[DS] and a submanifold of dimension n of the boundary of a strictly pseudoconvex domain with real analytic boundary which is (LMA^2) is real analytic [NR].

Here we use the contact structure of the boundary of a strictly pseudoconvex domain to study the foliations by complex-tangential submanifolds [BI]. These methods were used in the context of interpolation sets, peak sets or maximum modulus sets in [HT], [CC1], [CC2] and [DS]. We present the results from [BS] and [DS] about real analytic submanifolds of maximal dimension of the boundary of a strictly pseudoconvex domain with real analytic boundary which are (LMH) and we use them to obtain results in lower dimension ([BI]). Finally we present an approach from [NR] (matching of holomorphic and antiholomorphic functions along maximum modulus sets) and some examples from [NR], which prove that the situation is really complicated for curves transverse to the complex-tangent space which are not real analytic.

2. Preliminaries

a) *Symplectic structures.* A symplectic manifold is a couple (X, Ω), where X is a differentiable manifold of dimension $2n$ and Ω is a 2 closed form on X such that $\Omega^n \neq 0$ on X. A submanifold M of X is isotropic if $\Omega(\xi, \eta) = 0$ for all ξ, η tangent to M. If M is an isotropic submanifold of X, we have $\dim M \leq n$ and if $\dim M = n$ we say that M is lagrangian.

We shall use the following:

THEOREM 1 (Darboux-Weinstein theorem) [WEI]. *Let $(X_1, \Omega_1), (X_2, \Omega_2)$, two symplectic manifolds of the same dimension, $M_1 \subset X_1, M_2 \subset X_2$, submanifolds. Let $\varphi : M_1 \to M_2$ a diffeomorphism such that $\varphi^*(\Omega_2|M_2) = \Omega_1|M_1$. Then, for every $p \in M_1$ there exist a neighborhood V_1 in X_1, a diffeomorphism ψ on a neighborhood V_2 of $\varphi(p)$ in X_2 such that ψ is an extension of φ and $\psi^*(\Omega_2|V_2) = \Omega_1|V_1$.*

Theorem 1 is also true for real analytic objects.

b) *Contact structures.* A contact manifold is a couple (Z, ω), where Z is a differentiable manifold of dimension $2n + 1$, and ω is a 1 form on Z such that $\omega \wedge (d\omega)^n \neq 0$ on Z. There exists a unique vector field X_ω on Z (the characteristic vector field) such that $i(X_\omega)\omega = 1, i(X_\omega)d\omega = 0$, where $i(\xi)\eta$ is the left inner product of a differential form η by a vector field ξ.

A submanifold N of Z is isotropic if $\omega|N = 0$. If N is an isotropic submanifold of Z, we have $\dim N \leq n$ and if $\dim N = n$, we say that N is a Legendre manifold.

c) *Levi form.* Let D be a strictly pseudoconvex domain with C^2 boundary and ρ a strictly plurisubharmonic defining function for D. We denote by j the inclusion of ∂D in $\mathbb{C}^n$ and $\omega = j^*(\frac{1}{i}\partial\rho)$. Then the complex-tangent space to ∂D is $T^c(\partial D) = \ker \omega$ and ω is a contact form on ∂D.

If ξ, η are sections of $T^c(\partial D)$, the Levi form is defined by

$$\mathcal{L}(\xi, \eta) = \partial\overline{\partial}\rho(X, \overline{Y})$$

where $X, Y \in T^c(\partial D) \otimes \mathbb{C}, \xi = \operatorname{Re} X, \eta = \operatorname{Re} Y$. The form $\mathcal{L}(\xi, \eta)$ defines a hermitian metric on $T^c(\partial D)$ and we shall say that ξ and η are $\mathcal{L}$-orthogonal if $\mathcal{L}(\xi, \eta) = 0$.

A submanifold M of ∂D is complex-tangential if M is an isotropic submanifold of the contact manifold $(\partial D, \omega)$. Since $\partial\overline{\partial}\rho|\partial D = -id\omega$ a complex-tangential submanifold M is totally real and $\dim M \leq n - 1$ [HT], [BS]. Also we have $d\omega(\xi, \eta) = -\frac{1}{2} \operatorname{Im} \mathcal{L}(\xi, \eta)$.

