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LEBESGUE MEASURE AND MAPPINGS OF THE SOBOLEV CLASS $W^{1,n}$

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Abstract. We present a survey of the Lusin condition (N) for $W^{1,n}$-Sobolev mappings $f : G \to \mathbb{R}^n$ defined in a domain G of $\mathbb{R}^n$. Applications to the boundary behavior of conformal mappings are discussed.

1. Introduction. The change of the independent variable in integrals, i.e. formula (1.1) and its absolute value counterpart (1.2), is extremely useful in analysis and in its applications

$$\int_D u\big(f(x)\big) J_f(x)\,dx = \int_{f(D)} u(y)\mu(y,f,D)\,dy, \tag{1.1}$$

$$\int_A u\big(f(x)\big) |J_f(x)|\,dx = \int_{f(A)} u(y)N(y,f,A)\,dy. \tag{1.2}$$

Here J_f is the jacobian determinant of a mapping f, $\mu(y,f,D)$ is the topological degree of the triple (y,f,D), and $N(y,f,A) = \#(f^{-1}(y)\cap A)$ is the crude multiplicity function. On the real line conditions for (1.1) or (1.2) are well-known and a natural class of functions is the class of absolutely continuous functions. See the book [S] of S. Saks for this theory. In higher dimensional spaces (1.1) and (1.2) are problematic. In general, there are at least two conditions for the mapping f: The jacobian determinant J_f of $f : G \to \mathbb{R}^n$ must be integrable and f must satisfy the Lusin condition (N). This means that if $A \subset G$ and $|A| = 0$, then $|f(A)| = 0$. Here, and in the following, $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^n$. For a thorough discussion of the above conditions in connection with (1.1) and (1.2) see [RR, pp. 363–365]; cf. also a recent article [H] by P. Hajłasz.

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For a domain G in $\mathbb{R}^n$ we let $W^{1,p}(G)$, $p \geq 1$, denote the class of mappings $f : G \to \mathbb{R}^n$ such that the coordinate functions f_i of f and their first distributional derivatives $\partial_j f_i$ belong to $L^p(G)$. In $L^p(G)$ we use the standard L^p-norm $\| \;\|_{p,G} = \| \;\|_p$ and in $W^{1,p}(G)$ the norm

$$\|f\|_{1,p,G} = \|f\|_{1,p} = \| \,|f|\, \|_{p,G} + \| \,|\nabla f|\, \|_{p,G}$$

where $|\nabla f|$ denotes the usual matrix norm of the $n \times n$ matrix ∇f formed by the partial derivatives $\partial_j f_i$. Instead of $W^{1,p}(G)$ the local Sobolev space $W^{1,p}_{\mathrm{loc}}(G)$ could be used as well because the condition (N) has a local character. Since we are interested in the boundary behavior of Sobolev functions, the space $W^{1,p}(G)$ is an obvious choice.

If $f : G \to \mathbb{R}^n$ belongs to the Sobolev class $W^{1,n}(G)$, then J_f is integrable in G. Thus this class is natural for (1.1) and (1.2). In connection with the condition (N) the mapping f is usually assumed to be continuous: then the condition (N) is equivalent with the property that f maps measurable sets into measurable sets. However, in the class $W^{1,n}(G)$ continuity is partly superfluous because a mapping $f \in W^{1,n}(G)$ has an n-quasicontinuous version and this version provides a natural redefinition of f to study the condition (N).

