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ISOMETRIC IMBEDDINGS OF EUCLIDEAN SPACES INTO FINITE DIMENSIONAL l_p-SPACES

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Abstract. It is shown that l_2^n imbeds isometrically into $l_4^{n^2+1}$ provided that n is a prime power plus one, in the complex case. This and similar imbeddings are constructed using elementary techniques from number theory, combinatorics and coding theory. The imbeddings are related to existence of certain cubature formulas in numerical analysis.

1. General facts on imbeddings of l_2^n into l_p^N. As usual, l_p^N denotes $\mathbb{K}^N$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, equipped with the p-norm

$$\|x\|_p = \Big(\sum_{j=1}^{N} |x_j|^p\Big)^{1/p}, \quad x = (x_j)_{j=1}^N \in \mathbb{K}^N.$$

Dvoretzky's theorem, in the case of l_p-spaces, states that for any $\varepsilon > 0$ there is $c_\varepsilon \geq 1$ such that for any $n, N \in \mathbb{N}$ with $N \geq c_\varepsilon n$ if $p \leq 2$ and $N \geq c_\varepsilon n^{p/2}$ if $p > 2$, there is a subspace $Y_n \subset l_p^N$ of dimension $(Y_n) = n$ such that

$$d(Y_n, l_2^n) := \inf\{\|T\|\,\|T^{-1}\| \mid T : Y_n \longrightarrow l_2^n \text{ linear isomorphism}\} \leq 1 + \varepsilon.$$

I.e. for $p > 2$, l_2^n imbeds $(1+\varepsilon)$-isomorphically into l_p^N where $N = c_\varepsilon n^{p/2}$, cf. [FLM]. We study the case of $\varepsilon = 0$, i.e. the question of *isometric* imbeddings l_2^n into l_p^N for a given n and p with $N = N(n,p)$. We indicate isometric imbeddings by $l_2^n \hookrightarrow l_p^N$. They only exist if $p = 2k \in 2\mathbb{N}$ is an even integer, cf. [L] (probably a kind of folklore result). In this case of $p = 2k$ such isometric imbeddings exist and

$$N(n,k) := \min\{N \mid l_2^n \hookrightarrow l_{2k}^N \text{ imbeds isometrically}\}$$

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is finite. We would like to estimate $N(n,k)$ and find concrete imbeddings. The recent book by Reznick [R] and papers of Lyubich-Vaserstein [LV] and Seidel [S] are concerned with this problem, and together with the paper of Goethals-Seidel [GS] on spherical designs motivated this paper.

The formula

$$(x_1^2+x_2^2)^2 = \frac{8}{9}\left[x_1^4 + \left(\frac{\sqrt{3}x_2 - x_1}{2}\right)^4 + \left(\frac{\sqrt{3}x_2 + x_1}{2}\right)^4\right],\ x=(x_1,x_2)\in\mathbb{R}^2$$

yields an isometric imbedding $l_2^2 \hookrightarrow l_4^3$: the l_4-unit sphere in $\mathbb{R}^3$ contains a circular section. The formula

$$\Big(\sum_{i=1}^{4} x_i^2\Big)^2 = \frac{1}{6}\sum_{1\le i<j\le 4}[(x_i+x_j)^4 + (x_i-x_j)^4], \quad x=(x_i)\in\mathbb{R}^4,$$

giving an isometric imbedding $l_2^4 \hookrightarrow l_4^{12}$, was classically used in the context of Waring's problem, cf. [M]. We start with a characterization of imbeddings $l_2^n \hookrightarrow l_{2k}^N$. By $S^{n-1}\subset\mathbb{K}^n$ we denote the $(n-1)$-sphere in $\mathbb{R}^n$ or $\mathbb{C}^n$ and by $d\sigma$ the (rotation invariant, normalized) Haar measure on S^{n-1}. Let $\mathfrak{P}_{2k,n}^{hom}$ denote the space of homogeneous polynomials of degree $2k$ in n variables; in the complex case this means all polynomials $q(z_1,\dots,z_n,\bar z_1,\dots,\bar z_n)$ which are homogeneous of degree k in each set of variables $(z_1,\dots z_n)$ and $(\bar z_1,\dots \bar z_n)$. Further, $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{K}^n$.

