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SOME FUNCTORIAL PROPERTIES OF MICROLOCALIZATION FOR $\mathcal{D}$-MODULES

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Introduction. In several papers which appeared since the seventies, M. Kashiwara, T. Kawai and T. Oshima generalized the classical theory of differential equations with regular singularities in the framework of microlocal analysis and the theory of $\mathcal{D}$-modules.

These developments were parallel to the progress of the so-called microlocal theory of sheaves whose main tools are the specialization and microlocalization functors.

In this paper we deal with Kashiwara's notion of specialization and microlocalization for (not necessarily holonomic) $\mathcal{D}$-modules along a fixed submanifold Y.

In Theorem 1.1.1 we obtain the relation between specialization and nearby cycles for $\mathcal{D}$-modules using the normal deformation along Y, which is the analogue of Verdier's relation in sheaf theory. As a consequence, we give a precise meaning to the famous comparison theorems due to Kashiwara (in the eighties) for regular systems along Y (cf. [3], [13]).

We study the behaviour of microlocalization for $\mathcal{D}$-modules under formal tensor product (defined by Sato, Kawai, Kashiwara in [15]).

As a consequence we prove that the microlocalization of a $\mathcal{D}$-module M along Y only depends on the microdifferential system $\widetilde{M}$ obtained from M after tensoring by the sheaf of microdiferential operators. When the $\mathcal{D}$-module M is regular along Y we obtain a relation for the microcharacteristic varieties associated to $\Lambda = T^*_Y X$ (the conormal fiber bundle to Y).

Finally, we state some interesting properties of the bifunctor $\underline{\mu\mathrm{hom}}$ which was first introduced by Kashiwara-Kawai (cf. [6]) in the framework of regular holonomic systems, and generalized in [20] to a larger category, and we analyse some examples.

1. Microlocalization for $\mathcal{D}$-modules and $\mathcal{E}$-modules along a submanifold

1.1. *$\mathcal{D}$-modules.* Let X be an n-dimensional complex analytic manifold and $Y \subset X$

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a smooth d-codimensional submanifold. Let $\mathcal{D}_X$ be the sheaf of linear holomorphic differential operators of finite order on X, and

$$V_Y^k(\mathcal{D}_X) = \{P \in \mathcal{D}_X : PI^j \subset I^{j+k}, \forall j, j+k \geq 0\},$$

the so-called *V-filtration* on $\mathcal{D}_X$ with respect to Y, where I denotes the defining ideal of Y. Except in case of ambiguity we suppress Y. Let

$$\mathrm{gr}_V(\mathcal{D}_X) := \bigoplus_{k\in\mathbb{Z}} \frac{V^k(\mathcal{D}_X)}{V^{k+1}(\mathcal{D}_X)}$$

and $\tau : T_Y X \to Y$ the projection of the normal bundle on Y. Then $\tau_*\mathcal{D}_{[T_Y X]} \approx \mathrm{gr}_V(\mathcal{D}_X)$, where $\mathcal{D}_{[T_Y X]}$ denotes the algebraic linear differential operators on $T_Y X$ with respect to the fibers of τ.

Let θ be the Euler operator on $T_Y X$ (i.e. the vector field on $T_Y X$ describing the infinitesimal action of $\mathbb{C}^*$ on $T_Y X$). θ acts by the identity on I/I^2 and we shall keep the notation θ for any of its local representatives in $V^0(\mathcal{D}_X)$, which will be fixed henceforth. For example, if

$$Y = \{(x,t) \in \mathbb{C}^{n+d} : t \in \mathbb{C}^d,\ \ t = 0\}, \qquad t = (t_1, \ldots, t_d),$$

we may choose

$$\theta = \sum_{i=1}^{d} t_i \frac{\partial}{\partial t_i}.$$

Moreover, we shall often write "$\mathcal{D}$-module" instead of "$\mathcal{D}_X$-module".

Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. Then, locally, $\mathcal{M}$ admits a good V-filtration $\{\mathcal{M}^j\}_{j\in\mathbb{Z}}$, i.e.

(a) $\mathcal{M} = \bigcup_{j\in\mathbb{Z}} \mathcal{M}^j$;
(b) $\mathcal{M}^j \supset \mathcal{M}^{j+1}$, $\ \forall j$;
(c) $V^k(\mathcal{D}_X)\mathcal{M}^j \subset \mathcal{M}^{j+k}\ \ \forall j, k, \in \mathbb{Z}$;
(d) $V^k(\mathcal{D}_X)\mathcal{M}^j = \mathcal{M}^{j+k}\ \ \ \forall j \gg 0$ and $k \geq 0$ or $j \ll 0$ and $k \leq 0$;
(e) $\mathcal{M}^j$ is $V^0(\mathcal{D}_X)$-coherent.

We shall say that $\mathcal{M}$ is *specializable along* Y if for every good V-filtration $U^k(\mathcal{M})$ on $\mathcal{M}$ there exists a nontrivial polynomial $b(s) \in \mathbb{C}[s]$ such that

$$b(\theta - k)U^k(\mathcal{M}) \subset U^{k+1}(\mathcal{M});$$

such a $b(s)$ is also called a *Bernstein-Sato polynomial* for $U^*(\mathcal{M})$.

Let us endow $\mathbb{C}$ with the lexicographic order.

For every local nonvanishing section u of $\mathcal{M}$ there is a minimal polynomial $b_u(s)$, the *b-function* for u, such that $b_u(\theta)u \in V^1(\mathcal{D}_X)u$ and one sets order $u = \{\alpha \in \mathbb{C} : b(\alpha) = 0\}$.

