GEOMETRY IN NONLINEAR CONTROL
AND DIFFERENTIAL INCLUSIONS
BANACH CENTER PUBLICATIONS, VOLUME 32
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

SOLVABLE OPTIMAL CONTROL OF BROWNIAN MOTION IN SYMMETRIC SPACES AND SPHERICAL POLYNOMIALS

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1. Introduction. During the past three decades or so, a significant amount of research on stochastic control has been performed. However, only a relatively few examples of explicitly solvable stochastic control problems are available. The most notable example is the linear regular problem (e.g., [15]) though some other examples exist (e.g., [1, 2, 17, 21]). In recent years a family of nonlinear geometric examples has been given [6–13] that use spherical polynomials in a basic role. These examples are the control of a natural diffusion process in a family of well known manifolds with nonzero curvature. The spherical polynomials are eigenfunctions of the Laplace–Beltrami operator. These examples were motivated by the general theory that was developed in [4, 5].

To provide some insight into these solvable examples, an example in real hyperbolic three space is described [6, 11]. Real hyperbolic three space $\mathbb{H}^3(\mathbb{R})$ is probably the simplest three dimensional noncompact, irreducible Riemannian manifold that has nonzero curvature. A natural geometric model for $\mathbb{H}^3(\mathbb{R})$ is the unit ball

$$B_1(0) = \{y \in \mathbb{R}^3 : |y| < 1\} \tag{1.1}$$

where $|\cdot|$ is the usual Riemannian metric on $\mathbb{R}^3$. The space $B_1(0)$ with the Riemannian metric

$$ds^2 = 4(1 - |y|^2)^{-2}(dy_1^2 + dy_2^2 + dy_3^2) \tag{1.2}$$

is a complete Riemannian manifold with constant sectional curvature -1. More importantly for the approach here $\mathbb{H}^3(\mathbb{R})$ is a noncompact symmetric space, that

1991 *Mathematics Subject Classification*: Primary 93E20; Secondary 33C35.

Research partially supported by NSF Grant DMS-9305936.

The paper is in final form and no version of it will be published elsewhere.

is

$$\mathbb{H}^3(\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2) \tag{1.3}$$

where $\mathrm{SL}(2,\mathbb{C})$ is the Lie group of 2×2 matrices over $\mathbb{C}$ with determinant 1 and $\mathrm{SU}(2)$ is the (maximal compact) subgroup of unitary matrices in $\mathrm{SL}(2,\mathbb{C})$.

The global geodesic polar coordinates for $\mathbb{H}^3(\mathbb{R})$ at the origin, denoted by 0, are a useful coordinate system, that is, the map

$$\mathrm{Exp}_0 Y \mapsto (r,\theta_1,\theta_2)$$

where $Y \in T_0\mathbb{H}^3(\mathbb{R})$, Exp_0 is the exponential map at 0, $r = |Y|_0$ with $|\cdot|_0$ the Riemannian metric at 0 and (θ_1,θ_2) are some coordinates of the unit vector $Y/|Y|_0$. The Riemannian structure in this coordinate system is

$$ds^2 = dr^2 + (\sinh^2 r)d\sigma^2 \tag{1.4}$$

where $d\sigma^2$ is the usual Riemannian structure on the unit sphere in $T_0\mathbb{H}^3(\mathbb{R})$. The Laplace–Beltrami operator $\Delta_{\mathbb{H}^3(\mathbb{R})}$ in these coordinates is

$$\Delta_{\mathbb{H}^3(\mathbb{R})} = \frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \sinh^{-2} r\Delta_{S^2} \tag{1.5}$$

where Δ_{S^2} is the Laplace–Beltrami operator on the unit sphere in $T_0\mathbb{H}^3(\mathbb{R})$. The geodesic polar coordinates for $\mathbb{H}^3(\mathbb{R})$, and the Riemannian structure and the Laplace–Beltrami operator in this coordinate system are described in [19].

The stochastic control problem is the control of Brownian motion by a drift vector field so that this controlled diffusion process remains close to the origin. The cost functional has a term for the state being away from the origin that is an increasing function of the radial distance from the origin and a term for the use of control. This cost functional is

$$J(U) = E_{|Y(0)|_0}\int_0^T a\sinh^2\frac{|Y(t)|_0}{2} + \left(\cosh^2\frac{|Y(t)|_0}{2}\right)U^2(t)\,dt \tag{1.6}$$

where $(Y(t),\ t\ge 0)$ is the controlled diffusion in $\mathbb{H}^3(\mathbb{R})$ with the infinitesimal generator

$$\frac{1}{2}\Delta_{\mathbb{H}^3(\mathbb{R})} + u\frac{\partial}{\partial r}. \tag{1.7}$$

Since the cost functional only depends on $|Y(t)|_0$ and the control is only in the radial direction, it suffices to consider the radial part of this process $(X(t),\ t\ge 0)$ where $X(t) = |Y(t)|_0$. The process $(X(t),\ t\in[0,T])$ has the infinitesimal generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + u\frac{\partial}{\partial r} \tag{1.8}$$

and satisfies the stochastic differential equation

$$dX(t) = (\coth X(t) + U(t))dt + dB(t) \qquad X(0) = |Y(0)|_0 \tag{1.9}$$

where $(B(t),\ t \geq 0)$ is a real-valued standard Brownian motion. An admissible control at time t is a Borel measurable function of $X(t)$ such that the stochastic differential equation (1.9) has a unique strong solution.