Proposition 1. *Let $n,k,N\in\mathbb{N}$. The following are equivalent:*

(1) *There is an isometric imbedding $l_2^n\hookrightarrow l_{2k}^N$.*

(2) *There exist N points $x_1,\dots,x_N \in S^{n-1}$ and a probability sequence $(\mu_s)_{s=1}^N\subset\mathbb{R}^+$ $(\sum_{s=1}^{N}\mu_s=1)$ such that for all polynomials $p\in\mathfrak{P}_{2k,n}^{\rm hom}$*

$$\sum_{s=1}^{N}\mu_s p(x_s) = \int_{S^{n-1}} p(y)\ d\sigma(y). \tag{1.1}$$

(3) *There exist N points $x_1,\dots,x_N\in S^{n-1}$ and a probability sequence $(\mu_s)_{s=1}^N$ such that*

$$\sum_{s,t=1}^{N}\mu_s\mu_t\big|\langle x_s,x_t\rangle\big|^{2k} = \int_{S^{n-1}}\int_{S^{n-1}}\big|\langle x,y\rangle\big|^{2k}d\sigma(x)d\sigma(y) =: c_{nk}. \tag{1.2}$$

Since for $0\le l<k$ and $q\in\mathfrak{P}_{2l,n}^{\rm hom}$ the polynomial p defined by $p(x)=q(x)\langle x,x\rangle^{k-1}$ is in $\mathfrak{P}_{2k,n}^{\rm hom}$ with $q|_{S^{n-1}}=p|_{S^{n-1}}$, we have for $\mathbb{K}=\mathbb{R}$ that (1.1) holds for all *even* polynomials of degree $\le 2k$, provided (1)–(3) are true. If the points $(-x_1,\dots,-x_N)$ are added and the μ's divided by 2, formula (1.1) holds for all polynomials of degree $\le 2k+1$. Thus these points and weights constitute a *cubature formula* of degree $2k+1$ in n variables on $S^{n-1}\subset\mathbb{R}^n$. Concrete

imbeddings $l_2^n \hookrightarrow l_{2k}^N$ are thus equivalent to symmetric cubature formulas on S^{n-1}. For equal weights this is called a *spherical design*, cf. [GS].

The equivalence of (1) and (2) is in [R] and [LV], the equivalence of (2) and (3) follows from results of [GS]. We use the equivalence of (1) and (3) later to construct imbeddings $l_2^n \hookrightarrow l_{2k}^N$; if we have points (x_s) and weights (μ_s) with (1.2), the imbedding is given by $x \mapsto \big((\mu_s/c_{nk})^{1/2k}\langle x, x_s\rangle\big)_{s=1}^N$. To start, here is a direct elementary proof of Proposition 1.

Proof. (2) ⇒ (1). Let $x \in \mathbb{K}^n$ be fixed. Applying (2) to $p(y) = \big|\langle x, y\rangle\big|^{2k}$, we find, using rotation invariance of σ,

$$\sum_{s=1}^N \mu_s \big|\langle x, x_s\rangle\big|^{2k} = \int\limits_{S^{n-1}} \big|\langle x, y\rangle\big|^{2k} d\sigma(y) = \|x\|_2^{2k} \int\limits_{S^{n-1}} |y_1|^{2k} d\sigma(y) = c_{nk}\|x\|_2^{2k}.$$

Thus $x \mapsto \big((\mu_s/c_{nk})^{1/2k}\langle x, x_s\rangle\big)_{s=1}^N$ yields $l_2^n \hookrightarrow l_{2k}^N$.

(1) ⇒ (3). Any isometric imbedding $l_2^n \hookrightarrow l_{2k}^N$ has the form $x \mapsto \big(\langle x, z_s\rangle\big)_{s=1}^N$ with $z_s \in \mathbb{K}^n$. Define

$$x_s := z_s/\|z_s\|_2,\ \ \mu_s := \|z_s\|_2^{2k}\Big/\sum_{t=1}^N \|z_t\|_2^{2k}: \quad \sum_{s=1}^N \mu_s = 1.$$