3. Isotropic foliations

PROPOSITION 1 [BI]. *Let (X, ω) be a contact manifold of dimension $2n + 1$ and M an isotropic submanifold of dimension k, $0 \leq k \leq n$. Then, for every $p \in M$ there exist local coordinates $(x_0, \ldots, x_{2n})$ in a neighborhood of p such that $\omega = dx_0 + \sum_1^n x_i dx_{i+n}$ and $M = \{x_0 = x_{k+1} = \ldots = x_{2n} = 0\}$. In particular M is an intersection of Legendre submanifolds.*

Proof. Let $p \in M$. Since the characteristic vector field X_ω is transverse to M, we may find a neighborhood U of p such that $Y = U/X_\omega$ is a manifold and the restriction of the projection $\pi : U \to Y$ to M is an diffeomorphism onto $\pi(M \cap U)$. Then (Y, σ) is a symplectic manifold , where σ is the form induced by $d\omega$ on Y. Since $\pi(M \cap U)$ is isotropic, by theorem 1 we may extend a coordinate system of $\pi(M \cap U)$ to a coordinate system $(\widetilde{x}_1, \ldots, \widetilde{x}_{2n})$ on Y such that $\pi(M \cap U) = \{\widetilde{x}_{k+1} = \ldots = \widetilde{x}_{2n} = 0\}$ and $\sigma = \sum_1^n d\widetilde{x}_i \wedge d\widetilde{x}_{i+n}$. Then, if $\widetilde{x}_i = x_i \circ \pi$, the form $\omega - \sum_1^n x_i \wedge dx_{i+n}$ is closed and we may find x_0 such that $\omega = dx_0 + \sum_1^n x_i dx_{i+n}$ and $M = \{x_0 = x_{k+1} = \ldots = x_{2n} = 0\}$.

Now, M is the intersection of the Legendre manifolds $\{x_0 = x_{k+1} = \ldots = x_{k+n} = 0\}$ and $\{x_0 = x_{n+1} = \ldots = x_{2n} = 0\}$.

PROPOSITION 2 [BI]. *Let (X, ω) be a contact manifold of dimension $2n + 1$ and M a submanifold of dimension $k + 1$ transverse to* $\ker \omega$. *Then M admits a foliation by isotropic submanifolds of codimension 1 if and only if there exist local coordinates $(x_0, \ldots, x_{2n})$ such that $\omega = \varphi(dx_0 + \sum_1^n x_i dx_{i+n})$, with $\varphi \neq 0$ and $M = \{x_{k+1} = \ldots = x_{2n} = 0\}$. In particular, in this case, M is intersection of $n + 1$ dimensional submanifolds foliated by Legendre submanifolds.*

Proof. If ω is as in proposition 2, it is clear that the submanifolds $M_c = \{x_{k+1} = \ldots = x_{2n} = 0,\ x_0 = c\}$ give an isotropic foliation of codimension 1 of M, so we have only to prove the converse.

Let $p \in M$. By Frobenius theorem, there exist a neighborhood U of p and functions f, u on U such that $j^*(f\omega - du) = 0$ on U, where $j : M \to X$ is the inclusion. Since M is transverse to $\ker \omega$, there exists a vector field ξ tangent to M in a neighborhood of p such that $i(\xi) j^*(f\omega) = 1$.

We shall consider $\omega' = \varphi\omega$ such that the restriction of $X_{\omega'}$ to M is ξ.

For this we shall prove that there exists a function g in the neighborhood of p such that $g = 1$ on M and $i(\xi) d(gf\omega) = 0$ on M. Indeed, since $j^* d(f\omega) = 0$ and ξ

is tangent to M, we have $i(\xi)j^*d(f\omega) = 0$. It follows that $i(\xi)d(f\omega)$ is of the form

$$\sum \alpha_r du_r + u_r\beta_r$$

where α_r, β_r are 0 and respectively 1 forms defined in a neighborhood of p and $u_r = 0$ on M. We may take $g = 1 + \sum \alpha_r u_r$.