The Hamilton–Jacobi equation for the stochastic control problem (1.6–1.7) of diffusion type is well known (e.g., [15]) to be

$$0 = \frac{\partial W}{\partial s} + \min_{v \in \mathbb{R}} \left[\frac{1}{2} \frac{\partial^2 W}{\partial r^2} + \coth r \frac{\partial W}{\partial r} + v \frac{\partial W}{\partial r} + a \sinh^2 \frac{r}{2} + v^2 \cosh^2 \frac{r}{2} \right] \tag{1.10}$$

with the boundary condition

$$W(s, r) = 0, \qquad (s, r) \in \{T\} \times \mathbb{H}^3(\mathbb{R}).$$

A solution to (1.10) is

$$W(s, r) = g(s) \sinh^2 \frac{r}{2} + h(s) \tag{1.11}$$

where

$$g' + \frac{3}{2} g - \frac{1}{4} g^2 + a = 0, \qquad g(T) = 0, \tag{1.12}$$

$$h' + \frac{3}{4} g = 0, \qquad h(T) = 0. \tag{1.13}$$

It is not difficult to verify that (1.11) gives the admissible optimal control

$$U^*(s, y) = -\frac{1}{2} g(s) \tanh |y|_0 \tag{1.14}$$

where $s \in [0, T]$ and $y \in T_0\mathbb{H}^3(\mathbb{R})$. Since the terms of the stochastic differential equation with the control (1.14) are locally smooth it is only necessary to verify that the solution does not hit the origin to show that there is a unique strong solution. This verification is made by comparison with the stochastic differential equation for the so-called two dimensional Bessel process.

Since it is elementary that

$$\sinh^2 \frac{r}{2} = \frac{1}{2}(\cosh r - 1)$$

it follows easily that $f(r) = \sinh^2 \frac{r}{2}$ is an eigenfunction for the radial part, $\widetilde{\Delta}_{\mathbb{H}^3(\mathbb{R})}$, of $\Delta_{\mathbb{H}^3(\mathbb{R})}$ where

$$\widetilde{\Delta}_{\mathbb{H}^3(\mathbb{R})} = \frac{d^2}{dr^2} + 2 \coth r \frac{d}{dr}.$$

A basic aspect of the solvability of this stochastic control problem is that $\sinh^2 \frac{r}{2}$ is an eigenfunction of the radial part of the Laplace–Beltrami operator. These eigenfunctions are called spherical functions because in this case they are constant on spheres in $B_1(0)$.

This approach to stochastic control problems is not anomalous because it can be used to solve the scalar linear regulator problem.

2. Solvable stochastic control problems in noncompact symmetric spaces of rank one. To generalize the example in the symmetric space $\mathbb{H}^3(\mathbb{R})$ given in the Introduction two directions are followed. First, other symmetric spaces are considered and second other spherical functions are used. Since $\mathbb{H}^3(\mathbb{R})$ is an irreducible noncompact symmetric space of rank one, other such spaces are considered. In analogy with (1.3), a noncompact symmetric space can be described as G/K where G is a noncompact semisimple Lie group with finite center and K is a maximal compact subgroup of G [18]. The rank of G/K is the maximal dimension of a flat, totally geodesic submanifold. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the direct sum (or Cartan) decomposition of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{k}$ of K and its orthogonal complement with respect to the Killing form of $\mathfrak{g}$. For rank one symmetric spaces there is a restricted root $\alpha \in \Sigma$, the set of roots, such that 2α is the only other possible element in Σ. There is a one dimensional abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha} \tag{2.1}$$

where $\mathfrak{p}_\alpha$ and $\mathfrak{p}_{2\alpha}$ are the eigenspaces associated with α and 2α respectively. Define the integers p and q by the following equations

$$p = \dim \mathfrak{p}_\alpha, \tag{2.2}$$

$$q = \dim \mathfrak{p}_{2\alpha}. \tag{2.3}$$

E. Cartan [3] determined the values of $\mathfrak{p}$ and $\mathfrak{q}$ for rank one symmetric spaces (e.g., p. 532 [18]). There are three families of hyperbolic spaces from the real numbers, the complex numbers and the quaternions and one hyperbolic space from an exceptional Lie algebra associated with the Cayley numbers, that is,

$$\begin{aligned} &\mathbb{H}^n(\mathbb{R}), \quad p = n-1 \quad \text{and} \quad q = 0 \text{ for } n = 2, 3, \ldots, \\ &\mathbb{H}^n(\mathbb{C}), \quad p = n-2 \quad \text{and} \quad q = 1 \text{ for } n = 4, 6, \ldots, \\ &\mathbb{H}^n(\mathbb{H}), \quad p = n-4 \quad \text{and} \quad q = 3 \text{ for } n = 8, 12, \ldots \end{aligned}$$

and

$$\mathbb{H}^{16}(\mathrm{Cay}), \quad p = 8 \quad \text{and} \quad q = 7.$$

Let $(\theta_1, \ldots, \theta_{n-1})$ be Cartesian coordinates on an open subset of the unit sphere S^{n-1} in T_pG/K. The inverse of the mapping