Continuous $W^{1,n}(G)$ mappings for $n \geq 2$ need not satisfy the condition (N). Such an example was first constructed by L. Cesari [C] (and reinvented in [MM]); somewhat simpler examples were produced by M. Reimann [Rei] and Yu. G. Reshetnyak [Res1], see also [V]. The examples of Reimann and Reshetnyak are based on the Riemann mapping theorem. To be more precise, let $f : B(0,1) \to D$ be a Riemann mapping function sending the unit disk onto a Jordan domain D in $\mathbb{R}^2$ with $|\partial D| > 0$. Then f has, by the Carathéodory theorem, a homeomorphic extension to $\overline{B}(0,1)$. Define

$$f^*(z) = \begin{cases} f(z), & z \in \overline{B}(0,1), \\ f\Big(\dfrac{z}{|z|^2}\Big), & z \in B(0,2) \setminus \overline{B}(0,1). \end{cases} \tag{1.3}$$

It is not difficult to see that $f^* \in W^{1,2}(B(0,2))$; note that since f is conformal

$$\int_{B(0,1)} |\nabla f|^2 \, dx = \int_{B(0,1)} J_f \, dx = |D| < \infty$$

and since f is bounded, f certainly belongs to $W^{1,2}(B(0,1))$. Since $|\partial D| > 0$, the mapping f^* cannot satisfy (N). This extension method can be used to study the boundary behavior of mappings in the class $W^{1,2}(B(0,1))$; the same method applies to higher dimensional spaces as well.

Since a mapping $f \in W^{1,n}(G)$ need not satisfy (N), it is natural to look for minimal additional conditions. One condition is that f belongs to the higher Sobolev space, i.e. $f \in W^{1,p}(G)$ for some $p > n$. Then f not only satisfies (N) but also a stronger absolute continuity condition introduced by S. Banach: For every $\varepsilon > 0$ there is $\delta > 0$ such that $A \subset G$ with $|A| < \delta$ implies $|fA| < \varepsilon$.

See [BI, Lemma 8.1] for a simple proof. In Sections 2 and 3 we discuss some other conditions, in general weaker than $W^{1,p}(G)$, $p > n$, that also guarantee (N). Section 2 is devoted to topological methods; these methods were first used by Reshetnyak, see [Res2], to prove that a quasiregular mapping satisfies (N). In Section 3 we consider analytical methods. These have turned out to be the most powerful.

Quite recently there has been a considerable interest in properties of the jacobian determinant J_f of a mapping $f \in W^{1,n}(G)$. Especially, if $f \in W^{1,n}(G)$ and if $J_f \geq 0$ a.e., then by a result of Müller [Mu] J_f is not only integrable in G but belongs locally to the Zygmund class $L \log L(G)$. It is rather surprising that this result has little to do with the condition (N). The example in [MM] is such that the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, has the properties: (i) $f \in W^{1,n}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, (ii) f does not satisfy (N), and (iii) $J_f = 0$ a.e. in $\mathbb{R}^n$.

2. Topological methods. We first recall the basic properties of the topological degree of a mapping f. Let G be a domain in $\mathbb{R}^n$ and let $f : G \to \mathbb{R}^n$ be continuous.

The *topological degree* $\mu(y, f, D)$ of f at y is defined whenever $D \Subset G$ is a domain and $y \in \mathbb{R}^n \setminus f(\partial D)$. The degree $\mu(y, f, D)$ is integer valued and it has the following properties:

(i) $y \mapsto \mu(y, f, D)$ is constant in each component of $\mathbb{R}^n \setminus f(\partial D)$.

(ii) If $y \in f(D)$ and the restriction of f to $\overline{D}$ is one-to-one, then $|\mu(y, f, D)|=1$.

(iii) If $y \in D$ and id is the identity mapping, then $\mu(y, \mathrm{id}, D) = 1$.

(iv) If $\mu(y, f, D_i)$ is defined for $i = 1, \ldots, k$ and if $D_1, \ldots, D_k$ are mutually disjoint domains such that $f^{-1}(y) \cap D \subset \cup_{i=1}^k D_i \subset D$, then

$$\mu(y, f, D) = \sum_{i=1}^{k} \mu(y, f, D_i).$$

(v) If f and g are connected with a homotopy h_t, $0 \leq t \leq 1$, such that $\mu(y, h_t, D)$ is defined for $0 \leq t \leq 1$, then $\mu(y, f, D) = \mu(y, g, D)$.