Then, by taking $\varphi = fg$, since the characteristic vector field is the unique vector field η which satisfies $i(\eta)\omega' = 1$ and $i(\eta)d\omega' = 0$, we have $X_{\omega'} = \xi$ on a neighborhood of p in M. Since $i(\eta)\omega' = 1$ and $\omega' - du|M = f\omega - du|M = 0$ there exists an extension $\widetilde{u}$ of u such that $i(X_{\omega'})d\widetilde{u} = 1$ and we consider the foliation of X given by $\widetilde{u} = \text{constant}$.

Since $i(X_{\omega'})d\widetilde{u} = 1$, there exists a diffeomorphism $x \to (t, y)$ from a neighborhood V of p to $I \times Y$, where I is a real interval and Y is the manifold of orbits, such that $X_{\omega'}$ is transformed to $\frac{\partial}{\partial t}$. Since $d\omega'$ is an absolute integral invariant of $X_{\omega'}$, (Y, σ) is a symplectic manifold, where σ is the form induced by $d\omega'$. Finally, M is identified with $I \times M'$ where M' is an isotropic submanifold of Y and we may finish the proof by applying theorem 1 in the same way as in the proof of proposition 1.

4. Maximum modulus manifolds of maximal dimension. From now on, we shall denote by D a strictly pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$. If M is a real analytic submanifold of the boundary we shall denote by M^c a complexification of M.

PROPOSITION 3 [HS]. *Let M be a submanifold of $\partial\, D$ which is (LPH). Then M is complex-tangential. In particular M is totally real and* $\dim M \le n - 1$.

Proof. Let $p \in M$ and let ρ be a defining function for D in a neighborhood of p. Let $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$ be local coordinates in a neighborhood of p such that $p = 0$ and $\rho(z) = x_n + O(|z|^2)$. Let f be a holomorphic function in a neighborhood U of p such that $f = 0$ on $M \cap U$ and $\operatorname{Re} f < 0$ on $\overline{D} \cap U \backslash M$. By the Hopf lemma we have $\partial \operatorname{Re} f / \partial x_n(0) \neq 0$. Since the origin is a local maximum for $\operatorname{Re} f$ we have

$$\frac{\partial \operatorname{Re} f}{\partial x_j}(0) = \frac{\partial \operatorname{Re} f}{\partial y_j}(0) = 0, \quad 1 \le j \le n-1, \quad \frac{\partial \operatorname{Re} f}{\partial y_n}(0) = 0$$

and by the Cauchy-Riemann equations we have also

$$\frac{\partial \operatorname{Im} f}{\partial x_j}(0) = \frac{\partial \operatorname{Im} f}{\partial y_j}(0) = 0, \quad 1 \le j \le n-1, \quad \frac{\partial \operatorname{Im} f}{\partial x_n}(0) = 0, \quad \frac{\partial \operatorname{Im} f}{\partial y_n}(0) \neq 0.$$

It follows that $\Sigma = \{z \mid \rho(z) = \operatorname{Im} f(z) = 0\}$ is in a neighborhood of the origin a manifold of dimension $2n - 2$, $T_0(\Sigma) = \{z \mid z_n = 0\} = T_0^c(\partial D)$ and since $M \subset \Sigma$, M is complex-tangential.

THEOREM 2 [BS]. *Let M be a real analytic totally real submanifold of dimension $n - 1$ of ∂D. Then M is complex-tangential if and only if there exists M^c such that $M^c \cap \overline{D} = M$.*

Proof. Let $p \in M$ and let $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$ be holomorphic coordinates in a neighborhood of p such that $p = 0$ and $M = \{z \mid y_1 = \ldots = y_{n-1} = z_n = 0\}$. Let ρ be a strictly plurisubharmonic defining function for D in a neighborhood of p.

Let us suppose that $M^c \cap \overline{D} = M$. We have $\rho(z_1, \dots, z_{n-1}, 0) \geq 0$ and $\rho(x_1, \dots, x_{n-1}, 0) = 0$. In particular $\frac{\partial \rho}{\partial z_j}(x_1, \dots, x_{n-1}, 0) = 0$, $j = 1, \dots, n-1$ and M is complex-tangential.

Conversely, let M be a complex-tangential submanifold of ∂D.