$$(\theta_1, \ldots, \theta_{n-1}, r) \mapsto \mathrm{Exp}_p(r\theta_1, \ldots, r\theta_{n-1})$$

is a system of geodesic polar coordinates at $p \in G/K$ where Exp_p is the exponential map at p. The Laplace–Beltrami operator in these coordinates is

$$\Delta_{G/K} = \frac{\partial^2}{\partial r^2} + (\gamma p \coth \gamma r + 2\gamma q \coth 2\gamma r)\frac{\partial}{\partial r} + \Delta_{S^{n-1}} \tag{2.4}$$

where $\gamma = (2p+8q)^{-1/2}$, p and q are given in (2.2, 2.3) and $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on S^{n-1} in T_pG/K. The radial part $\widetilde{\Delta}_{G/K}$ of the Laplace–

Beltrami operator is

$$\widetilde{\Delta}_{G/K} = \frac{\partial^2}{\partial r^2} + (\gamma p \coth \gamma r + 2\gamma q \coth 2\gamma r)\frac{\partial}{\partial r}. \tag{2.5}$$

More details about these hyperbolic spaces and their Laplace–Beltrami operators can be found in [19].

To determine the spherical functions consider the eigenvalue problem (p. 302 [16])

$$\widetilde{\Delta}_{G/K}\varphi_\lambda = \lambda_\Delta \varphi_\lambda \tag{2.6}$$

where $\lambda_\Delta = -(\langle l,l\rangle + \langle \varrho,\varrho\rangle)$, $\langle\cdot,\cdot\rangle$ is the Killing form and ϱ is one half of the sum of the positive restricted roots with their multiplicities. This eigenvalue problem reduces to the differential equation

$$z(z-1)\frac{d^2\varphi_\lambda}{dz^2} + [(a+b+1)z - c]\frac{d\varphi_\lambda}{dz} + ab\varphi_\lambda = 0 \tag{2.7}$$

where $z = -\sinh^2\gamma r$, $\langle l,l\rangle + \langle\varrho,\varrho\rangle = \frac{1}{2}(p+4q)^{-1}[\lambda(H_0)^2 + \varrho(H_0)^2]$, $\alpha(H_0) = 1$, $\langle H_0, H_0\rangle = 2(p+4q)$, $a = \frac{1}{2}[p+2q+2i\lambda(H_0)]$, $b = \frac{1}{2}[p+2q-2i\lambda(H_0)]$ and $c = \frac{1}{2}(p+q+1)$. A power series solution u_1 that is regular at the origin is the hypergeometric function F, that is,

$$u_1(z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!} = F(a,b,c,z) \tag{2.8}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}. \tag{2.9}$$

and Γ is the gamma function. If a (or b) is a negative integer $-m$ then the hypergeometric function is a polynomial over $\mathbb{R}$ expressed as

$$F(-m,b,c,z) = \sum_{n=0}^{m} \frac{(-m)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{2.10}$$

and each term in the polynomial is positive for $z < 0$.

Consider the controlled diffusion process $(Y(t),\ t \ge 0)$ with the infinitesimal generator

$$\frac{1}{2}\Delta_{G/K} + u\frac{\partial}{\partial r} \tag{2.11}$$

and the cost functional

$$J_m(U) = E_{|Y(0)|_0}\int_0^T [F(-m,b,c,-\sinh^2\gamma|Y(t)|_0) + f(|Y(t)|_0)U^2(t)]dt \tag{2.12}$$

where

$$f(x) = \frac{[F_x(-m,b,c,-\sinh^2\gamma x)]^2}{F(-m,b,c,-\sinh^2\gamma x)},$$
$$F_x(-m,b,c,-\sinh^2\gamma x) = \frac{\partial}{\partial x} F(-m,b,c,-\sinh^2\gamma x), \tag{2.13}$$

and $0 < T \le T_1$ ([9]).

An admissible control at time t is a Borel measurable function of $X(t) = |Y(t)|_0$ such that there is a unique strong solution of the stochastic differential equation

$$dX(t) = \left[\frac{1}{2}(\gamma p \coth \gamma X(t) + 2\gamma q \coth 2\gamma X(t)) + U(t, X(t))\right] dt + dB(t),$$
$$X(0) = |Y(0)|, \tag{2.14}$$

where $(B(t),\, t \ge 0)$ is a real-valued standard Brownian motion.

The above stochastic control problem is explicitly solvable [9].

THEOREM 2.1. *Let the dimension of* G/K *be at least three. For each* $m \in \mathbb{N}$ *the stochastic control problem* (2.11, 2.12) *has an optimal control* U^* *that in geodesic polar coordinates at the origin is*

$$U^*(t,x) = \frac{-1}{2} \frac{F(-m,b,c,-\sinh^2\gamma x)}{F_x(-m,b,c,-\sinh^2\gamma x)} g(t) \tag{2.15}$$

where g *is the unique positive solution of the Riccati equation*

$$g' - \frac{1}{2}(p+4q)^{-1}\left[\lambda(H_0)^2 + \left(\frac{1}{2}p + q\right)^2\right] g - \frac{1}{2}g^2 + 1 = 0,$$
$$g(T) = 0. \tag{2.16}$$

For the simplest nontrivial symmetric space, the real hyperbolic plane $\mathbb{H}^2(\mathbb{R})$, solvable stochastic control problems can be formulated and solved in analogy with the higher dimensional symmetric spaces. An unusual property in this case is that the controlled diffusion can hit the origin which is a singularity for the stochastic differential equation but nevertheless there is a unique strong solution for the optimal system [10].