If G is a domain and if for all domains $D \Subset G$ and $y \in f(D) \setminus f(\partial D)$ we have $\mu(y, f, D) > 0$, then f is called *sense-preserving*. If $\mu(y, f, D) < 0$ for all such y and D, then f is called *sense-reversing*.

The standard reference for the topological degree is the monograph by Radó and Reichelderfer [RR]. Bojarski and Iwaniec [BI] and Reshetnyak [Res2] present a different, more analytic approach to the topological degree. If $f : G \to \mathbb{R}^n$ is differentiable at $x_0 \in G$ and $J_f(x_0) \neq 0$, then there exists a neighborhood D of x_0 such that $\mu(y, f, D) = \operatorname{sign} J_f(x_0)$ for all $y \in f(D)$. Thus the above definition of a sense-preserving mapping is an extension of the more familiar case when f is differentiable.

The basic idea for studying the condition (N) in the class $W^{1,n}(G)$ is to use approximation. Formula (1.1) certainly holds for smooth mappings and passing to the limit one hopes to get it for the limit mapping as well. This method was used by Reshetnyak [Res2, pp. 179–181]. He called a continuous mapping $f : G \to \mathbb{R}^n$ *stable* if it satisfies the following condition (S): Let $D \Subset G$ be a domain. Then for every point $y \in f(D) \setminus f(\partial D)$ there is an $\varepsilon > 0$ such that for every continuous mapping $\varphi : D \to \mathbb{R}^n$ with $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in D$ the set $\varphi(D)$ contains y.

2.1. Theorem [Res2, Theorem 6.2]. *Let G be a domain in $\mathbb{R}^n$ and $f : G \to \mathbb{R}^n$ a continuous mapping of the class $W^{1,n}(G)$. If f is stable, then f has the property (N).*

Reshetnyak also presented a simple condition ensuring the condition (S). Let $f : G \to \mathbb{R}^n$ be a continuous mapping. Assume that for every domain $D \Subset G$ and any point $y \in f(D)$ not in $f(\partial D)$ the degree $\mu(y, f, D)$ is nonzero. Then f is a stable mapping. Indeed, assume that f satisfies the given condition. Take an arbitrary domain $D \Subset G$ and a point $y \in f(D) \setminus f(\partial D)$. Let $\varepsilon = \operatorname{dist}(y, f(\partial D))$. Then $\varepsilon > 0$. Let $\varphi : D \to \mathbb{R}^n$ be a continuous mapping such that $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in D$. The mapping φ is homotopic to f as a mapping of the pair $(D, \partial D)$ into the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. The required homotopy is given by the mapping

$$\varphi_t(x) = (1 - t)\, f(x) + t\varphi(x), \quad t \in [0, 1].$$

This implies that $y \notin \varphi(\partial D)$ and $\mu(y, \varphi, D) = \mu(y, f, D) \neq 0$, and allows us to conclude that $y \in f(D)$, which is what was required to prove. See (v).

Since every sense-preserving (or sense-reversing) mapping satisfies the above condition, we have

2.2. Corollary. *Every sense-preserving mapping $f : G \to \mathbb{R}^n$ of the class $W^{1,n}(G)$ satisfies (N).*

Every homeomorphism $f : G \to \mathbb{R}^n$ is either sense-preserving or sense-reversing. Hence Corollary 2.2 yields

2.3. Corollary [Res2, Corollary 1, p. 182]. *Every homeomorphism $f : G \to \mathbb{R}^n$ of the class $W^{1,n}(G)$ satisfies (N).*

Note that Corollary 2.3 is not true for homeomorphisms in the class $W^{1,p}$, $p < n$. See [P].

The approximation method was strengthened in [MZ]. Let $f : G \to \mathbb{R}^n$ be a mapping with partial derivative a.e. We say that f satisfies *Sard's condition* (SA) if $J_f = 0$ a.e. in an open set $A \subset G$ implies $|fA| = 0$.

