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GLOBAL EXISTENCE OF SOLUTIONS OF PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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In [1], H. Amann derived an a priori bound for solutions of parabolic problems with nonlinear boundary conditions in the Sobolev space $W_p^s(\Omega, \mathbb{R}^N)$ ($s \geq 1$, $p > n$, $\Omega \subset \mathbb{R}^n$ bounded). The result ([1, Theorem 15.2]) is based on the assumptions of an apriori estimate for the solutions in some weaker norm (in $W_{p_0}^{s_0}(\Omega, \mathbb{R}^N)$, $s > s_0$, $p_0 \geq 1$) and of suitable growth conditions for the local nonlinearities arising in the problem. However, the proof of this result contains some discrepancies (the choice of r in the proof does not match the assumptions in [1, Lemma 15.1]) and the result itself is not correct in the case $n = 1$: the growth of the function g arising in the boundary condition has to be controlled by the power $1 + p_0/(n - s_0p_0)$ also in this case. The aim of this paper is to give a correct proof of a modification of the result mentioned above and to show that the growth assumption is optimal for $n = 1$.

The idea of our proof is the same as that in [1]. For the sake of simplicity we consider only the special case $s_0 = 0$. On the other hand, unlike [1] we do not assume $p > n$. We consider the problem

$$\text{(P)} \qquad \begin{cases} u_t + \mathcal{A}u = f(x,t,u,\nabla u) & \text{in } \Omega \times (0,T), \\ \mathcal{B}u = g(x,t,u) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

where $0 < T \leq \infty$, Ω is a bounded domain in $\mathbb{R}^n$ of class C^2, $u : \Omega \times [0,T) \to \mathbb{R}^N$, $\mathcal{A}u = (-\Delta u_1, \ldots, -\Delta u_N)$, $\mathcal{B}u = \partial u/\partial n$ is the derivative with respect to the outer normal on the boundary $\partial\Omega$ (the generalization to more complicated, non-autonomous

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operators $\mathcal{A}$, $\mathcal{B}$ as in [1] is straightforward), f, g are C^1 functions with

$$\begin{aligned}|\partial_t f(x,t,\xi,\eta)| &\le C(1+|\xi|^{2\nu_1+1}+|\eta|^{\nu_2+1}),\\|\partial_\xi f(x,t,\xi,\eta)| &\le C(1+|\xi|^{2\nu_1}+|\eta|^{\nu_2+\min(1,\nu_2)}),\\|\partial_\eta f(x,t,\xi,\eta)| &\le C(1+|\xi|^{\nu_1}+|\eta|^{\nu_2}),\\|\partial_t g(x,t,\xi)| &\le C(1+|\xi|^{\nu_1+1}),\\|\partial_\xi g(x,t,\xi)| &\le C(1+|\xi|^{\nu_1}),\end{aligned}$$

for some $\nu_1 < p/(n-p)$ (if $p<n$), $\nu_2 < p/n$ and $p>1$ (the assumptions concerning the smoothness of f, g in t and x can be relaxed; see e.g. [1, p. 255] for sufficient assumptions in the case $p>n$).

If $u_0 \in W_p^s(\Omega,\mathbb{R}^N)$, $s\in[1,1+1/p)$, then the theory developed in [1] guarantees the existence of a unique maximal solution of (P) in $W_p^s(\Omega,\mathbb{R}^N)$. Moreover, $u(t)\in W_p^{s+\varepsilon}(\Omega,\mathbb{R}^N)$ for some $\varepsilon>0$ and any $t>0$ and a simple bootstrap argument together with standard imbedding theorems show that $u(t)\in W_{\tilde p}^{\tilde s}(\Omega,\mathbb{R}^N)$ for any $\tilde p \ge p$, $\tilde s < 1+1/\tilde p$ and $t>0$. The solution fulfils a variation-of-constants formula of the form