3. Solvable stochastic control problems in compact symmetric spaces of rank one. Since many models of physical phenomena are described in compact spaces, it is natural to investigate stochastic control problems in compact symmetric spaces. While compact symmetric spaces are simpler in some respects than noncompact symmetric spaces, there are other aspects that make these compact spaces more complicated for stochastic control than their noncompact counterparts, e.g., typically the lack of global triviality, the existence of conjugate points and spherical polynomials lack a global monotonicity.

If M is a compact symmetric space of rank one then $M \simeq G/K$ where G is the identity component of the group of isometries of M and K is the isotropy subgroup of G at $0 \in M$ called the origin. Let L be the diameter of M, that is, the maximal distance between any two points. If $x \in M$ then let A_x be the set of points of M that are a distance L from x. A_x is a submanifold of M that is called the antipodal manifold associated with x.

The following is a complete list of the irreducible compact symmetric spaces of rank one and their corresponding antipodal manifolds (p. 167 [19]): i) spheres S^n for $n = 1, 2, \dots$ and A_0 is a point, ii) real projective spaces $\mathbb{P}^n(\mathbb{R})$ for $n = 2, 3, \dots$ and $A_0 = \mathbb{P}^{n-1}(\mathbb{R})$, iii) complex projective spaces $\mathbb{P}^n(\mathbb{C})$ for $n = 4, 6, \dots$ and $A_0 = \mathbb{P}^{n-2}(\mathbb{C})$, iv) quaternion projective spaces $\mathbb{P}^n(\mathbb{H})$ for $n = 8, 12, \dots$ and $A_0 = \mathbb{P}^{n-4}(\mathbb{H})$ and v) the Cayley plane $\mathbb{P}^{16}(\mathrm{Cay})$ and $A_0 = S^8$. The values of p and q in (2.2, 2.3) for these compact spaces are the same as their dual hyperbolic spaces. In the Killing form metric the diameter of a compact symmetric space of rank one L satisfies the following equation

$$L^2 = p\frac{\pi^2}{2} + 2q\pi^2. \tag{3.1}$$

Let $(\theta_1, \dots, \theta_{n-1})$ be Cartesian coordinates on an open subset of the unit sphere $S_1(0)$ in T_xM where M is a compact symmetric space of rank one. The mapping $\mathrm{Exp}_x : T_xM \to M$ is a diffeomorphism of the ball $B_L(0) = \{y \in T_xM : |y| < L\}$ onto the open set $M \setminus A_x$. The inverse of the mapping

$$(\theta_1, \dots, \theta_{n-1}, r) \mapsto \mathrm{Exp}_x(r\theta_1, \dots, r\theta_n)$$

is a system of geodesic polar coordinates at $x \in M$ where $r \in (0, L)$, $r = |Y|$ and $Y \in B_L(0)$.

In analogy (or duality) to (2.4) the Laplace–Beltrami operator for a compact symmetric space of rank one is

$$\varDelta_M = \frac{\partial^2}{\partial r^2} + (p\gamma \cot(\gamma r) + 2q\gamma \cot 2\gamma r)\frac{\partial}{\partial r} + \varDelta_{S_r} \tag{3.2}$$

where $\varDelta_{S_r}$ is the Laplace–Beltrami operator on $S_r(0)$, the sphere in M with center 0 and radius r. The radial part of $\varDelta_M$ is

$$\widetilde{\varDelta}_M = \frac{\partial^2}{\partial r^2} + (p\gamma \cot \gamma r + 2q\gamma \cot 2\gamma r)\frac{\partial}{\partial r}. \tag{3.3}$$

Let G be given by the following equality

$$G(m, r) = F\left(\frac{1}{2}p + q + m, -m, \frac{1}{2}(p + q + 1), \sin^2 \gamma r\right)$$

where F is the hypergeometric function (2.8) and $m \in \mathbb{Z}^+$. There is a maximal interval $[0, \delta]$ where $G(m, \cdot)$ is strictly decreasing for $r \in [0, \delta]$. Choose $c > 0$ such that

$$\sqrt{c}L \le \delta \tag{3.4}$$

where L is the diameter of the compact symmetric space of rank one. Let $k_0(m,c)$ be chosen such that

$$G(m,\sqrt{c}L)+k_0(m,c)=0. \tag{3.5}$$

Define $\widetilde{G}$ by the equation

$$\widetilde{G}(m,r)=G(m,r)+k_0(m,c) \tag{3.6}$$

The controlled diffusion $(Y(t),\, t\geq 0)$ has the infinitesimal generator

$$\frac{1}{2}c\Delta_M+u\frac{\partial}{\partial r} \tag{3.7}$$

and the cost functional is for suitable $T>0$ [8]

$$J_m(U)=E_{|Y(0)|}\int_0^T(-\widetilde{G}(m,|Y(t)|)+h(m,|Y(t)|)U^2(t))dt \tag{3.8}$$

where

$$h(m,r)=\frac{[\widetilde{G}_r(m,r)]^2}{\widetilde{G}(m,r)}. \tag{3.9}$$

Let $X(t)=|Y(t)|$ be the radial part of the controlled diffusion. The process $(X(t),\, t\geq 0)$ satisfies

$$\begin{aligned} dX(t)&=\left[\frac{c}{2}\gamma p\cot\gamma X(t)+c\gamma q\ \cot 2\gamma X(t)+U(t)\right]dt+\sqrt{c}dB(t),\\ X(0)&=|Y(0)|, \end{aligned} \tag{3.10}$$

where $(B(t),\, t\geq 0)$ is a real-valued standard Brownian motion. An admissible control at time t is a Borel measurable function of $X(t)$ such that (3.10) has one and only one strong solution.