The *canonical V-filtration on* $\mathcal{M}$ is defined as follows: for $\alpha \in \mathbb{C}$, $V^\alpha(\mathcal{M})$ is the sheaf of germs of sections u such that $\mathrm{order}\, u \subset \{\beta \in \mathbb{C} : \beta \geq \alpha\}$.

We denote by $V^{>\alpha}(\mathcal{M})$ the sheaf of germs of sections such that $\mathrm{order}\, u \subset \{\beta \in \mathbb{C} : \beta > \alpha\}$. We define the *specialization of* $\mathcal{M}$ *along* Y by

$$\underline{\nu}_Y(\mathcal{M}) := \mathcal{D}_{T_Y X} \underset{\mathcal{D}_{[T_Y X]}}{\otimes} \tau^{-1}\left(\bigoplus_{j\in\mathbb{Z}} \frac{V^j(\mathcal{M})}{V^{j+1}(\mathcal{M})}\right)$$

(cf. [13] for more details).

Therefore, the characteristic variety of $\underline{\nu}_Y(\mathcal{M})$ is contained in the canonical hypersurface of $T^*(T_Y X)$, which is the characteristic variety of θ.

One defines the $\mathcal{D}_Y$-module of *nearby cycles of* $\mathcal{M}$ *along* Y as

$$\psi_Y(\mathcal{M}) := \frac{V^0(\mathcal{M})}{V^1(\mathcal{M})} = \bigoplus_{0<\alpha\leq 1} \frac{V^\alpha}{V^{>\alpha}} = \mathrm{gr}^0(\mathcal{M}).$$

It is obvious that the complex

$$\mathrm{Sol}(\underline{\nu}_Y(\mathcal{M})) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{T_Y X}}(\underline{\nu}_Y(\mathcal{M}), \mathcal{O}_{T_Y X})$$

is monodromic, i.e., its cohomology groups are locally constant on the orbits of the action of $\mathbb{C}^*$.

Let us briefly recall the real and complex normal deformations of a real (resp. complex) manifold X along a submanifold Y, which we will denote respectively by $\widetilde{X}^{\mathbb{R}}$ and $\widetilde{X}$ (for more details see [8] and [21]).

1. Let X be a real differentiable manifold. Then $\widetilde{X}^{\mathbb{R}}$ is a real differentiable manifold together with canonical morphisms

$$\widetilde{X}^{\mathbb{R}} \xrightarrow{p} X \quad \text{and} \quad \widetilde{X}^{\mathbb{R}} \xrightarrow{q} \mathbb{R}$$

such that if we consider local coordinates (x, y) in X, with Y being defined by $x = (x_1, \dots, x_d) = 0$, one obtains a system of local coordinates on $\widetilde{X}^{\mathbb{R}}$, (x', y', c), $c \in \mathbb{R}$, such that $p(x', y', c) = (x'c, y')$ and $c(x', y', c) = c$.

Let Ω be the open subset of $\widetilde{X}^{\mathbb{R}}$, $\Omega = c^{-1}(\mathbb{R}^+)$, and consider the commutative diagram of morphisms

$$\begin{tikzcd}
T_Y X \arrow[r, "s"] \arrow[d, "\tau"'] & \widetilde{X}^{\mathbb{R}} \arrow[d, "p"] & \Omega \arrow[l, hook', "j"'] \arrow[dl, "\tilde{p}"] \\
Y \arrow[r, "i"'] & X &
\end{tikzcd}$$

Then for any $F^{\cdot} \in \mathrm{Obj}\, D^b(X)$ one defines *Sato's specialization*

$$\nu_Y(F^{\cdot}) := s^{-1}\mathbb{R}_{j*}\tilde{p}^{-1}F^{\cdot}.$$

2. Let X be a complex analytic manifold. Then one considers the complex construction analogous to the preceding one, i.e., with $\mathbb{R}$ replaced by $\mathbb{C}$.

One defines *Verdier's specialization* in $D^b(X)$, as follows:

(a) First of all let us recall *Deligne's nearby cycle functor* associated to a holomorphic function $f : X \to \mathbb{C}$, $\psi_f(\cdot)$. Let $(\widetilde{\mathbb{C}}, p)$ be a universal covering of $\mathbb{C} \setminus \{0\}$, and $\widetilde{X}'$ the fiber product $\widetilde{X}' = X \times_{\mathbb{C}} \widetilde{\mathbb{C}}$ with $\tilde{p} : \widetilde{X}' \to X$ and $\tilde{f} : \widetilde{X}' \to \mathbb{C}$ the canonical projections. Consider the diagram

$$\begin{tikzcd}
 & \widetilde{X}' \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{p}"] & \widetilde{\mathbb{C}} \arrow[d, "p"] \\
Y \arrow[r, "i"'] & X \arrow[r, "f"'] & \mathbb{C}
\end{tikzcd}$$

Then for $F^{\cdot} \in \mathrm{Obj}\, D^b(X)$ one sets

$$\psi_f(F^{\cdot}) = i^{-1}\mathbb{R}\tilde{p}_*\tilde{p}^{-1}(F^{\cdot}).$$

(b) Let us consider the following diagram:

$$\begin{array}{ccccc} c^{-1}(0)= & T_Y X & \longrightarrow & \widetilde{X} & \underset{c}{\longrightarrow} & \mathbb{C} \\ & \tau\downarrow & & \searrow^{\pi} \quad \downarrow p & & \\ & Y & \underset{i}{\longrightarrow} & X & & \end{array}$$

Then *Verdier's specialization functor* $\nu_Y^{\mathbb{C}}(\cdot)$ is defined as follows, for any $F^\cdot \in \operatorname{Obj} D^b(X)$:

$$\nu_Y^{\mathbb{C}}(F^\cdot) = \psi_c(p^* F^\cdot).$$

For a complex of left $\mathcal{D}_{\widetilde{X}}$-modules $F^\cdot$ set

$$F^\cdot[c^{-1}] := \theta_{\widetilde{X}}[c^{-1}] \underset{\theta_{\tilde{X}}}{\otimes} F^\cdot,$$

the localization of $F^\cdot$ along $\widetilde{Y} := c^{-1}(0)$.