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(s,u(s))\,ds,$$

where A is an operator associated with the differential operators $\mathcal{A}$, $\mathcal{B}$ and F is a map induced by the nonlinear functions f, g (see [1, p. 244] for details). The results of [1, Section 12] imply also that this solution is global if the map F fulfils an estimate of the type

$$\|F(t,u(t))\|_{W_{\mathcal{B}}^{s'/2-1}} \le c(t)\big(1+\|u(t)\|^{\varepsilon}_{W_{\mathcal{B}}^{s/2}}\big)$$

for some $\varepsilon<1$, $s<s'<1+1/p$ and a nondecreasing function $c:\mathbb{R}^+\to\mathbb{R}^+$ (where $W_{\mathcal{B}}^{s/2} = W_p^s(\Omega,\mathbb{R}^N)$ and the extrapolation space $W_{\mathcal{B}}^{s'/2-1}$ can be viewed as the dual of the space $W_q^{2-s'}(\Omega,\mathbb{R}^N)$ with $1/p+1/q=1$; see [1]). Moreover, if $T=\infty$ and $c:\mathbb{R}^+\to\mathbb{R}^+$ is bounded then $u:[0,T)\to W_p^s(\Omega,\mathbb{R}^N)$ is bounded.

Our main result is the following modification of [1, Theorem 15.2]. By $\|\cdot\|_{s,p}$ or $\|\cdot\|_p$ we denote the norm in $W_p^s(\Omega,\mathbb{R}^N)$ or $L_p(\Omega,\mathbb{R}^N)$, respectively.

THEOREM. *Let* $p_0\ge 1$, $p>\max(1,p_0(n-1)/(p_0+n))$, $\hat\lambda_1 < 1+1/p$,

$$\begin{aligned}1 \le \hat\lambda_j &< 1+\frac{p_0(2-j)}{n+jp_0}, \qquad j=0,1,\\1\le\hat\lambda &< 1+\frac{p_0}{n},\\|f(x,t,\xi,\eta)| &\le C(1+|\xi|^{\hat\lambda_0}+|\eta|^{\hat\lambda_1}),\\|g(y,t,\xi)| &\le C(1+|\xi|^{\hat\lambda})\end{aligned}$$

for $x\in\overline{\Omega}$, $y\in\partial\Omega$, $t\in[0,T)$ *and* $(\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^{Nn}$. *Let* $u_0\in W_p^1(\Omega,\mathbb{R}^N)$ *and let* u *be the corresponding maximal solution of* (P) *with the maximal existence time* $T_{\max}\le T$. *Let* $c:\mathbb{R}^+\to\mathbb{R}^+$ *be a nondecreasing function and let* $\|u(t)\|_{p_0}\le c(t)$ *for any* $t\in[0,T_{\max})$. *Then* $T_{\max}=T$ *and* $\sup_{t\in[t_1,t_2)}\|u(t)\|_{s,p}<\infty$ *for any* $s<1+1/p$, $t_1>0$ *and* $t_2\le T$, $t_2<\infty$ (*or* $t_2=\infty$ *if* $T=\infty$ *and* $c:\mathbb{R}^+\to\mathbb{R}^+$ *is bounded*).

Remark 1. The assumption $\hat{\lambda}_1 < 1+1/p$ seems to be of technical nature: it is due to the fact that we work in the space $W^s_p(\Omega,\mathbb{R}^N)$ with $s < 1+1/p$ which is required by the nonlinear boundary conditions. If we consider e.g. homogeneous Dirichlet boundary conditions then one can use the variation-of-constants formula and corresponding estimates in the space $W^s_p(\Omega,\mathbb{R}^N)$ for any $s<2$ and the assumption $\hat{\lambda}_1 < 1+1/p$ becomes unnecessary (cf. also [2, Theorem 5.3]).

The proof of the Theorem is based on the following three lemmas.