The solution to the stochastic control problem (3.7, 3.8) is given in the following theorem [7, 8].

THEOREM 3.1. *The stochastic control problem described by* (3.7, 3.8) *has an optimal control* U^* *that in geodesic polar coordinates at the origin is*

$$U^*(s,r)=-\frac{1}{2}\frac{\widetilde{G}(m,r)}{\widetilde{G}_r(m,r)}g(s) \tag{3.11}$$

where g *is the unique positive solution of*

$$g'+\frac{c}{2}\gamma^2(4m^2+2mp+4mq)g-\frac{1}{4}g^2-1=0,\qquad g(T)=0, \tag{3.12}$$

and U^* *is extended by continuity to be zero on the antipodal manifold.*

4. Solvable stochastic control problems in noncompact symmetric spaces of higher rank. The stochastic control problems in Section 2 are generalized to all classical noncompact symmetric spaces of arbitrary rank. It is important to describe the Laplace–Beltrami operator in "good" coordinates so that the eigenvalue problem for spherical functions can be solved to give a fairly explicit construction of the spherical polynomials. These coordinates arise quite naturally from the rank one case and some other geometrical considerations.

Initially the symmetric cones are described [14]. Let $X = \mathbb{R}^n$ be a real Euclidean space with inner product $\langle\cdot,\cdot\rangle$. An open proper cone $L \subset X$ with vertex at 0 is called self-dual if its closure $\overline{\Lambda}$ satisfies

$$\overline{\Lambda} = \{x \in X : \langle x, y\rangle \geq 0 \text{ for all } y \in \overline{\Lambda}\}. \tag{4.1}$$

A self-dual cone Λ is called symmetric if its linear automorphism group

$$G_\Lambda = \{g \in \mathrm{GL}(X) : g(\Lambda) = \Lambda\} \tag{4.2}$$

acts transitively on Λ. Let $e \in \Lambda$ be the unit element of X that is considered as a Jordan algebra. The stabilizer subgroup is $K_\Lambda := \{g \in G_\Lambda : g(e) = e\}$.

The matrix cones are one family of these symmetric cones. Let $\mathbb{K}$ be one of the division algebras: $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. For an integer $r \geq 1$ let

$$\Lambda_{\mathbb{K}} := H_r^+(\mathbb{K})$$

denote the set of all positive definite, self-adjoint, $(r \times r)$-matrices with entries in $\mathbb{K}$. Then $\Lambda = \Lambda_K$ is a symmetric cone with $G_\Lambda = \mathrm{GL}_r(\mathbb{K})$ acting on Λ as $(g, x) \mapsto gxg^*$ for $x \in \Lambda$ and $g \in G_\Lambda$. The stabilizer subgroup K_Λ is $U_r(\mathbb{K})$, the unitary matrices over $\mathbb{K}$.

Another family of examples are the forward light cones. For $n \geq 3$ define Λ_n as

$$\Lambda_n = \{x \in \mathbb{R}^n : x_1 > (x_2^2 + \ldots + x_n^2)^{1/2}\}$$

Λ_n is a symmetric cone with $G_\Lambda = \mathbb{R}_+ \times \mathrm{SO}_0(n-1, 1)$ and $K_\Lambda = \mathrm{SO}(n-1)$.

Consider the positive Weyl chamber, $\mathfrak{a}_\Lambda^+$,

$$\mathfrak{a}_\Lambda^+ = \left\{ \sum t_i H_i^\Lambda : t_1 > \ldots > t_r \right\} \tag{4.3}$$

where (H_i^Λ) is a suitable basis [13] of the abelian subspace of the Lie algebra and r is the rank. Consider the associated coordinates $x_1 > \ldots > x_r > 0$ defined by

$$x_i = \exp(t_i) \tag{4.4}$$

for $i = 1, \ldots, r$. The radial part $\widetilde{\Delta}_\Lambda$ of the Laplace–Beltrami operator on Λ is

$$\begin{aligned} \widetilde{\Delta}_\Lambda &= \sum_{i=1}^r \left(x_i^2 \frac{\partial^2}{\partial x_i^2} + \left(1 - \frac{d}{2}(r-1)\right) x_i \frac{\partial}{\partial x_i} \right) + d \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^r \left(\frac{\partial^2}{\partial t_i^2} - \frac{d}{2}(r-1) \frac{\partial}{\partial t_i} \right) + d \sum_{i \neq j} \frac{1}{1 - \exp(t_j - t_i)} \frac{\partial}{\partial t_i} \end{aligned} \tag{4.5}$$

where $d \in \mathbb{Z}^+$ is determined by Λ [13].