We denote $z' = (z_1, \dots, z_{n-1}), z' = x' + iy'$. Since M is complex-tangential, we have $\frac{\partial \rho}{\partial z_j}(x', 0) = 0, j = 1, \dots, n-1$. So

$$\rho(z', 0) = \frac{1}{2} \sum_{j,k=1}^{n-1} \frac{\partial^2 \rho}{\partial y_j \partial y_k}(x', 0) y_j y_k + O(|y|^3).$$

Since $\frac{\partial \rho}{\partial z_j}(x', 0) = 0$, we have also $\frac{\partial^2 \rho}{\partial x_j \partial x_k}(x', 0) = \frac{\partial^2 \rho}{\partial x_j \partial y_k}(x', 0) = 0$, for every $j, k = 1, \dots, n-1$, so $\frac{\partial^2 \rho}{\partial y_j \partial y_k}(0) = 4 \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(0)$. Since ρ is strictly plurisubharmonic, there exits $m > 0$ such that

$$\sum_{j,k=1}^{n-1} \frac{\partial^2 \rho}{\partial y_j \partial y_k}(x', 0) y_j y_k \geq m|y'|^2$$

for every (x', y') in a neighborhood of the origin, so $\rho(z', 0) \geq 0$ and $\rho(z', 0) = 0$ if and only if $y' = 0$ for $|y'|$ small enough.

PROPOSITION 4 [I2]. *Let M be a submanifold of ∂D which is (LMH). Than M is totally real. If we suppose that M is transverse to $T^c(\partial D)$, then M admits a foliation by complex-tangential submanifolds of codimension* 1.

Proof. Let $p \in M$ and f a holomorphic function in the neighborhood U of p such that $|f| = 1$ on $M \cap U$ and $|f| < 1$ on $\overline{D} \cap U \backslash M$. Let $z = (z', z_n), z' = (z_1, \dots, z_{n-1})$, $z_j = x_j + iy_j$ be holomorphic coordinates in a neighborhood of p such that $p = 0$ and D has a strictly plurisubharmonic defining function $\rho = x_n + h(z', y_n)$, where h vanishes to second order at the origin.

Let $g = log f$, with g holomorphic in the neighborhood of p and $\operatorname{Im} g(p) = 0$. As in the proof of proposition 3 we have

$$\frac{\partial \operatorname{Re} g}{\partial x_j}(0) = \frac{\partial \operatorname{Re} g}{\partial y_j}(0) = \frac{\partial \operatorname{Im} g}{\partial x_j}(0) = \frac{\partial \operatorname{Im} g}{\partial y_j}(0) = 0, \quad 1 \leq j \leq n-1,$$
$$\frac{\partial \operatorname{Re} g}{\partial y_n}(0) = \frac{\partial \operatorname{Im} g}{\partial x_n}(0) = 0, \quad \frac{\partial \operatorname{Re} g}{\partial x_n}(0) \neq 0, \quad \frac{\partial \operatorname{Im} g}{\partial y_n}(0) \neq 0.$$

So we may consider the holomorphic change of coordinates near 0 given by $w_n = g$, $w_n = u_n + iv_n$ and we have $h(z', v_n) \geq 0$ and $h(z', v_n) = 0$ if $(z', v_n) \in M$. It follows that $grad h(z', v_n) = 0$ if $(z', v_n) \in M$.

Since $h(z',0)$ is strictly plurisubharmonic, by [HW] there exists a complex linear change of the coordinates z' such that

$$h(z',0)=\sum_{1}^{n-1}(1+\lambda_j)x_j^2+(1-\lambda_j)y_j^2+O(|z'|^3),\qquad \text{with } \lambda_j\ge 0.$$

It follows that $\{z\mid \rho=0,\frac{\partial\rho}{\partial x_j}(z)=0, j=1,\dots n-1\}$ is a totally real manifold of dimension n in a neighborhood of the origin which contains M. Suppose that M is not complex-tangential. Than the set $M_a=\{z\in M\mid \operatorname{Im} g=a\}$ is a manifold of codimension 1 in a neighborhood of the origin for every a small enough. But M_a is (LPH) for the function $F=e^{-ia}(f+e^{ia})/2$ [DS] and by proposition 3 it follows that M admits a foliation by complex-tangential submanifolds of codimension 1.