One says that $\mathcal{M}$ is *regular along* Y if $\mathcal{M}$ is specializable and if there exists an $\mathcal{O}_X$-coherent submodule $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{D}_X\mathcal{M}_0$, and a nontrivial polynomial $b(s) \in \mathbb{C}[s]$ such that

$$b(\theta)\mathcal{M}_0 \subset [V^1(\mathcal{D}_X) \cap \mathcal{D}_X(m)]\mathcal{M}_0,$$

where $\{\mathcal{D}_X(k)\}_{k\geq 0}$ denotes the filtration on $\mathcal{D}_X$ by the order, and m is the degree of $b(s)$.

The following Theorem 1.1.1 is the analogue of Verdier's specialization. For any coherent $\mathcal{D}_X$-module $\mathcal{M}$ let

$$\mathcal{M}^0 = \mathcal{H}^0(\mathcal{D}_{\widetilde{X}\to X} \overset{\mathbb{L}}{\underset{p^{-1}\mathcal{D}_X}{\otimes}} \mathcal{M}).$$

Theorem 1.1.1 ([20]). *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then*:

1) $\mathcal{H}^k(^{\mathbb{L}}p^*\mathcal{M}[c^{-1}]) = 0$, $\forall k \neq 0$;

2) $\mathcal{M}^0[c^{-1}]$ *is coherent and regular along* $\widetilde{Y}$;

3) *suppose that $\mathcal{M}$ is specializable along Y and consider $M^0[c^{-1}]$ endowed with the canonical V-filtration. Then the $\mathcal{D}_{\widetilde{Y}}$-modules* $\operatorname{gr}^0(M^0[c^{-1}])$ *and* $\underline{\nu}_Y(\mathcal{M})$ *are naturally isomorphic.*

Let us rapidly recall the *formal Fourier transform* (for more details, see [10], [2]).

Let $E \overset{\pi}{\longrightarrow} Y$ be a complex holomorphic vector bundle and denote by $\mathcal{D}_{[E]} \subset \mathcal{D}_E$ the sheaf of algebraic differential operators with respect to the fibers of π. Let us denote by $\operatorname{Mon}(\mathcal{D}_E)$ or $\operatorname{Mon}(\mathcal{D}_{[E]})$ the abelian category whose objects are the $\mathcal{D}_E$-coherent modules (or $\mathcal{D}_{[E]}$-modules) locally generated by sections u such that there exists a nontrivial $b_u(s) \in \mathbb{C}[s]$ satisfying $b_u(\theta)u = 0$.

Let $E^* \overset{\tilde{\pi}}{\longrightarrow} Y$ be the dual vector bundle and $\Omega_{[E/Y]}$ the sheaf of algebraic relative differential forms of maximal degree, with respect to $\pi : E \to Y$.

One defines the *formal Fourier transform* $\underline{F}$ as an isomorphism of sheaves on Y:

$$\Omega_{[E/Y]} \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{[E]} \underset{\mathcal{O}_Y}{\otimes} \Omega_{[E/Y]}^{\otimes -1} \overset{\underline{F}}{\longrightarrow} \mathcal{D}_{[E^*]}$$

given by

$$d\tau \otimes P(y, D_y) \otimes d\tau^{\otimes -1} \mapsto P(y, D_y),$$

$$d\tau \otimes \tau_j \otimes d\tau^{\otimes -1} \mapsto \frac{\partial}{\partial \xi_j},$$

$$d\tau \otimes \frac{\partial}{\partial \tau_j} \otimes d\tau^{\otimes -1} \mapsto -\xi_j,$$

where τ_j denotes the variables on the fibers of E, y the variables on Y, ξ_j the variables on the fibers of E^*, for a given trivialization of E. Let now $\mathcal{M}$ be a $\mathcal{D}_{[E]}$-coherent module. Then $\underline{F}$ induces an exact functor of $\mathrm{Mon}\,(\mathcal{D}_{[E]})$ in $\mathrm{Mon}\,(\mathcal{D}_{[E^*]})$ given by

$$\underline{F}(\mathcal{M}) := \Omega_{[E/Y]} \underset{\mathcal{O}_Y}{\otimes} \mathcal{M},$$

where $\underline{F}$ is regarded as a $\mathcal{D}_{[E^*]}$-module.

Now, if $\mathcal{M}$ is a monodromic $\mathcal{D}_E$-module one defines

$$\underline{F}(\mathcal{M}) := \mathcal{D}_{E^*} \underset{\mathcal{D}_{[E^*]}}{\otimes} \underline{F}(\mathcal{M}'),$$

where $\mathcal{M}'$ is a monodromic $\mathcal{D}_{[E]}$-submodule generating $\mathcal{M}$ since it does not depend on the choice of $\mathcal{M}'$ satisfying these conditions.

Let now $\mathcal{M}$ be a $\mathcal{D}_X$-module specializable along Y. One may define the *microlocalization of* $\mathcal{M}$ *along* Y by $\underline{\mu}_Y(\mathcal{M}) := \underline{F}(\underline{\nu}_Y(\mathcal{M}))$.