Let $\mathbf{m} = (m_1, \ldots, m_r)$ be a partition, that is, $m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$. A spherical polynomial $\widetilde{\varphi}_{\mathbf{m}}^\Lambda$ is a symmetric polynomial in the coordinates $(x_1, \ldots, x_r)$

so it can be expressed as

$$\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}(x) = \sum_{\mu \le \mathbf{m}} c_{\mathbf{m}}^{\mu} q_{\mu}(x) \tag{4.6}$$

where the sum is over all the partitions $\mu \le \mathbf{m}$ in the reverse lexicographic ordering [20], the $c_{\mathbf{m}}^{\mu}$ are coefficients and q_μ is the monomial symmetric function associated with μ that is obtained by summing over all distinct permutations of $\mu_1, \ldots, \mu_r$. Using the definition of the monomial symmetric functions it follows that

$$\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}(x) = \sum a_{\mathbf{m}}(\mathbf{i}) x_1^{i_1} \ldots x_r^{i_r}$$

where $\mathbf{i} = (i_1, \ldots, i_r)$ and $a_{\mathbf{m}}(i) = a_{\mathbf{m}}(\mathbf{j})$ if $\mathbf{j}$ is a distinct permutation of $\mathbf{i}$. For $j \in \{1, \ldots, r\}$ define $a_{\mathbf{m}}^j(\mathbf{i})$ as

$$a_{\mathbf{m}}^j(\mathbf{i}) = \frac{i_j}{|\mathbf{m}|} a_{\mathbf{m}}(\mathbf{i})$$

and

$$\widetilde{\varphi}_{\mathbf{m}j}^{\Lambda} = \sum a_{\mathbf{m}}^j(\mathbf{i}) x_1^{i_1} \ldots x_r^{i_r} \tag{4.7}$$

where the sum is over distinct r-tuples such that $a_{\mathbf{m}}^j(\mathbf{i}) \ne 0$. It follows easily that

$$\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}(x) = \sum_{j=1}^{r} \widetilde{\varphi}_{\mathbf{m}j}^{\Lambda}(x).$$

The spherical polynomials for the real matrix cones first developed in the statistical literature in multivariate analysis. The following result [12, 13] is important for the use of $\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}$ in a stochastic control problem because it provides a monotonicity of $\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}$ along rays so that $\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}$ can be used in a cost functional.

Proposition 4.1. *The coefficients $c_{\mathbf{m}}^{\mu}$ of the spherical polynomial $\widetilde{\varphi}_{\mathbf{m}}^{\Lambda}$ in* (4.5) *are nonnegative.*

The controlled diffusion has the infinitesimal generator

$$\frac{1}{2}\Delta_\Lambda + \sum_{j=1}^{r} u_j(s, \mathbf{t}) \frac{\partial}{\partial t_j} \tag{4.8}$$

where $\mathbf{t} = (t_1, \ldots, t_r)$ are the coordinates (4.3) and Δ_Λ is the Laplace–Beltrami operator. The cost functional for the stochastic control problem for the partition $\mathbf{m}$ is

$$J_{\mathbf{m}}^{\Lambda}(U) = E_{X(0)} \int_0^T \widehat{\varphi}_{\mathbf{m}}^{\Lambda}(X(s)) + \sum_{j=1}^{r} f_j(X(s)) U_j^2(s) ds \tag{4.9}$$

where

$$\widehat{\varphi}_{\mathbf{m}}^{\Lambda}(\mathbf{t}) = \widehat{\varphi}_{\mathbf{m}}^{\Lambda}(e^{t_1}, \ldots, e^{t_r}), \tag{4.10}$$

$$\widehat{\varphi}_{\mathbf{m}j}^{\Lambda}(\mathbf{t}) = \widehat{\varphi}_{\mathbf{m}j}^{\Lambda}(e^{t_1}, \ldots, e^{t_r}), \tag{4.11}$$

$$f_j(\mathbf{t}) = \frac{[D_j \widehat{\varphi}^{\Lambda}_{\mathbf{m}}(\mathbf{t})]^2}{\widehat{\varphi}^{\Lambda}_{\mathbf{m}j}(\mathbf{t})}, \tag{4.12}$$

$$D_j = \frac{\partial}{\partial t_j} \tag{4.13}$$

and $(X(t),\ t \in [0,T])$ is the radial part of the controlled diffusion with the infinitesimal generator (4.8). The process $(X(t),\ t \in [0,T])$ satisfies the family of stochastic differential equations

$$\begin{aligned} &dX_k(s) \\ &= \Big[-\frac{d}{4}(r-1) + \frac{d}{2}\sum_{j\neq k} \frac{1}{1-\exp(X_j(s)-X_k(s))} + U_k(s, X(s))\Big] ds + dB_k(s) \end{aligned} \tag{4.14}$$

for $k = 1, \ldots, r$, $X(0) = (X_1(0), \ldots, X_r(0))$, $X_1(0) > \ldots > X_r(0)$, and $(B_1(s), \ldots$ $\ldots, B_r(s); s \geq 0)$ is a standard r-dimensional Brownian motion.

An admissible control at time t is a Borel measurable function of $X(t)$ such that the family of stochastic differential equations has one and only one strong solution.

The solution of the stochastic control problem (4.8, 4.9) is described in the following result [13].