Let m denote the Haar measure on the orthogonal (unitary in the complex case) group $O(n)$ which by $U \in O(n) \mapsto Ue \in S^{n-1}$ for fixed $e \in S^{n-1}$ induces σ on S^{n-1}. Using the assumption and the $O(n)$-invariance of $\|\cdot\|_2$, we find for $x \in \mathbb{K}^n$ and $U \in O(n)$

$$\begin{aligned}
\Big(\sum_{t=1}^N \|z_t\|^{2k}\Big)\Big(\sum_{s=1}^N \mu_s \big|\langle x, x_s\rangle\big|^{2k}\Big) &= \sum_{s=1}^N \big|\langle x, z_s\rangle\big|^{2k} \\
= \langle x, x\rangle^k = \langle U^* x, U^* x\rangle^k &= \sum_{s=1}^N \big|\langle x, Uz_s\rangle\big|^{2k} \\
= \sum_{s=1}^N \int\limits_{O(n)} \big|\langle x, Uz_s\rangle\big|^{2k} dm(U) &= \sum_{s=1}^N \|z_s\|^{2k} \int\limits_{S^{n-1}} \big|\langle x, y\rangle\big|^{2k} d\sigma(y) \\
&= \Big(\sum_{s=1}^N \|z_s\|^{2k}\Big) \cdot c_{nk} \cdot \langle x, x\rangle^k.
\end{aligned}$$

This yields for $x = x_t$

$$c_{nk} = c_{nk}\langle x_t, x_t\rangle^k = \sum_{s=1}^N \mu_s \big|\langle x_t, x_s\rangle\big|^{2k}$$

indepedently of $t \in \{1, \ldots, N\}$. Multiply by μ_t and sum over t to get (3).

(3) ⇒ (2). For $x \in \mathbb{K}^n$, $l \in \mathbb{N}$, let $x^{\otimes l} = x \otimes \ldots \otimes x \in \mathbb{K}^{n^l}$ denote the l-fold tensor product. Then $\langle x^{\otimes l}, y^{\otimes l}\rangle = \langle x, y\rangle^l$ for $x, y \in \mathbb{K}^n$. As in [GS] or [KT], consider

$$\xi := \sum_{s=1}^{N} \mu_s x_s^{\otimes k} \otimes \bar{x}_s^{\otimes k} - \int\limits_{S^{n-1}} x^{\otimes k} \otimes \bar{x}^{\otimes k} d\sigma(x) \in \mathbb{K}^{n^{2k}}.$$

The rotation invariance of σ yields after an elementary calculation

$$0 \le \langle \xi, \xi\rangle = \sum_{s,t=1}^{N} \mu_s \mu_t \big|\langle x_s, x_t\rangle\big|^{2k} - c_{nk}.$$

By assumption, the right hand side is zero. Thus $\xi = 0$ in $\mathbb{K}^{n^{2k}}$. This means, written in coordinates, that all monomials of degree $2k$ in $\mathfrak{P}_{2k,n}^{\mathrm{hom}}$ are integrated exactly by the cubature formula (x_s, μ_s), i.e. (1.1) holds for all monomials and thus for all $p \in \mathfrak{P}_{2k,n}^{\mathrm{hom}}$. ■

Corollary 1. *If $l_2^n \hookrightarrow l_{2k}^N$, also $l_2^n \hookrightarrow l_{2l}^N$ for $1 \le l \le k$.*

Proof. As noted before, if (1.1) holds for $p \in \mathfrak{P}_{2k,n}^{\mathrm{hom}}$, it also holds for $p \in \mathfrak{P}_{2l,n}^{\mathrm{hom}}$, $1 \le l \le k$. ■

Corollary 2. [R, LV]. *For $\mathbb{K} = \mathbb{R}$, $l_2^2 \hookrightarrow l_{2k}^{k+1}$.*

Proof. Take $x_s := \exp(\pi i(s + 1/2)/(k + 1)) \in \mathbb{C} = \mathbb{R}^2$ for $s = 0, \ldots, k$ and $\mu_s = (k + 1)^{-1}$. Then (3) holds, since for any $t = 0, \ldots, k$

$$\frac{1}{k+1} \sum_{s=0}^{k} \big|\langle x_t, x_s\rangle\big|^{2k} = \frac{1}{k+1} \sum_{j=0}^{k} \cos\Big(\frac{\pi j}{k+1}\Big)^{2k}$$
$$\stackrel{(*)}{=} \frac{1}{2\pi} \int_0^{2\pi} (\cos x)^{2k} dx = \int\limits_{S^1} \big|\langle x, e_1\rangle\big|^{2k} d\sigma(x) = c_{2k}.$$

Here $(*)$ is true since all trigonometric polynomials of degree $2k$ are integrated exactly by a coresponding formula; this is easily checked for the exponentials e^{-ilx}, $|l| \le 2k$. ■