2.4. Theorem [MZ, Theorem 3.12]. *Suppose that $f \in W^{1,n}(G)$ is continuous with $J_f \geq 0$ a.e. Then the conditions (N) and (SA) are equivalent.*

2.5. Corollary [MZ, Corollary 3.13]. *If $f \in W^{1,n}(G)$ is continuous with $J_f > 0$ a.e., then f satisfies (N).*

Either Corollary 2.2 or Theorem 2.4 can be used to prove that every continuous, discrete, and open mapping $f \in W^{1,n}(G)$ satisfies (N). In particular, every quasiregular mapping has the property (N), see [Res2, p. 182].

It seems that the approximation method is difficult to use to study the condition (N) for mappings whose jacobian determinant may change sign. However, using a special property of the plane Martio and Ziemer [MZ] showed that every continuous and open mapping of the class $W^{1,2}$ in the plane has the property (N). For the general result in this direction see Corollary 3.2 below.

3. Analytical methods. In [MM] it was observed that analytical methods, Gehring's lemma (Lemma 3.3 below) and certain covering arguments, provide a proper tool to study the condition (N) for mappings with arbitrary sign of the jacobian determinant. There are two main results, Theorems 3.4 and 3.6 below.

A continuous mapping $f : G \to \mathbb{R}^n$ is called *M-pseudomonotone* if for all balls $B(x,r) \Subset G$

$$\operatorname{diam}\big(fB(x,r)\big) \le M \operatorname{diam}\big(f\partial B(x,r)\big).$$

3.1. Theorem [MM, Theorem A]. *Suppose that $f : G \to \mathbb{R}^n$ is a pseudomonotone mapping of the class $W^{1,n}(G)$. Then f satisfies (N).*

Since every continuous and open mapping is 1-pseudomonotone, Theorem 3.1 yields

3.2. Corollary [MM, Corollary B]. *A continuous and open mapping of the class $W^{1,n}$ satisfies the condition (N).*

The proof of Theorem 3.1 is based on Gehring's lemma, which is a version of Sobolev's imbedding theorem on $(n-1)$-dimensional spheres. For $n = 2$ this lemma is immediate.

3.3. Gehring's lemma [G], [Res2, Lemma 4.3]. *Suppose that $f : G \to \mathbb{R}^n$ is a $W^{1,n}(G)$-mapping. Then for each $x \in G$ and a.e. $r > 0$*

$$\operatorname{diam}\big(f\partial B(x,r)\big)^n \le cr \int\limits_{\partial B(x,r)} |\nabla f(y)|^n \, dS(y)$$

whenever $B(x,r) \Subset G$. The constant c depends only on n.

Since smooth mappings satisfy (N), and even $W^{1,p}$-mappings, $p > n$, have the same property, it is natural to ask which additional smoothness conditions for $W^{1,n}$-mappings imply (N). It is rather surprising that Hölder continuity suffices.

3.4. Theorem [MM, Theorem C]. *Every locally Hölder continuous $W^{1,n}$-mapping satisfies the condition (N).*

Here the local Hölder continuity of a mapping $f : G \to \mathbb{R}^n$ is understood in the weakest possible sense: For each compact set $K \subset G$ there is $\alpha > 0$ and $M < \infty$ such that

$$|f(x) - f(y)| \le M\,|x-y|^{\alpha}$$

for all x, $y \in G$.