Let us recall the symplectic (nonhomogeneous) isomorphism of vector bundles

$$T^*(E) \xrightarrow{\Phi_E} T^*(E^*)$$

introduced in [8]. In local symplectic coordinates $(y,\tau;\xi,\eta)$ on T^*E and $(y,\tilde{\tau};\xi,\tilde{\eta})$ on T^*E^* one has

$$\phi_E(y,\tau;\xi,\eta) = (y,\eta;\xi,-\tau).$$

It is straightforward that

$$\mathrm{Car}(\underline{\mu}_Y(\mathcal{M})) = \mathrm{Car}(\underline{\nu}_Y(\mathcal{M}))$$

using the identification by ϕ_E. If $P \in \mathcal{D}_{[E]}$ and $\widehat{P}$ is its image by $\underline{F}$ in $\mathcal{D}_{[E^*]}$ one has $\sigma(\widehat{P})\phi_E = \sigma(P)$ (σ denotes the principal symbol).

Let F be the *Fourier-Sato transform* in sheaf theory. One has

$$F(\mathrm{Sol}(\underline{\nu}_Y(\mathcal{M}))) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{T^*_Y X}}(\underline{\mu}_Y(\mathcal{M}), \mathcal{O}_{T^*_Y X})[-d].$$

THEOREM 1.1.2 ([3]). *Let* $\mathcal{M}$ *be a regular* $\mathcal{D}$*-module along* Y. *Then one defines natural isomorphisms in* $D^b(T_Y X)$ (*resp. in* $D^b(T^*_Y X)$):

(a) $$\nu_Y(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{T_Y X}}(\underline{\nu}_Y(\mathcal{M}), \mathcal{O}_{T_Y X})$$

(*resp.*

(b) $$\mu_Y(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{T^*_Y X}}(\underline{\mu}_Y(\mathcal{M}), \mathcal{O}_{T^*_Y X}).$$

Idea of the proof. (a) One starts by proving the natural isomorphism

$$\nu_Y(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M}, \nu_Y(\mathcal{O}_X))$$

which is easy. Then, by theorem 7.2 of [6], one gets that the natural morphism

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \nu_Y^{\mathbb{C}}(\mathcal{O}_X)) \to \mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \nu_Y(\mathcal{O}_X))$$

is an isomorphism.

Let $\tilde{p}$ denote the restriction of $p : \widetilde{X} \to X$ to $\widetilde{X} \setminus \widetilde{Y}$. Since $\widetilde{Y} = c^{-1}(0)$ is a smooth hypersurface defined by a global equation, we may use the results in [13] and obtain

$$\begin{aligned}
\nu_Y(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) &\xleftarrow{\sim} \nu_Y^{\mathbb{C}}(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \\
&= \psi_c(p^{-1}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \\
&\xrightarrow{\sim} \psi_c(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\widetilde{X}}}({}^{\mathbb{L}}p^*\mathcal{M}[c^{-1}], \mathcal{O}_{\widetilde{X}})) \\
&\simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{T_YX}}(\psi_c(\mathcal{M}^0[c^{-1}]), \mathcal{O}_{T_YX}) \\
&\simeq \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{T_YX}}(\underline{\nu}_Y(\mathcal{M}), \mathcal{O}_{T_YX}),
\end{aligned}$$

where we use the fact that, for any object $F^{\cdot}$ of $D^b(X)$, $\psi_c(F^{\cdot})$ only depends on the behaviour of $F^{\cdot}$ outside $c = 0$ and $\tilde{p}$ is smooth.

(b) is analogous to (a).

1.2. *$\mathcal{E}$-modules.* Let X be a complex analytic manifold and $\mathcal{E}_X$ the sheaf of microdifferential operators of finite order on T^*X. Let Λ be a smooth homogeneous lagrangian submanifold of $\overset{\circ}{T}{}^*X$, the cotangent vector bundle with the zero section deleted. Let $\overset{\circ}{\pi} : \overset{\circ}{T}{}^*X \to X$ be the projection. Let $\mathcal{E}_\Lambda$ be the subsheaf of rings of microdifferential operators generated by $\mathcal{E}_X(0)$, and

$$\mathcal{I}_\Lambda = \{P \in \mathcal{E}_X(1) : \sigma_1(P)|_\Lambda = 0\}.$$

Here $\mathcal{E}_X(k)$ denotes the sheaf of microdifferential operators of order k and σ_1 denotes the symbol of order one. One sets

$$\mathcal{E}_\Lambda(m) = \mathcal{E}_\Lambda\, \mathcal{E}_X(m)$$

(filtration on $\mathcal{E}_X$ with respect to Y), and

$$\mathcal{E}_{\Lambda,m} = \mathcal{E}_\Lambda \cap \mathcal{E}_X(m)$$

(filtration on $\mathcal{E}_\Lambda$ by the order). When $\Lambda = \overset{\circ}{T}{}^*_YX$, one has

$$\mathcal{E}_\Lambda(m)|_{T^*_YY} = V_Y^{-m}(\mathcal{D}_X)|_Y, \qquad \forall m \geq 0.$$

Let us choose a section (locally) $\theta \in \mathcal{I}_\Lambda$ satisfying

$$d\sigma_1(\theta) = -\omega \bmod I_\Lambda \Omega^1_{T^*X}, \qquad \sigma_0(\theta) = \frac{1}{2} \sum_j \frac{\partial^2 \sigma_1(\theta)}{\partial x_j \partial \xi_j} \bmod I_\Lambda,$$

where (x, ξ) stands for a system of canonical coordinates on T^*X, ω the canonical one-form on T^*X, I_Λ the defining ideal of Λ. θ is well defined $\bmod\, \mathcal{E}_\Lambda(-1)$.