Example 1. Let D be the domain

$$D=\Big\{z=(z_1,\dots,z_n)\in\mathbb{C}^n\ \Big|\ \operatorname{Re} z_n+\sum_{i=1}^{n-1}(\operatorname{Re} z_i)^2+((\operatorname{Im} z_n+(\operatorname{Im} z_1)^2)^4<0\Big\}$$

and $M=\{z\mid \operatorname{Re} z_1=\ldots=\operatorname{Re} z_n=0, \operatorname{Im} z_n=-(\operatorname{Im} z_1)^2\}$. D is a strictly pseudoconvex domain with real analytic boundary and M is a totally real submanifold of dimension $n-1$ of ∂D. We have $\operatorname{Re} z_n\le 0$ on $\overline{D}$ and $\operatorname{Re} z_n=0$ on $\overline{D}$ if and only if and only if $z\in M$. So M is (LMH) for the function $\exp(z_n)$. But

$$T_0(M)=\{z\mid \operatorname{Re} z_1=\ldots=\operatorname{Re} z_{n-1}=z_n=0\}\subset T_0^c(\partial D)=\{z|z_n=0\}$$

and M is transverse to $T_z^c(\partial D)$ if $\operatorname{Im} z_1\ne 0$.

Corollary 1 [DS]. *Let M be an n-dimensional real analytic submanifold of ∂D which is (LMH). Then M is totally real and admits a real analytic foliation of codimension 1 by complex-tangential submanifolds.*

Theorem 3 [DS]. *Let M be an n-dimensional real analytic totally real submanifold of ∂D. Then M is (LMH) if and only if M admits a real analytic foliation of codimension 1 by complex-tangential submanifolds.*

Proof. By corollary 1 we have only to prove the converse. Let $z\in M$ and suppose that $M_a=\{z\in M\mid \varphi(z)=a\}$ give a foliation of M by complex-tangential submanifolds, where φ is real analytic in a neighborhood of z in M. Since M is totally real there exists a holomorphic extension $\widetilde{\varphi}$ of φ in a neighborhood of z in $\mathbb{C}^n$. Let $\Sigma=\{z\mid \operatorname{Im}\widetilde{\varphi}(z)=0\}$. Since $d\operatorname{Re}\widetilde{\varphi}(z)\ne 0$, by the Cauchy-Riemann equations, Σ is a hypersurface in a neighborhood U of z which contains M. Also if $a\in\mathbb{R}$, $\Sigma_a=\{z\in U\mid \widetilde{\varphi}(z)=a\}$ is a complex submanifold of Σ which is a complexification of M_a. By proposition 4, we have $\Sigma_a\cap\overline{D}=M_a$, so $\Sigma\cap\overline{D}\cap U=M\cap U$. It follows that $\operatorname{Im}\widetilde{\varphi}$ has constant sign on $D\cap U$, so $M\cap U$ is (LMH) for one of the functions $\exp(\pm i\widetilde{\varphi}(z))$.

Example 2 [DS]. Let B_2 be the unit ball in $\mathbb{C}^2$, $\alpha_1,\alpha_2\in\mathbb{R}$ such that $\alpha_1^2+\alpha_2^2=1$ and $\gamma=\alpha_1^2/\alpha_2^2$ is irrational and $M=\{z=(z_1,z_2)||z_1|=\alpha_1,|z_2|=\alpha_2\}$. M is a totally real 2-dimensional submanifold of ∂B_2 which is foliated by the complex-tangential curves $M_c = \{z = (z_1, z_2) \mid z_1 = \alpha_1 e^{-ci\alpha_2^2 t},\ z_2 = \alpha_2 e^{ci\alpha_1^2 t}\}$. Since M_c is dense in M, it follows that M is not a global maximum modulus set. But M is (LMH) for the function $f(z_1, z_2) = z_1^2 z_2^\gamma$.

5. Complex-tangential foliations and diagonalizable Levi form

LEMMA 1 [BI]. *Let M be a k-dimensional submanifold of ∂D. Let $p \in M$ such that M is not complex-tangential in a neighborhood of p. Then $T(M) \cap T^c(\partial D)$ is a distribution of rank $k-1$ in a neighborhood of p.*

Proof. Since M is not complex-tangential, we have $\dim T_z(M) \cap T_z^c(\partial D) < k$ for z in a neighborhood of p and since both $T_z(M)$ and $T_z^c(\partial D)$ are subspaces of $T_z(\partial D)$ we have $\dim T_z(M) \cap T_z^c(\partial D) = k-1$ for z in a neighborhood of p.