The proof of Theorem 3.4 uses Theorem 3.6 below and a recent result of J. Malý:

3.5. Theorem [M]. *Let $g \in W^{1,n}(\mathbb{R}^n)$, $\varepsilon > 0$ and $1 < p < n$. Then there exist a Hölder continuous mapping $h \in W^{1,n}(\mathbb{R}^n)$ and a set E such that $\operatorname{Cap}_{1,p} E < \varepsilon$ and $g = h$ in $\mathbb{R}^n \setminus E$.*

Note that usually mappings in the class $W^{1,n}(G)$ are defined up to the set of Lebesgue measure zero. However, there is always an n-quasicontinuous representative, see [HKM, Ch. 4], which is defined up to the set of $(1, n)$-capacity zero. In Theorem 3.5 such a representative for g is used. $\operatorname{Cap}_{1,p}$ refers to the usual Sobolev p-capacity, i.e.

$$\operatorname{Cap}_{1,p} E = \inf \int_{\mathbb{R}^n} (|u|^p + |\nabla u|^p)\, dx$$

where the infimum is taken over all real valued functions $u \in W^{1,p}(\mathbb{R}^n)$ such that $u \geq 1$ a.e. in a neighborhood of E.

Since the condition (N) does not hold for $W^{1,n}$-mappings, it is natural to ask in which extent it fails. For example, if $f : G \to \mathbb{R}^n$ is a $W^{1,n}(G)$-mapping, is it possible to pick a small set $A \subset G$ such that $f|G \setminus A$ satisfies (N)? An example, see [JM], shows that the set A cannot be chosen to be of $(1, n)$-capacity zero. However, the following two results show that a slightly larger set A suffices.

3.6. Theorem [MM, Theorem 6]. *Let $f : G \to \mathbb{R}^n$ be a mapping of the class $W^{1,n}(G)$. Then there is a set $A \subset G$ of Hausdorff dimension zero such that $f|G \setminus A$ satisfies (N).*

In Theorem 3.6 we again use the n-quasicontinuous representative for f. If no such representative is used, then Theorem 3.6 still implies the following result of P. Hajłasz:

3.7. Corollary [H, Theorem 2]. *Suppose that $f \in W^{1,n}(G)$. Then we can redefine f on a set of Lebesgue measure zero in such a way that the new mapping satisfies (N).*

In Corollary 3.7 it suffices to assume that $f \in W^{1,1}(G)$, see [H, p. 94].

Another criterion for the smallness of an exceptional set A can be expressed in capacitary terms.

Let G be a domain in $\mathbb{R}^n$ and let C be a relatively closed subset of G. For $x \in G$ and $0 < r < d(x, \partial G)$ we write

$$\operatorname{cap}_n(x, C, r) = \operatorname{cap}_n\big(C \cap \overline{B}(x, r), B(x, 2r)\big)$$

where

$$\operatorname{cap}_n(F, A) = \inf_u \int_A |\nabla u|^n\, dx$$

is the usual n-capacity of the condenser (F, A), i.e. the infimum is taken over all functions $u \in C_0^\infty(A)$ such that $u \geq 1$ on F. We say that C has the *lower*

n-capacity density zero in G if

$$\varliminf_{r\to 0} \operatorname{cap}_n(x,C,r) = 0$$

for all $x \in G$.

3.8. Theorem [MM, Theorem F]. *Suppose that $f : G \to \mathbb{R}^n$ is a continuous mapping of the class $W^{1,n}(G)$. If A is a relatively closed subset of G with $|A| = 0$, then there are closed subsets C_i, $i = 1, 2, \ldots$, of A such that each C_i has the lower n-capacity density zero in G and*

$$|f(A \setminus \cup C_i)| = 0.$$

The proof for Theorem 3.8 uses the following result: If $f \in C(G) \cap W^{1,n}(G)$, then there is a relatively closed subset C of G such that

$$|f(G \setminus C)| \le c \int_G |\nabla f|^n \, dx$$

and C has the lower n-capacity zero in G; the constant c depends only on n. This result is based on Gehring's lemma, Lemma 3.3.