Let $\mathcal{M}$ be a coherent left $\mathcal{E}_X$-module.

Definition 1.2.1. One says that $\mathcal{M}$ is *microlocalizable along Λ* if, locally, one has:

(a) there is a coherent $\mathcal{E}_\Lambda$ submodule $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{E}_X \mathcal{M}_0$,
(b) there is a nontrivial $b(s) \in \mathbb{C}[s]$ such that $b(\theta)\mathcal{M}_0 \subset \mathcal{E}_X(-1)\mathcal{M}_0$.

Remark 1.2.2. We may suppose that the zeros of $b(s)$ do not differ by an integer.

For a microlocalizable $\mathcal{E}_X$-module $\mathcal{M}$ one sets

$$\mathrm{gr}_\Lambda(\mathcal{M}) := \bigoplus_k \frac{\mathcal{E}_\Lambda(k)\mathcal{M}_0}{\mathcal{E}_\Lambda(k-1)\mathcal{M}_0},$$

where $\mathcal{M}_0$ is chosen in the conditions (a), (b) and remark 1.2.2. If $\mathcal{M}_0$ is replaced by another $\mathcal{M}'_0$ in the same conditions, we obtain a natural isomorphism of graded modules.

DEFINITION 1.2.1′. We shall say that $\mathcal{M}$ is *regular along* Λ if there exists a coherent $\mathcal{E}_X(0)$-submodule $\mathcal{M}_0$ such that $\mathcal{M}_0$ generates $\mathcal{M}$ and a nontrivial polynomial $b(s)$ such that

$$b(\theta)\mathcal{M}_0 \subset \mathcal{E}_X(-1)\mathcal{E}_{\Lambda,m+1}\mathcal{M}_0,$$

where m is the degree of $b(s)$. This notion is different from those in [5], [7] (which imply in particular that $\mathrm{Car}(\mathcal{M}) \subset \Lambda$) but is the same as that introduced in [16].

Locally on Λ we may choose an invertible $\mathcal{O}_\Lambda$-module $\mathcal{L}$ such that

$$\mathcal{L} \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{L} \simeq \Omega^h_\Lambda \underset{\overset{\circ}{\pi}{}^{-1}\mathcal{O}_X}{\otimes} \overset{\circ}{\pi}{}^{-1}\Omega_X^{\otimes -1},$$

where Ω^h_Λ denotes the sheaf of homogeneous differential forms on Λ of maximal degree.

A remarkable fact is that $\mathrm{gr}_\Lambda(\mathcal{E}_X)$ is isomorphic to the sheaf $\mathcal{A}$ of homogeneous differential operators on $\mathcal{L}$ (cf. [5], [7]). Consequently, we may identify $\mathcal{D}^h_\Lambda$, the sheaf of homogeneous differential operators on Λ, and the sheaf

$$\mathcal{L}^{\otimes -1} \underset{\mathcal{O}^h_\Lambda}{\otimes} \mathrm{gr}_\Lambda(\mathcal{E}_X) \underset{\mathcal{O}^h_\Lambda}{\otimes} \mathcal{L},$$

where $\mathcal{O}^h_\Lambda$ denotes the sheaf of homogeneous holomorphic functions on Λ.

Let $\mathcal{M}$ be a microlocalizable $\mathcal{E}_X$-module along Y.

DEFINITION 1.2.3. The *microlocalization of* $\mathcal{M}$ *along* Λ, $\underline{\mu}_\Lambda(\mathcal{M})$, is the $\mathcal{D}_\Lambda$-coherent module

$$\underline{\mu}_\Lambda(\mathcal{M}) = \mathcal{D}_\Lambda \underset{\mathcal{D}^h_\Lambda}{\otimes} (\mathcal{L}^{\otimes -1} \underset{\mathcal{O}^h_\Lambda}{\otimes} \mathrm{gr}_\Lambda(\mathcal{M})),$$

where $\mathrm{gr}_\Lambda(\mathcal{M})$ is defined by choosing a coherent $\mathcal{E}_\Lambda$-submodule $\mathcal{M}_0$ as in Definition 1.2.1.

If we take another $\mathcal{M}'_0$ in the same condition, we obtain two isomorphic $\mathcal{D}_\Lambda$-modules, as well as if we choose a different $\mathcal{O}_\Lambda$-invertible module $\mathcal{L}'$ with the same property as $\mathcal{L}$.

R e m a r k 1.2.4. Let $Y \subset X$ be a smooth submanifold, $\Lambda = \overset{\circ}{T}{}^*_Y X$ and $\mathcal{M}$ be a specializable $\mathcal{D}$-module. It is straightforward that $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ is microlocalizable along Λ and, choosing $\mathcal{L} = \Omega^h_{\Lambda|Y}$ one has

$$\begin{aligned}
\underline{\mu}_Y(\mathcal{M})|_\Lambda &= \mathcal{D}_\Lambda \underset{\mathcal{D}_{[\Lambda]}}{\otimes} (\Omega^{\otimes -1}_{[\Lambda|Y]} \underset{\overset{\circ}{\pi}{}^{-1}\mathcal{O}_Y}{\otimes} \overset{\circ}{\pi}{}^{-1}\mathrm{gr}_Y(\mathcal{M}))\\
&= \mathcal{D}_\Lambda \underset{\mathcal{D}^h_\Lambda}{\otimes} (\Omega^{h\,\otimes -1}_{\Lambda|Y} \underset{\mathcal{O}^h_\Lambda}{\otimes} \mathrm{gr}_\Lambda(\mathcal{E}_X) \underset{\pi^{-1}\mathcal{O}_Y}{\otimes} \overset{\circ}{\pi}{}^{-1}\mathrm{gr}_Y(\mathcal{M}))\\
&= \underline{\mu}_\Lambda(\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{M}).
\end{aligned}$$

Let X and X' be two complex analytic manifolds. Let us recall the *formal tensor product* of a $\mathcal{D}_X$-module $\mathcal{M}$ and a $\mathcal{D}'_X$-module $\mathcal{M}'$ as defined in [15]:

$$\mathcal{M} \boxtimes \mathcal{M}' = \mathcal{D}_{X\times X'} \underset{p_1^{-1}\mathcal{D}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{D}_{X'}}{\otimes} (p_1^{-1}\mathcal{M} \otimes_{\mathbb{C}} p_2^{-1}\mathcal{M}'),$$

where $p_1 : X \times X' \to X$ and $p_2 : X \times X' \to X'$ are the projections.