Corollary 3. *If $l_2^n(\mathbb{C}) \hookrightarrow l_{2k}^N(\mathbb{C})$, then $l_2^{2n}(\mathbb{R}) \hookrightarrow l_{2k}^{N(k+1)}(\mathbb{R})$.*

Proof. $l_2^{2n}(\mathbb{R}) \equiv l_2^n(\mathbb{C}) \hookrightarrow l_{2k}^N(\mathbb{C}) \equiv l_{2k}^N(l_2^2(\mathbb{R}))$
$\hookrightarrow l_{2k}^N(l_{2k}^{k+1}(\mathbb{R})) = l_{2k}^{N(k+1)}(\mathbb{R})$. ■

Corollary 3 is useful in the context of the cubature formula, since points $(x_s) \subseteq \mathbb{C}^n$, (μ_s) with (3) can often be constructed using *complex* exponentials (see below) which then, by using Corollary 3 and the explicit construction of Corollary 2, can be translated into explicit cubature formulae for $S^{2n-1}(\mathbb{R})$.

2. Estimates on the dimension number $N(n,k)$**.** Since $\dim \mathfrak{P}^{\mathrm{hom}}_{2k,n} \sim n^{2k}$, Proposition 1 would seem to indicate that $N \sim n^{2k}$ points are needed for cubature formula (1.1) and thus for any imbedding $l_2^n \hookrightarrow l_{2k}^N$. On the other hand, Dvoretzky's theorem, by putting naively $\varepsilon = 0$ for $p = 2k$, would suggest that only $N \sim n^{p/2} = n^k$ are needed. These orders are, in fact, upper and lower bounds for $N(n,k)$:

PROPOSITION 2. *For any* $2 \le n,k \in \mathbb{N}$, $L(n,k) \le N(n,k) \le U(n,k)$, *where*

$$L(n,k) := \begin{cases} \binom{n+k-1}{k}, & \mathbb{K} = \mathbb{R} \\ \binom{n+[\frac{k+1}{2}]-1}{[\frac{k+1}{2}]}\binom{n+[\frac{k}{2}]-1}{[\frac{k}{2}]}, & \mathbb{K} = \mathbb{C}, \end{cases}$$

$$U(n,k) := \begin{cases} \binom{n+2k-1}{2k}, & \mathbb{K} = \mathbb{R} \\ \binom{n+k-1}{k}^2, & \mathbb{K} = \mathbb{C}. \end{cases}$$

Clearly, $L(n,k) \sim n^k$, $U(n,k) \sim n^{2k}$, up to constants depending on k. In the real case, this can be found in [LV] and [R]; the upper bound is classical, see [M].

P r o o f f o r $\mathbb{K} = \mathbb{R}$. Hilbert's formula for $x \in R^n$

$$\int\limits_{S^{n-1}} \left|\langle x,y\rangle\right|^{2k} d\sigma(y) = c_{nk}\|x\|_2^{2k}$$

shows that $\|\cdot\|^{2k}$ is in the (closed) convex hull of the polynomials $\{\langle\cdot,y\rangle^{2k} \bigm| y \in S^{n-1}\}$ in the positive cone of $\mathfrak{P}^{\mathrm{hom}}_{2k,n}$. By Carathéodory's theorem, $N(n,k)$ is thus bounded from above by $\dim \mathfrak{P}^{\mathrm{hom}}_{2k,n} = \binom{n+2k-1}{2k}$.

As for the lower bound, if N were $< \binom{n+k-1}{k} = \dim \mathfrak{P}^{\mathrm{hom}}_{k,n}$, for any given set $(x_s)_{s=1}^N \subset S^{n-1}$ there would be a non-zero $p \in \mathfrak{P}^{\mathrm{hom}}_{k,n}$ with $p(x_s) = 0$ for all $s \in \{1,\ldots,N\}$. Then $p^2 \in \mathfrak{P}^{\mathrm{hom}}_{2k,n}$ and for any probability sequence (μ_s)

$$\sum_{s=1}^N \mu_s p(x_s)^2 = 0 \ne \int\limits_{S^{n-1}} p(y)^2 d\sigma(y).$$

Hence (2) and thus (1) of Proposition 1 is violated. ∎

We show below that for $k = 2$, i.e. imbeddings into l_4, the lower bound gives the right order of growth (n^2), solving a problem on the last page of Reznick's book [R]. Thus the lower bound seems to be more interesting one to investigate. We need some specific polynomials for this purpose: given $\alpha,\beta > -1$, the Jacobi polynomials $P_k^{(\alpha,\beta)}$ are the k-th order orthogonal polynomials on $(-1,1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, normalized by $P_k^{(\alpha,\beta)}(1) = 1$. For