4. Applications. We consider the boundary behavior of conformal mappings in the unit disk.

If $f : B(0,1) \to \mathbb{R}^2$ is a bounded conformal mapping of the unit disk into $\mathbb{R}^2$, then f has a finite Dirichlet integral, i.e.

$$\int_{B(0,1)} |\nabla f|^2 \, dx = \int_{B(0,1)} |f'|^2 \, dx = \int_{B(0,1)} J_f \, dx = |fB(0,1)| < \infty.$$

Consequently f belongs to $W^{1,2}\big(B(0,1)\big)$ and using (1.3) we see that f has a $W^{1,2}\big(B(0,2)\big)$-extension f^*. In general, f^* need not be continuous but if $\partial fB(0,1)$ is a Jordan curve or if f is Hölder continuous, then f^* is continuous.

The behavior of f^* on $\partial B(0,1)$ has been studied a lot. We consider two results.

4.1. Theorem [NP]. *If $f : B(0,1) \to \mathbb{R}^2$ is a Hölder continuous conformal mapping, then $|f^* \partial B(0,1)| = 0$.*

4.2. Theorem [JM]. *If $f : B(0,1) \to \mathbb{R}^2$ is a conformal mapping, then there is a set $A \subset \partial B(0,1)$ such that the Hausdorff dimension of A is zero and $|f^*\big(\partial B(0,1) \setminus A\big)| = 0$.*

The proof of Theorem 4.1 is based on the modulus method and Theorem 4.2 follows from careful estimates for harmonic measure. These estimates also provide a proof for Theorem 4.1, see [JM].

Now Theorem 3.4 and Theorem 3.6 together with the extension method (1.3) show that Theorems 4.1 and 4.2 hold for arbitrary $W^{1,2}$-mappings. These results have immediate extensions to higher dimensional euclidean spaces as well as to analytic functions with a finite Dirichlet integral, see [MM].

References

[BI] B. Bojarski and T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 257–324.

[C] L. Cesari, *Sulle transformazioni continue*, Ann. Mat. Pura Appl. 21 (1941), 157–188.

[G] F. W. Gehring, *Rings and quasiconformal mappings in space*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 98–105.

[H] P. Hajłasz, *Change of variables formula under minimal assumptions*, Colloq. Math. 114 (1993), 93–101.

[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, Oxford 1993.

[JM] P. Jones and N. G. Makarov, *Density properties of harmonic measure*, preprint.

[M] J. Malý, *Hölder type quasicontinuity*, to appear.

[MM] J. Malý and O. Martio, *Lusin's condition (N) and mappings of the class* $W^{1,n}$, to appear.

[MZ] O. Martio and W. P. Ziemer, *Lusin's condition (N) and mappings with non-negative Jacobians*, Michigan Math. J., to appear.

[Mu] S. Müller, *Higher integrability of determinants and weak convergence in* L^1, J. Reine Angew. Math. 412 (1990), 20–34.

[NP] R. Näkki and B. Palka, *Boundary angles, cusps and conformal mappings*, Complex Variables Theory Appl. 5 (1986), 165–180.

[P] S. P. Ponomarev, *Examples of homeomorphisms in the class* $ACTL^p$ *which do not satisfy the absolute continuity condition of Banach*, Dokl. Akad. Nauk SSSR 201 (1971), 1053–1054 (in Russian).

[RR] T. Radó and P. V. Reichelderfer, *Continuous Transformations in Analysis*, Springer 1955.

[Rei] H. M. Reimann, *On the absolute continuity of a surface representation*, Comment. Math. Helv. 46 (1971), 44–47.

[Res1] Yu. G. Reshetnyak, *The condition (N) for* $W^1_{n,\mathrm{loc}}$ *space mappings*, Sibirsk. Mat. Zh. 28 (1987), 149–153 (in Russian).

[Res2] —, *Space Mappings with Bounded Distortion*, Transl. Math. Monographs 73, Amer. Math. Soc., 1989.

[S] S. Saks, *Theory of the Integral*, Warsaw, 1937.

[V] J. Väisälä, *Quasiconformal maps and positive boundary measure*, Analysis 9 (1989), 205–216.