Proposition 1.2.5. *Let X, X' be two complex analytic manifolds.*

(a) *Let $Y \subset X$ and $Y' \subset X'$ be two smooth submanifolds, $\mathcal{M}$ a $\mathcal{D}_X$-module specializable (resp. regular) along Y, and $\mathcal{M}'$ specializable (resp. regular) along Y'. Then $\mathcal{M}\boxtimes\mathcal{M}'$ is specializable (resp. regular) along $Y \times Y'$ and there is a natural isomorphism*

$$\underline{\mu}_Y(\mathcal{M}) \boxtimes \underline{\mu}_{Y'}(\mathcal{M}') \simeq \underline{\mu}_{Y\times Y'}(\mathcal{M} \boxtimes \mathcal{M}').$$

(b) *The same statement holds with Y replaced by a homogeneous lagrangian submanifold Λ of $\overset{\circ}{T}{}^*X$, Y' by a homogeneous lagrangian submanifold Λ' in $\overset{\circ}{T}{}^*X'$, and $\mathcal{M}$ (resp. $\mathcal{M}'$) a left $\mathcal{E}_X$-module (resp. a left $\mathcal{E}_{X'}$-module) microlocalizable (resp. regular) along Λ (resp. along Λ').*

Proof. We will only prove (a) since (b) is similar. Let us fix representatives θ, θ' of the Euler fields respectively on T_YX and $T_{Y'}X'$.

Let us consider a system $\{u_i\}_{i\in I}$ (resp. $\{v_j\}_{j\in J}$) of local generators of $\mathcal{M}$ (resp. of $\mathcal{M}'$). Then $\mathcal{M} \boxtimes \mathcal{M}'$ is generated by the images $u_i \boxtimes v_j$. Let

$$b_{u_i}(\theta) = \prod_k (\theta - \alpha_{ik})^{p_{ik}}$$

be a b-function for u_i and

$$b_{u'_j}(\theta') = \prod_{k'} (\theta' - \beta_{jk'})^{q_{jk'}}$$

a b-function for u'_j. Then an easy calculation entails that

$$b_{ij}(\theta+\theta') = \prod_{k,k'} (\theta + \theta' - \alpha_{ik} - \beta_{jk'})^{p_{ik}+q_{jk'}} \tag{$*$}$$

is a b-function for $u_i \boxtimes u'_j$ hence the first assertion of (a) is proved. Now, relation $(*)$ entails that $V^\alpha(\mathcal{M}\boxtimes\mathcal{M}')$ is equal to

$$\sum_{\ell+\ell'=\alpha} V^0_\triangle(\mathcal{D}_{X\times X'}) \underset{p_1^{-1}V^0_Y(\mathcal{D}_X)\otimes_{\mathbb{C}} p_2^{-1}V^0_{Y'}(\mathcal{D}_{X'})}{\otimes} (p_1^{-1}V^\ell(\mathcal{M}) \underset{\mathbb{C}}{\otimes} p_2^{-1}V^{\ell'}(\mathcal{M}')),$$

where we consider the canonical V-filtrations respectively with respect to $Y\times Y'$, Y and Y', and $I_{Y\times Y'}$ denotes the defining ideal of $Y\times Y'$ in $X \times X'$.

An easy verification shows that $V^0_\triangle(\mathcal{D}_{X\times X'})$ is flat with respect to $p_1^{-1}V^0_Y(\mathcal{D}_X) \otimes_{\mathbb{C}} p_2^{-1}V^0_{Y'}(\mathcal{D}_{X'})$.

Hence

$$\underline{\nu}_{Y\times Y'}(\mathcal{M}\boxtimes\mathcal{M}') \simeq \underline{\nu}_Y(\mathcal{M}) \boxtimes \underline{\nu}_Y(\mathcal{M}')$$

and to finish we only have to remark that the formal Fourier transform is compatible with formal tensor products of monodromic $\mathcal{D}$-modules. ∎

Corollary 1.2.6. *Let $\mathcal{M}$ be specializable along Y. Then:*

(a) *$\underline{\mu}_Y(\mathcal{M})$ only depends on $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}\,|_{T^*_YX}$,*

(b) *$\operatorname{supp}\underline{\mu}_Y(\mathcal{M}) \subset \operatorname{Car}(\mathcal{M}) \cap T^*_YX$.*

Proof. (b) is an immediate consequence of remark 1.2.4 as well as (a) when we restrict to $\overset{\circ}{T}{}^*_YX$. Now let $p \in T^*_YY$. We will use the dummy variable trick (cf. [5]).