$n \geq 2$ and $k \in \mathbb{N}$, define the k-th order polynomial $C_{n,k}$ by

$$C_{n,k}(x) = \begin{cases} P_k^{(\frac{n-1}{2},\frac{n-1}{2})}(x), & \mathbb{K} = \mathbb{R}, \\ P_{k/2}^{(n-1,0)}(2x^2-1), & \mathbb{K} = \mathbb{C}, \quad k \text{ even} \\ xP_{(k-1)/2}^{(n-1,1)}(2x^2-1), & \mathbb{K} = \mathbb{C}, \quad k \text{ odd.} \end{cases}$$

The relevance of these polynomials here comes from the fact that they are related to positive define functions on S^{n-1} via the addition formula for spherical harmonics.

Proposition 3. *Let $2 \leq n, k \in \mathbb{N}$ and assume that $L(n,k) = N(n,k) =: N$. Then there exist $(x_s)_{s=1}^N \subset S^{n-1}$ such that* (1.1), (1.2) *hold with $\mu_s = 1/N$, and for any $1 \leq s \neq t \leq N$, the number $|\langle x_s, x_t\rangle|$ is a zero of the polynomial $C_{n,k}$. For $k = 2$, the coincidence*

$$N = L(n,2) = N(n,2) = \begin{cases} n(n+1)/2 & (\mathbb{R}) \\ n^2 & (\mathbb{C}) \end{cases}$$

is equivalent to the existence of N ***"equiangular"*** *lines/points $(x_s)_{s=1}^N$ with*

$$|\langle x_s, x_t\rangle| = \begin{cases} 1/\sqrt{n+2} & (\mathbb{R}) \\ 1/\sqrt{n+1} & (\mathbb{C}), \end{cases}$$

which is the maximal possible number.

We do not give the proof here; it can be given by modifying the techniques of Delsarte-Goethals-Seidel [DGS] in the real case, who estimate the number of points of spherical designs (where the μ_s are all equal but also odd order polynomials are integrated exactly). In the complex case we also refer to Bannai [B] and Hoggar [Ho]. It is known that such configurations do not exist if $k > 5$, $n > 2$. However, some examples of such systems are known which satisfy (3) of Proposition 1 with $\mu_s = 1/N$. In particular, using equiangular points one gets imbeddings with best possible $N = N(n,2)$:

$$\mathbb{R}: \quad l_2^2 \hookrightarrow l_4^3,\ l_2^3 \hookrightarrow l_4^6, l_2^7 \hookrightarrow l_4^{28},\ l_2^{23} \hookrightarrow l_4^{276}.$$
$$\mathbb{C}: \quad l_2^2 \hookrightarrow l_4^4,\ l_2^3 \hookrightarrow l_4^9,\ l_2^8 \hookrightarrow l_4^{64}.$$

Further imbeddings from "system with few angles" are

$$\mathbb{R}: \quad l_2^2 \hookrightarrow l_6^4,\ l_2^8 \hookrightarrow l_6^{120},\ l_2^{23} \hookrightarrow l_6^{2300},\ l_2^2 \hookrightarrow l_{10}^6,\ l_2^{24} \hookrightarrow l_{10}^{98280}.$$
$$\mathbb{C}: \quad l_2^2 \hookrightarrow l_6^6,\ l_2^4 \hookrightarrow l_6^{40},\ l_2^6 \hookrightarrow l_6^{126}.$$

The real case imbeddings were given already in [R], [LV], the example $l_2^{24} \hookrightarrow l_{10}^{98280}$ related to the Leech lattice being quite spectacular. To find imbeddings with $N(n,2) \approx L(n,2)$, we look for an almost maximal number of almost equiangular vectors in $\mathbb{K}^n$. The main result of this paper is

PROPOSITION 4.