Theorem 4.2. *The stochastic control problem described by* (4.8, 4.9) *has an optimal control*

$$U^*_j(s, \mathbf{t}) = \frac{-\widehat{\varphi}^{\Lambda}_{\mathbf{m}j}(\mathbf{t})}{2D_j \widehat{\varphi}^{\Lambda}_{\mathbf{m}}(\mathbf{t})} g(s) \tag{4.15}$$

where $j = 1, \ldots, r$, $s \in [0,T]$, $\mathbf{t} \in \mathfrak{a}^+_{\Lambda}$, $D_j = \partial/\partial t_j$, g *is the unique positive solution of*

$$g' - \frac{1}{2}\chi^{\Lambda}_{\mathbf{m}} g - \frac{1}{4} g^2 + 1 = 0, \qquad g(T) = 0, \tag{4.16}$$

and $\chi^{\Lambda}_{\mathbf{m}} = \sum m_i(m_i + \frac{d}{2}(r+1-2i))$ *is the eigenvalue of* $\widehat{\varphi}^{\Lambda}_{\mathbf{m}}$ *for* $\widetilde{\Delta}_{\Lambda}$.

The other family of noncompact symmetric spaces are called symmetric balls [14]. Let $Z \simeq \mathbb{C}^n$ be a complex vector space with a norm $|\cdot|$. Let

$$\Omega = \{z \in Z : |z| < 1\} \tag{4.17}$$

be the open unit ball in Z. Ω is called symmetric if its holomorphic automorphism group

$$G_{\Omega} = \{g : \Omega \to \Omega \text{ biholomorphic}\} \tag{4.18}$$

acts transitively on Ω. Since Ω is a bounded domain, G_{Ω} is a finite dimensional real Lie group. Let $0 \in \Omega$ be the origin. The stabilizer subgroup of G_{Ω} is

$$K_{\Omega} = \{g \in G_{\Omega} : g(0) = 0\}. \tag{4.19}$$

One family of examples of symmetric balls is the hyperbolic matrix balls. Let $p, q \geq 1$ and define

$$\Omega_{p,q} = \{z \in \mathbb{C}^{p\times q} : \mathrm{Spec}(zz^*) < 1\}$$

that is the open unit ball of complex $(p \times q)$ matrices with respect to the operator norm. Then $\Omega = \Omega_{p,q}$ is a symmetric ball where $G_\Omega = \mathrm{SU}(p, q)/\text{center}$ acting by Moebius transformations and $K_\Omega = S(U(p) \times U(q))$ acting by $z \mapsto uzv^*$ for $z \in \Omega$, $u \in U(p)$, $v \in U(q)$ and $\det(u)\det(v) = 1$.

Another family of examples is the Lie balls. For $n \geq 3$ define Ω_n as

$$\Omega_n = \{z \in \mathbb{C}^n : z\cdot \overline{z} < 1 + |z\cdot \overline{z}|^2 < 2\}$$

where $z \cdot w$ is the dot product.

Consider the coordinates $t_1 > \ldots > t_r > 0$ on the positive Weyl chamber $\mathfrak{a}_\Omega^+$ and the associated coordinates $x_1 > \ldots > x_r > 0$ defined by $x_i = \sinh^2 t_i$ for $= 1, \ldots, r$. The radial part $\widetilde{\Delta}_\Omega$ of the Laplace–Beltrami operator Δ_Ω [13] is

$$\begin{aligned}
(4.20)\quad \widetilde{\Delta}_\Omega = 4\Bigg\{ &\sum_{i=1}^{r}\left(x_i(1+x_i)\frac{\partial^2}{\partial x_i^2} + \left(\frac{1+b}{2} + c + (1+c+b)x_i\right)\frac{\partial}{\partial x_i}\right)\\
&+ d\sum_{i\neq j}\frac{x_i(1+x_i)}{x_i - x_j}\frac{\partial}{\partial x_i}\Bigg\}\\
= &\sum_{i=1}^{r}\left(\frac{\partial^2}{\partial t_i^2} + (2c\coth(2t_i) + b\coth(t_i))\frac{\partial}{\partial t_i}\right)\\
&+ 2d\sum_{i\neq j}\frac{\sinh(t_i)\cosh(t_i)}{\sinh^2(t_i) - \sinh^2(t_j)}\frac{\partial}{\partial t_i}
\end{aligned}$$

where b, c and d are integers that are determined from the symmetric ball.

Let $\widehat{\varphi}_{\mathbf{m}}^{\Omega}(x)$ be the spherical polynomial for the partition $\mathbf{m}$ in the coordinates $x_i = \sinh^2(t_i)$. It can be expressed as

$$\varphi_{\mathbf{m}}^{\Omega}(x) = \sum_{\mu\leq \mathbf{m}} c_m^\mu q_\mu(x) \tag{4.21}$$

where q_μ is the monomial symmetric function for the partition μ and $c_{\mathbf{m}}^\mu \geq 0$. Using the definition of the monomial symmetric functions it follows that

$$\widetilde{\varphi}_{\mathbf{m}}^{\Omega}(x) = \sum a_{\mathbf{m}}(i) x_1^{i_1}\ldots x_r^{i_r}$$