THEOREM 4 [BI]. *Let M be a submanifold of ∂D and $p \in M$ such that M is not complex-tangential in a neighborhood of p. The following are equivalent*:

a) *There exists a neighborhood of p where M admits a foliation by complex-tangential submanifolds of codimension* 1.

b) *There exists a neighborhood U of p such that $\mathcal{L}(\xi, \eta) \in \mathbb{R}$ for any sections ξ, η over U of the bundle $T(M) \cap T^c(\partial D)$.*

Proof. By Frobenius theorem a) is valid if and only if the distribution defined by $T(M) \cap T^c(\partial D)$ is integrable. With the notations of 1c) this happens if and only if $j^*(d\omega) = \varphi \wedge j^*(\omega)$. Since $d\omega(\xi, \eta) = -\frac{1}{2} \operatorname{Im} \mathcal{L}(\xi, \eta)$ for any sections ξ, η of $\ker \omega$ we obtain the result.

By theorem 2, proposition 2 and theorem 3 we obtain:

COROLLARY 2 [BI]. *Let M be a real analytic totally real submanifold of ∂D which is not complex-tangential at any point. Then M is (LMH) if and only if $\mathcal{L}(\xi, \eta) \in \mathbb{R}$ for every ξ, η sections of $T(M) \cap T^c(\partial D)$. This is always true if* $\dim M \le 2$.

COROLLARY 3 [BI]. *Let M be a real analytic totally real n-dimensional submanifold of ∂D. M is (LMH) if and only if for every $p \in M$ there exists a complex $\mathcal{L}$-orthogonal frame of $T^c(\partial D)$ in a neighborhood of p which generates $T(M) \cap T^c(\partial D)$ over $\mathbb{R}$.*

EXAMPLE 3 [I1]. Let $D = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 \mid 2 \operatorname{Re} z_3 + |z_1|^2 + |z_2|^2 + |z_3|^2 < 0\}$ and $M = \{z \in \partial D \mid \operatorname{Re} z_2 = 2 \operatorname{Im} z_1,\ \operatorname{Re} z_1 = \operatorname{Im} z_2\}$. D is isomorphic with the unit ball in $\mathbb{C}^3$ and M is a real analytic totally real submanifold of dimension 3 of ∂D. The vector space $T_0(M) \cap T_0^c(\partial D)$ is generated by

$$\xi = \operatorname{Re}\left[\left(\frac{\partial}{\partial z_1}\right)_0 - i\left(\frac{\partial}{\partial z_2}\right)_0\right] \quad \text{and} \quad \eta = \operatorname{Re}\left[2\left(\frac{\partial}{\partial z_2}\right)_0 - i\left(\frac{\partial}{\partial z_1}\right)_0\right].$$

We see that $\mathcal{L}_0(\xi, \eta) \notin \mathbb{R}$, so M does not admit a foliation by complex tangential submanifolds.

Theorem 4 [BI]. *A real analytic submanifold M of ∂D is (LPH) if and only if M is complex-tangential.*

Proof. If $\dim M = n-1$, it follows by proposition 2 that M is the leaf of an n-dimensional submanifold M' of ∂D which admits a foliation by complex-tangential submanifolds. By theorem 3 M' is (LMH) and by the proof of proposition 4 it follows that M is (LPH). The general case follows by applying proposition 2.

6. Real analyticity for smooth maximum modulus manifolds

Theorem 5 [NR]. *Let D be a strictly pseudoconvex domain with real analytic boundary and E a (LMA^2). Then, for every $\zeta \in E$, there exist a neighborhood U of ζ and a C^1 map G on U, holomorphic in $D \cap U$, such that $G(z) = \overline{z}$ on E.*

Proof. Let us suppose that in a neighborhood V of z we have $D \cap V = \{z \in V \mid \rho(z) < 0\}$ where ρ is strictly plurisubharmonic on V. Let $\Sigma = \{(z, \xi) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} \mid z \in \partial D \cap V, \xi = [\partial\rho(z)]\}$ where $\mathbb{CP}^{n-1}$ is the complex projective space of dimension $n-1$, and $[\partial\rho(z)]$ is the point in $\mathbb{CP}^{n-1}$ which has homogeneous coordinates $(\partial\rho/\partial z_1, \ldots, \partial\rho/\partial z_n)$. By [WEB] it follows that Σ is a real analytic totally real submanifold of dimension $2n-1$ of $\mathbb{C}^n \times \mathbb{CP}^{n-1}$. We denote by $\chi = (\chi_1, \chi_2)$ the antiholomorphic reflection across Σ.