(a) *Let q be a prime power and $n = q + 1$. Then there exists an imbedding $l_2^n \hookrightarrow l_4^{n^2+1}$ over the complex numbers.*

(b) *Let $n = q$ be an odd prime power. Then there exists an imbedding $l_2^n \hookrightarrow l_4^{n^2+n}$ over the complex numbers which can be given explicitly in terms of exponential vectors.*

(c) *Let $n = 4^m$, $m \in \mathbb{N}$. Then the Kerdock code yields an isometric imbedding $l_2^n \hookrightarrow l_4^{n(n+2)/2}$ over the real numbers.*

For the proof of (a) we use a classical result of Singer on B_2-sequences, cf. [HR]:

LEMMA. *For q and n as in* (a) *there exist integers $0 \le d_1 < \ldots < d_n < M$, $M := q^2 + q + 1 = n^2 - n + 1$ such that all numbers from 1 to $M - 1$ show up as residues* mod M *of the differences $d_i - d_j$ $(i \ne j)$ exactly once,*

$$\{(d_i - d_j)(M) \mid i \ne j\} = \{1, \ldots, M - 1\}.$$

Proof of Proposition 4.

(a) Take $d_1 < \ldots < d_n$ as in the lemma and define

$$x_s := \frac{1}{\sqrt{n}}\Big(\exp\big(\frac{2\pi i}{M} d_j s\big)\Big)_{j=1}^{n} \in S^{n-1}(\mathbb{C}),\ s = 1, \ldots, M,$$
$$x_s := e_{s-M} \text{ (unit vectors)},\ s = M+1, \ldots, N := n^2 + 1.$$

For $1 \le s \ne t \le M$, the vectors are equiangular, $|\langle x_s, x_t\rangle| = \sqrt{n-1}/n$ $(\Theta := s - t)$:

$$n^2 |\langle x_s, x_t\rangle|^2 = \sum_{j,k=1}^{n} \exp\Big(\frac{2\pi i}{M}(d_j - d_k)\Theta\Big)$$
$$= \Big(\sum_{j=k} + \sum_{j\ne k}\Big) \exp\Big(\frac{2\pi i}{M}(d_j - d_k)\Theta\Big) = n + \sum_{l=1}^{M-1} \exp\Big(\frac{2\pi i}{M} l\Theta\Big) = n - 1.$$

For $1 \le s \le M$, let $\mu_s = \mu^{(1)} = \frac{n}{n+1}\frac{1}{M}$. For $M < s \le N$, let $\mu_s = \mu^{(2)} = \frac{n}{n+1}\frac{1}{n^2}$. Then $\sum_{s=1}^{N} \mu_s = 1$ and for $1 \le t \le M$,

$$\sum_{s=1}^{N} \mu_s |\langle x_t, x_s\rangle|^4 = \mu^{(1)}\Big(1 + (M-1)\frac{(n-1)^2}{n^4}\Big) + \mu^{(2)}\Big(n \cdot \frac{1}{n^2}\Big) = \frac{2}{n(n+1)},$$

and for $M < t \le N$,

$$\sum_{s=1}^{N} \mu_s |\langle x_t, x_s\rangle|^4 = \mu^{(1)}\Big(M\frac{1}{n^2}\Big) + \mu^{(2)} = \frac{2}{n(n+1)},$$

so that

$$\sum_{s,t=1}^{N} \mu_s\mu_t |\langle x_t, x_s\rangle|^4 = \frac{2}{n(n+1)} = c_{n,2}(\mathbb{C}) = \int_{S^{n-1}(\mathbb{C})} |y_1|^4 d\sigma(y).$$

The last fact $c_{n,2} = \frac{2}{n(n+1)}$ can be checked by a direct calculation using polar coordinates.

(b) Let n be an odd prime power. Identify $s = (s_1, s_2) \in \{1, \ldots, n\}^2$ with $s \in \{1, \ldots, n^2\}$ and define

$$\begin{aligned} x_s &:= \frac{1}{\sqrt{n}} \Big(\exp\Big(\frac{2\pi i}{n}(s_1 j + s_2 j^2)\Big)\Big)_{j=1}^n \in S^{n-1}(\mathbb{C}), \ s = 1, \ldots, n^2 \\ x_s &:= e_{s-n^2}, \qquad\qquad n^2 < s \le n^2 + n =: N. \end{aligned}$$