Let $\mathcal{N}$ be

$$\frac{\mathcal{D}_{\mathbb{C}}}{\mathcal{D}_{\mathbb{C}}t} = \mathcal{D}_{\mathbb{C}}\delta(t),$$

where t is a coordinate on $\mathbb{C}$. Then one has

$$\underline{\mu}_{Y\times\{0\}}(\mathcal{M}\,\underline{\boxtimes}\,\mathcal{N}) \simeq \underline{\mu}_Y(\mathcal{M})\,\underline{\boxtimes}\,\underline{\mu}_{\{0\}}(\mathcal{N}) \simeq \underline{\mu}_Y(\mathcal{M})\,\underline{\boxtimes}\,\mathcal{O}_{T^*_{\{0\}}\mathbb{C}}.$$

Let $\Lambda = T^*_Y X$ and $\Lambda' = \overset{\circ}{T}{}^*_{\{0\}}\mathbb{C}$ and $j : \Lambda \to \Lambda\times\Lambda'$ the inclusion $q \mapsto (q,(0,1))$. One easily checks the isomorphisms

$$(**)\qquad j^{-1}\mathcal{H}om_{\mathcal{D}_{\Lambda\times\Lambda'}}(\mathcal{D}_\Lambda\,\underline{\boxtimes}\,\mathcal{O}_{\Lambda'}, \underline{\mu}_Y(\mathcal{M})\,\underline{\boxtimes}\,\mathcal{O}_{\Lambda'}) \simeq j^{-1}(\mathbb{C}_{\Lambda'}\,\underline{\boxtimes}\,\underline{\mu}_Y(\mathcal{M})) \simeq \underline{\mu}_Y(\mathcal{M})$$

Let $\mathcal{M}'$ be another specializable $\mathcal{D}$-module along Y, Ω an open subset of $T^*_Y X$ containing p, and suppose

$$\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}|_\Omega \simeq \mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}'|_\Omega.$$

Let π denote the projection $T^*(X\times\mathbb{C}) \to X\times\mathbb{C}$. Then

$$\mathcal{E}_{X\times\mathbb{C}} \underset{\pi^{-1}(\mathcal{D}_{X\times\mathbb{C}})}{\otimes} (\mathcal{M}\,\underline{\boxtimes}\,\mathcal{N})|_{\Omega\times\Lambda'} \simeq \mathcal{E}_{X\times\mathbb{C}} \underset{\pi^{-1}(\mathcal{D}_{X\times\mathbb{C}})}{\otimes} (\mathcal{M}'\,\underline{\boxtimes}\,\mathcal{N})|_{\Omega\times\Lambda'}$$

hence

$$\underline{\mu}_{Y\times\{0\}}(\mathcal{M}\,\underline{\boxtimes}\,\mathcal{N})|_{\Omega\times\Lambda'} \simeq \underline{\mu}_{Y\times\{0\}}(\mathcal{M}'\,\underline{\boxtimes}\,\mathcal{N})|_{\Omega\times\Lambda'},$$
$$\underline{\mu}_Y(\mathcal{M})\,\underline{\boxtimes}\,\mathcal{O}_{\Lambda'}|_{\Omega\times\Lambda'} \simeq \underline{\mu}_Y(\mathcal{M}')\,\underline{\boxtimes}\,\mathcal{O}_{\Lambda'}|_{\Omega\times\Lambda'}$$

which completes the proof by $(**)$. ∎

Let $\mathcal{E}^{\mathbb{R}}_X$ be the sheaf of microlocal operators on T^*X and if N is a left $\mathcal{E}_X$-module set

$$\mathcal{N}^{\mathbb{R}} := \mathcal{E}^{\mathbb{R}}_X \underset{\mathcal{E}_X}{\otimes} N.$$

The microlocal analogue of Theorem 1.1.2 is the following:

THEOREM 1.2.7 [6]. *Let $\mathcal{M}$ be a regular $\mathcal{E}_X$-module along the homogeneous lagrangian submanifold Λ and $\mathcal{L}$ a simple holonomic $\mathcal{E}_X$-module supported by Λ. Then there is a natural isomorphism*:

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{L}^{\mathbb{R}})|_\Lambda \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_\Lambda}(\underline{\mu}_\Lambda(\mathcal{M}),\mathcal{O}_\Lambda).$$

1.3. *Applications to microcharacteristic varieties.* Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module and Λ a smooth lagrangian homogeneous submanifold of $\overset{\circ}{T}{}^*X$. Let $C_\Lambda(\mathcal{M})$ be the microcharacteristic variety of $\mathcal{M}$ along Λ (i.e. the normal cone of $\operatorname{supp}\mathcal{M}$ along Λ [9]), $\widehat{C}_\Lambda(\mathcal{M})$ be the formal microcharacteristic variety and $C^1_\Lambda(\mathcal{M})$ the 1-microcharacteristic variety (see [10] and [17]).

The following inclusions are well known:

$$\widehat{C}_\Lambda(\mathcal{M}) \subset C^1_\Lambda(\mathcal{M}), \qquad C_\Lambda(\mathcal{M}) \subset C^1_\Lambda(\mathcal{M}).$$

When $\Lambda = \overset{\circ}{T}{}^*_Y X$ and $\mathcal{M}$ is of the form $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{N}$, where $\mathcal{N}$ is a regular $\mathcal{D}$-module along Y, one has

$$\begin{aligned}\widehat{C}_\Lambda(\mathcal{M}) &= SS(\mathbb{R}\mathcal{H}om_{\mathcal{D}_\Lambda}(\underline{\mu}_Y(\mathcal{N})|_\Lambda,\mathcal{O}_\Lambda) = SS(\mu_Y(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{O}_X))|_\Lambda)\\ &\subset C_\Lambda(SS(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{O}_X)) = C_\Lambda(\mathcal{M}).\end{aligned}$$

Here SS stands for "microsupport" (cf. [8]). Hence $\widehat{C}_\Lambda(\mathcal{M}) \subset C_\Lambda(\mathcal{M})$ if $\mathcal{M}$ is regular along Y.