where $i = (i_1, \ldots, i_r)$ and the sum is over distinct r-tuples such that $a_m(\mathbf{i}) \neq 0$. For $j \in \{1, \ldots, r\}$ define $a_{\mathbf{m}}^j(\mathbf{i})$ as

$$a_{\mathbf{m}}^j(\mathbf{i}) = \frac{i_j}{\sum_{k=1}^r i_k} a_{\mathbf{m}}(\mathbf{i})$$

and

$$\widetilde{\varphi}_{\mathbf{m}j}^{\Omega} = \sum a_{\mathbf{m}}^j(\mathbf{i}) x_1^{i_1}\ldots x_r^{i_r} \tag{4.22}$$

and the sum is over distinct r-tuples such that $a^j_{\mathbf{m}}(\mathbf{i}) \neq 0$. It easily follows that

$$\widetilde{\varphi}^{\Omega}_{\mathbf{m}}(x) = \sum_{j=1}^{r} \widetilde{\varphi}^{\Omega}_{\mathbf{m}j}(x) + c^{\Omega}_{\mathbf{m}}$$

where $c^{\Omega}_{\mathbf{m}}$ is the constant term in $\widetilde{\varphi}^{\Omega}_{\mathbf{m}}$. These spherical polynomials can be constructed in an explicit way using the spherical polynomials for the symmetric cones [13].

The controlled diffusion has the infinitesimal generator

$$\frac{1}{2}\Delta_{\Omega} + \sum_{j=1}^{r} u_j(s,\mathbf{t})\frac{\partial}{\partial t_j} \tag{4.23}$$

where $\mathbf{t} = (t_1, \ldots, t_r)$ and Δ_{Ω} is the Laplace–Beltrami operator on Ω.

The cost functional for the stochastic control problem for the partition $\mathbf{m}$ is

$$J^{\Omega}_{\mathbf{m}}(U) = E_{X(0)} \int_0^T \widehat{\varphi}^{\Omega}_{\mathbf{m}}(X(s)) + \sum_{j=1}^{r} f_j(X(s))U^2(s)ds \tag{4.24}$$

where

$$\widehat{\varphi}^{\Omega}_{\mathbf{m}}(\mathbf{t}) = \widetilde{\varphi}^{\Omega}_{\mathbf{m}}(\sinh^2 t_1, \ldots, \sinh^2 t_r), \tag{4.25}$$

$$\widehat{\varphi}^{\Omega}_{\mathbf{m}j}(\mathbf{t}) = \widetilde{\varphi}^{\Omega}_{\mathbf{m}j}(\sinh^2 t_1, \ldots, \sinh^2 t_j), \tag{4.26}$$

$$f_j(\mathbf{t}) = \frac{[D_j\widehat{\varphi}^{\Omega}_{\mathbf{m}}(\mathbf{t})]^2}{\widehat{\varphi}^{\Omega}_{\mathbf{m}j}(\mathbf{t})}, \tag{4.27}$$

$$D_j = \frac{\partial}{\partial t_j}, \tag{4.28}$$

and $(X(t),\ t \in [0,T])$ is the radial part of the controlled diffusion with the infinitesimal generator (4.23). The process $(X(t),\ t \in [0,T])$ satisfies the family of stochastic differential equations

$$\begin{aligned} dX_k(s) = \Big[& 2c\coth 2X_k(s) + b\coth X_k(s) \\ & + 2d\sum_{k\neq j} \frac{\sinh X_k(s)\cosh X_k(s)}{\sinh^2 X_k(s) - \sinh^2 X_j(s)} + U_k(s, X(s))\Big] ds + dB_k(s) \end{aligned} \tag{4.29}$$

for $k = 1, \ldots, r$, $X(0) = (X_1(0), \ldots, X_r(0)), X_1(0) > \ldots > X_r(0) > 0$, $X(t) \in \mathfrak{a}^+_{\Omega}$ and $(B_1(s), \ldots, B_r(s);\ s \geq 0)$ is a standard r-dimensional Brownian motion.

An admissible control at time t is a Borel measurable function of $X(t)$ such that the family of stochastic differential equations (4.29) has one and only one strong solution.

The solution of the stochastic control problem (4.23, 4.24) is described in the following result [13].

THEOREM 4.3. *The stochastic control problem described by* (4.23, 4.24) *has an optimal control*

$$U_j^*(s,\mathbf{t}) = \frac{-\widehat{\varphi}^{\Omega}_{\mathbf{m}j}(\mathbf{t})}{2D_j\widehat{\varphi}^{\Omega}_{\mathbf{m}}(\mathbf{t})}g(s) \tag{4.30}$$

where $j = 1,\ldots,r$, $s \in [0,T]$, $\mathbf{t} \in \mathfrak{a}^+_\Omega$, $D_j = \partial/\partial t_j$ *and* g *is the unique positive solution of*

$$g' + \frac{1}{2}\chi^{\Omega}_{\mathbf{m}} g - \frac{1}{4}g^2 + 1 = 0, \qquad g(T) = 0, \tag{4.31}$$

where $\chi^{\Omega}_{\mathbf{m}} = \sum_i m_i(m_i + d(r-i) + 2c + b)$ *is the eigenvalue of* $\widetilde{\varphi}^{\Omega}_{\mathbf{m}}$ *for* $\widetilde{\Delta}_\Omega$.

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