For $s = (s_1, s_2),\ t = (t_1, t_2) \in \{1, \ldots, n\}^2,\ \Theta_1 := s_1 - t_1,\ \Theta_2 := s_2 - t_2 \neq 0$

$$\begin{aligned} n^2 \big|\langle x_s, x_t\rangle\big|^2 &= \sum_{j,k=1}^n \exp\Big(\frac{2\pi i}{n}\{\Theta_1(j-k) + \Theta_2(j^2 - k^2)\}\Big) \\ &= \sum_{j,l=1}^n \exp\Big(\frac{2\pi i}{n}\{\Theta_1 l + \Theta_2 l(2j - l)\}\Big) \\ &= \sum_{j,l=1}^n \exp\Big(\frac{2\pi i}{n}\{\Theta_2(l - l_0(j))^2 - \Theta_2(j - j_0)^2\}\Big) \\ &= \Big|\sum_{l=1}^n \exp\Big(\frac{2\pi i}{n}\Theta_2 l^2\Big)\Big|^2 \end{aligned}$$

where $l_0(j) := j - s_1/(2s_2)$ and $j_0 := s_1/(2s_2)$ are calculated in the field $\mathbb{F}_n$ with n elements. By well-known facts on Gaussian sums for *odd* numbers n, cf. [H], the latter square equals n. Thus

$$\big|\langle x_s, x_t\rangle\big| = \begin{cases} 1 & s = t \\ 0 & s_2 = t_2,\ s_1 \neq t_1 \\ 1/\sqrt{n} & \text{else} \end{cases} \qquad s, t \le n^2.$$

This time, we let $\mu_s = \frac{1}{N}$ for all $1 \le s \le N$. Similarly as above, one finds again

$$\sum_{s,t=1}^N \mu_s \mu_t \big|\langle x_s, x_t\rangle\big|^4 = \frac{2}{n(n+1)} = c_{n,2}(\mathbb{C}).$$

The resulting isometric imbedding $l_2^n \hookrightarrow l_4^N(\mathbb{C})$ is given explicitly by

$$e_j \mapsto \Big(\frac{1}{\sqrt{n}} \exp\Big(\frac{2\pi i}{n}(s_1 j + s_2 j^2)\Big)_{s_1, s_2 = 1, \ldots, n}, e_j \Big),$$

up to a homothetic factor.

(c) Here we use the vectors of the Kerdock code [MS], identifying opposite vectors and replacing the zeros in the code words $(0, 1, \ldots)$ by -1's, thus obtaining vectors $x_s = \frac{1}{\sqrt{n}}(\pm 1, \ldots, \pm 1) \in \mathbb{R}^n$, $n = 4^m$. The code has n^2 code words of minimal distance $d = \frac{n - \sqrt{n}}{2}$; we thus get $n^2/2$ vectors x_s with $\big|\langle x_s, x_t\rangle\big| \le \big|\frac{n-2d}{n}\big| = \frac{1}{\sqrt{n}}$, $s \neq t$. Actually, for a fixed s, and $t \neq s$, the value $\frac{1}{\sqrt{n}}$ is attained

$\frac{n^2}{2} - n$ times; the value 0 occurs $(n-1)$ times, cf. [MS]. Again we add the unit vectors $e_1, \ldots, e_n$ to these points in $S^{n-1}(\mathbb{R})$ to find $N = n(n+2)/2$ points x_s such that with $\mu_s = \frac{1}{N}$

$$\sum_{s,t=1}^{N} \mu_s \mu_t \big|\langle x_s, x_t\rangle\big|^4 = \frac{3}{n(n+2)} = c_{n,2}(\mathbb{R}) = \int\limits_{S^{n-1}} |y_1|^4 d\sigma(y)$$

by a similar calculation as before.

Remark. From $l_2^n \hookrightarrow l_4^{n^2+1}(\mathbb{C})$ one gets $l_2^m \hookrightarrow l_4^{3/4m^2+3}(\mathbb{R})$ by Corollary 3, where $m = 2q+2$, $q =$ prime power. This yields a cubature formula of degree 5 on S^{m-1} with $\frac{3}{2}m^2 + 6$ points. Integrating the radius by Gaussian quadrature with 3 points, one also gets cubature formulas of degree 5 on the full unit ball $B_m \subset \mathbb{R}^m$ with $\frac{9}{2}m^2 + 18$ points. The Kerdock code in $\mathbb{R}^n$, $n = 4^m$, yields formulas on $S^{n-1}(B_n)$ of degree 5 with $n(n+2)$ $(3n(n+2))$ points.

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