1.4. *Application to* $\mu\underline{\text{hom}}$. Let us consider the category $\mathcal{C}$ whose objects are pairs of coherent $\mathcal{D}_X$-modules such that $\mathcal{M}\boxtimes\mathcal{N}^*$ is specializable along $\triangle$, the diagonal of $X\times X$. Here

$$\mathcal{N}^* = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{D}_X)[n]\otimes\Omega_X^{\otimes -1},$$

with $n = \dim X$. One defines the bifunctor $\mu\underline{\text{hom}}$ from $\mathcal{C}$ to $\mathrm{Mod}_c(\mathcal{D}_{T^*X})$ by $\mu\underline{\text{hom}}(\mathcal{M},\mathcal{N}) = \underline{\mu}_\triangle(\mathcal{M}\boxtimes\mathcal{N}^*)$, where we identify T^*X and $T^*_\triangle(X\times X)$ by the first projection.

THEOREM 1.4.1 [20]. *Let* $(\mathcal{M},\mathcal{N})$ *be an object of* $\mathcal{C}$ *such that* $(\mathcal{M}\boxtimes\mathcal{N}^*)$ *is regular along* $\triangle$. *Then one has a natural isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X\otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M},\mathcal{E}_X^{\mathbb{R}}\otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{N}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_{T^*X}}(\mu\underline{\text{hom}}(\mathcal{M},\mathcal{N}),\mathcal{O}_{T^*X}).$$

From the results in Section 1, we conclude that for $(\mathcal{M},\mathcal{N})\in\mathrm{Obj}\,\mathcal{C}$,

(i) supp $\mu\underline{\text{hom}}(\mathcal{M},\mathcal{N})\subset \mathrm{Car}\,\mathcal{M}\times\mathrm{Car}\,\mathcal{N}$,
(ii) $\mu\underline{\text{hom}}(\mathcal{M},\mathcal{N})$ only depends on $\mathcal{E}_X\otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M}$ and $\mathcal{E}_X\otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{N}$.

EXAMPLE 1.4.2. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module specializable along

$$Y = \{(t,x)\in\mathbb{C}^{1+d} : t=0\}\subset X=\mathbb{C}^{1+d},$$

$\Lambda = T^*_YX$, and

$$\mathcal{N} = B_{Y|X} = \frac{\mathcal{D}_X}{\mathcal{D}_Xt+\mathcal{D}_XD_{x_1}+\ldots+\mathcal{D}_XD_{x_d}} = \mathcal{D}_X\delta(t).$$

Then $(\mathcal{M},\mathcal{N}^*)$ is an object of $\mathcal{C}$ and we will check that

$$\mu\underline{\text{hom}}(\mathcal{M},\mathcal{N}^*)\simeq\mathcal{D}_{T^*X\leftarrow\Lambda}\otimes_{\mathcal{D}_\Lambda}\underline{\mu}_Y(\mathcal{M}).$$

Let u be a local section of $\mathcal{M}$ and $b_u(s)$ be a b-function for u. Then

$$b_u((t-t')D'_t+(x_1-x'_1)D_{x_1}+\ldots+(x_d-x'_d)Dx_d$$

is a b-function for $\delta(t)\boxtimes u$ where we consider $(t,x_1,\cdots,x_d,t',x'_1,\cdots,x'_d)$ as a local coordinate system on $X\times X$. Hence $B_{Y|X}\boxtimes\mathcal{M}$ is specializable along $\triangle\subset X\times X$. Furthermore,

$$B_{Y|X}\boxtimes\mathcal{M}\simeq\frac{\mathcal{D}_{X\times X}}{\mathcal{D}_{X\times X}t+\mathcal{D}_{X\times X}\mathcal{D}_{x_1}+\ldots+\mathcal{D}_{X\times X}\mathcal{D}_{x_d}}\underset{p_2^{-1}\mathcal{D}_X}{\otimes}p_2^{-1}\mathcal{M}$$

and $V^\alpha_\triangle(B_{Y|X}\boxtimes\mathcal{M})$ is the image on $B_{Y|X}\boxtimes\mathcal{M}$ of

$$\sum_{k+i=\alpha}V^i_\triangle\left(\frac{\mathcal{D}_{X\times X}}{\mathcal{D}_{X\times X}t+\mathcal{D}_{X\times X}\mathcal{D}_{x_1}+\ldots+\mathcal{D}_{X\times X}\mathcal{D}_{x_d}}\right)\underset{p_2^{-1}V^0_Y(\mathcal{D}_X)}{\otimes}p_2^{-1}V^k_Y(\mathcal{M}),$$

where we consider the canonical V-filtrations with respect to $\triangle$ and Y.

At this point we only have to remark that the Fourier transform of

$$\frac{\mathcal{D}_{T_\triangle(X\times X)}}{\mathcal{D}_{T_\triangle(X\times X)}t+\mathcal{D}_{T_\triangle(X\times X)}\mathcal{D}_{\tilde{x}_1}+\ldots+\mathcal{D}_{T_\triangle(X\times X)}\mathcal{D}_{\tilde{x}_d}}$$

is precisely $\mathcal{D}_{T^*X}\leftarrow\Lambda$ where we consider the equations

$$\tilde{t}=t-t'=0,\quad \tilde{x}_1=x_1-x'_1=0,\ \ldots,\ \tilde{x}_d=x_d-x'_d=0$$

as the equations of $\triangle$ in $X\times X$ and identify T^*X and $T^*_\triangle(X\times X)$ by the first projection.

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