Let $z \in E$ and f a holomorphic function in a neighborhood U of $z, U \subset V$, such that $|f| = 1$ on $E \cap U$ and $|f| < 1$ on $\overline{D} \cap U \backslash E$. By the Hopf lemma we have $[\partial\rho(z)] = [\partial f(z)]$ for every $z \in E$. We denote $G(z) = \overline{\chi_1(z, [\partial f(z)])}$. Since χ is antiholomorphic and $\chi(\Sigma) = \Sigma$ it follows that G is a C^1 map on U, holomorphic in $D \cap U$, and if $z \in E$, we have

$$G(z) = \overline{\chi_1(z, [\partial f(z)])} = \overline{\chi_1(z, [\partial \rho(z)]} = \overline{z}.$$

Corollary 4 [NR]. *Let D be a strictly pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$ and M a C^1 submanifold of dimension n of ∂D which is (LMA^2). Then M is real analytic.*

Proof. Let $p \in E$. By theorem 5 there exist a neighborhood U of p and a C^1 map G on U, holomorphic on $D \cap U$ such that $G(z) = \overline{z}$. Then G is a C^1 diffeomorphism in a neighborhood of p and the maps $G(z)$ and $F(z) = \overline{G^{-1}(\overline{z})}$ are extensions of the restriction of $\overline{z}$ to M. But F and G are holomorphic on opposite wedges with edge M, so by the edge of the wedge theorem for C^1 manifolds [R], it follows that the restriction of $\overline{z}$ to M has a holomorphic extension $\Phi = (\Phi_1, \ldots, \Phi_n)$ to some neighborhood of p. Then from the $2n$ equations

$$\operatorname{Re}\Phi_j = \operatorname{Re} z_j, \quad \operatorname{Im}\Phi_j = -\operatorname{Im} z_j, \quad j = 1, \ldots, n$$

we can extract n independent equations which define the n dimensional manifold M.

Example 4 [NR]. This example will give a smooth curve transverse to the complex-tangent space in the boundary of the unit ball B_2 in $\mathbb{C}^2$ which is locally a maximum modulus set for $A^\infty(B_2)$ and it is not real analytic.

Let D be the unit disk in $\mathbb{C}$, $h \in A^\infty(D)$, such that $h(z) - (1-z)$ vanishes to infinite order at 1 but $h(z) - (1-z)$ does not vanish identically. If $\varepsilon > 0$ is small enough, the set $\Gamma_\varepsilon = \{(z_1, z_2) \in \partial B_2 \mid \overline{z}_2 = \varepsilon h(z_1)\}$ is a smooth curve in ∂B_2 which has contact to infinite order at $(1,0)$ with the circle $\{|z_1| = 1, z_2 = 0\}$. So, Γ_ε is not real analytic. The curve Γ_ε can be parametrized by $(z_1, \theta(z_1))$ where z_1 belongs to a smooth simple closed curve γ_ε in the z_1 plane. We consider a smooth extension of θ to the bounded component Ω_ε of $\mathbb{C}\backslash\gamma_\varepsilon$ that we still denote by θ and we shall denote $\Omega = \{(z_1, \theta(z_1)) \mid z_1 \in \Omega_\varepsilon\}$. Let us consider the holomorphic vector field

$$Z = \frac{-\varepsilon h(z_1)}{1 - \varepsilon z_2 h(z_1)} \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}.$$

Since $\overline{z}_2 = \varepsilon h(z_1)$ and $\overline{z}_1 = (1 - z_2 \varepsilon h(z_1))/z_1$ on Γ_ε, Z is tangent to ∂B_2 along Γ_ε and it defines a holomorphic foliation of the ball near $(1,0)$. For ε small enough, Z is close to $\partial/\partial z_2$, so we may consider new real variables (x_1, y_1, x_2, y_2) in a neighborhood of $(1,0)$ such that $Z = \partial/\partial x_2 + i\partial/\partial y_2$.

Since for ε small the change of variables is close to the identity and Z is tangent to ∂B_2 at the points of Γ_ε, the points in Ω may be parametrized by (x_1, y_1), so each leaf of the foliation defined by Z has a unique point of intersection with Ω. So there exists a retraction G of the neighborhood of $(1,0)$ in $\overline{B}_2$ in a neighborhood of $(1,0)$ in $\overline{\Omega}$ such that the points in $\overline{B}_2\backslash\Gamma_\varepsilon$ correspond to points in Ω. Using G, we can define an almost complex structure J on $\overline{\Omega}$ induced by the complex structure in $\mathbb{C}^2$. This structure is integrable because of the complex dimension 1. So there exists a conformal transformation f from (Ω, J) to D which extends smoothly to $\overline{D}$. Then in a neighborhood of $(1,0)$, we have $|f \circ G| < 1$ on $\overline{B}_2\backslash\Gamma_\varepsilon$ and $|f \circ G| = 1$ on Γ_ε.

Example 5 [NR]. We shall give an example of a smooth curve Γ in the boundary of the unit ball B_2 in $\mathbb{C}^2$ such that the restriction of $\overline{z}$ to Γ has a holomorphic extension to B_2, but Γ is not (LMA^2).

Let D be the unit disk in $\mathbb{C}$ and $h \in A^\infty(D)$ such that h vanishes to infinite order at $(1,0)$. Let $\Gamma = \{(z_1, z_2) \in \partial B_2 \mid \overline{z}_2 = 2z_2 + h(z_1)\}$. Since Γ has contact to infinite order at $(1,0)$ with $\{(z_1, z_2) \mid |z_1| = 1,\ z_2 = 0\}$, it follows that Γ is not real analytic. On Γ we have

$$\overline{z}_1 = \frac{1 - 2z_2^2 - z_2 h(z_1)}{z_1}$$

so the restriction of $\overline{z}$ to Γ has a holomorphic extension in a neighborhood of $(1,0)$.

Let us suppose that there exist a neighborhood U of $(1,0)$ and $F \in A^2(B_2 \cap U)$ such that $|F| = 1$ on $\Gamma \cap U$ and $|F| < 1$ on $(\overline{B}_2\backslash\Gamma) \cap U$. Using the Hopf lemma as in the proof of proposition 3, we have $\frac{\partial F}{\partial z_1}(1,0) \neq 0$, $\frac{\partial F}{\partial z_2}(1,0) = 0$ and $(\frac{\partial F}{\partial z_2}(z), -\frac{\partial F}{\partial z_1}(z)) \in T_z^c(\partial B_2)$ if $z \in \Gamma$. So, if $z \in \Gamma$, we have

$$\frac{\frac{\partial F}{\partial z_2}}{\frac{\partial F}{\partial z_1}} = \frac{\overline{z}_2}{\overline{z}_1} = \frac{z_1(2z_2 + h(z_1))}{1 - 2z_2^2 - z_2 h(z_1)}.$$

Let $\lambda(t) = (\lambda_1(t), \lambda_2(t))$ be the solution of the Cauchy problem

$$\frac{d\lambda}{dt} = \left(-\frac{\frac{\partial F}{\partial z_2}}{\frac{\partial F}{\partial z_1}}(\lambda(t)), 1 \right), \qquad \lambda(0) = (1, 0).$$

By the form above of $\frac{\partial F}{\partial z_2} / \frac{\partial F}{\partial z_1}$, since Γ is not real analytic, we have $\lambda_1(t) = 1 - t^2 + o(t^2)$. Since $\lambda_2(t) = t$, we have $|\lambda(t)| < 1$ for t small enough, $t \neq 0$, so λ is a curve in $\overline{B}_2$ through $(1, 0)$ and $\lambda(t) \in B_2$ if $t \neq 0$. But $\frac{d}{dt}(F \circ \lambda) = 0$, so F is constant on λ. It follows that $|F| = 1$ for some points in the ball, which is a contradiction.

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