Lemma 1. *Let* $p_0,\lambda,r\ge 1$, $\lambda r>1$, $p>1$, $s,\sigma\in[0,2]$, $s>0$ *and*

$$1+p_0(1/r-1/p)<\lambda<1+p_0\frac{s-\sigma+n(1/r-1/p)}{n+\sigma p_0}. \tag{A}$$

Then there exists $\varepsilon\in(0,1)$ *such that*

$$\|u\|^{\lambda}_{\sigma,r\lambda}\le C\|u\|^{\lambda-\varepsilon}_{p_0}\|u\|^{\varepsilon}_{s,p}\qquad \textit{for any } u\in W^s_p(\Omega,\mathbb{R}^N)\cap L_{p_0}(\Omega,\mathbb{R}^N).$$

Proof. The proof follows from [1, Lemma 15.1] by choosing ε sufficiently close to 1, $s_0=0$ and observing that the assumption $r\ge p\ge p_0$ in [1] can be relaxed to the assumption $1/(\lambda r)<(1-1/\lambda)/p_0+(1/\lambda)/p$ (cf. [2, Proposition 4.1]) which is equivalent to $\lambda>1+p_0(1/r-1/p)$. ∎

Lemma 2. *Let* $p_0\ge 1$, $p>\max(1,p_0(n-1)/(p_0+n))$, $1\le\hat{\lambda}<1+p_0/n$. *If* $s\in[1,1+1/p)$ *is sufficiently close to* $1+1/p$ *then there exist* $r\ge 1$ *and* $\lambda\ge\hat{\lambda}$ *such that* $r>p(n-1)/(n-p(s-1))$, $r\lambda<p(n-1)/(n-sp)$ (*if* $n>sp$) *and* (A) *is fulfilled with* $\sigma=1/(\lambda r)$.

Proof. If $n>1$ choose $s\in[1,1+1/p)$ such that $s>\max(2-n+n/p,1/n+1/p)$. Then $\tilde{r}:=p(n-1)/(n-p(s-1))>1$. Choose $r>\tilde{r}$ such that $r(1+p_0/n)<p(n-1)/(n-sp)$ (if $n>sp$) and $\lambda_{\max}(r)>\max(\hat{\lambda},\lambda_{\min}(r))$, where

$$\lambda_{\min}(r):=1+p_0(1/r-1/p)\quad\text{and}\quad\lambda_{\max}(r):=\lambda_{\min}(r)+(p_0/n)(s-1/r).$$

This is possible since $\tilde{r}(1+p_0/n)<p(n-1)/(n-sp)$ (if $n>sp$) and $\lambda_{\max}(\tilde{r})=1+p_0/n>\lambda_{\min}(\tilde{r})$. If $n=1$ and $r>1$ is arbitrary then $\lambda_{\max}(r)=1+p_0(s-1/p)>\max(\hat{\lambda},\lambda_{\min}(r))$ if s is sufficiently close to $1+1/p$.

Now for any $n\ge 1$ choose $\lambda\in(\max(\hat{\lambda},\lambda_{\min}(r)),\lambda_{\max}(r))$. This choice guarantees (A) with $\sigma=1/(\lambda r)$ since the second inequality in (A) is equivalent to $\lambda<\lambda_{\max}(r)$ in this case. ∎

Lemma 3. *Let* $p_0\ge 1$, $p>\max(1,p_0(n-1)/(p_0+n))$, $1\le\hat{\lambda}_0<1+2p_0/n$, $s\in[1,1+1/p)$, $s'\in(s,1+1/p)$. *Put* $r=pn/(n+(2-s')p)$ *if* $n>1$, $r=1$ *if* $n=1$. *If* $s\in[1,1+1/p)$ *is sufficiently close to* $1+1/p$ *then there exists* $\lambda_0>\hat{\lambda}_0$ *such that* $r\lambda_0<pn/(n-sp)$ (*if* $n>sp$) *and* (A) *is fulfilled with* $\sigma=0$ *and* λ *replaced by* λ_0. *If, moreover,* $1\le\hat{\lambda}_1<1+\min(p_0/(n+p_0),1/p)$ *then there exist* $R\ge r$ *and* $\lambda_1>\hat{\lambda}_1$ *such that* $R\lambda_1<pn/(n-(s-1)p)$ *and* (A) *is fulfilled with* $\sigma=1$, r *replaced by* R *and* λ *replaced by* λ_1.

Proof. Denote $\lambda_{\min}=1+p_0(1/r-1/p)$, $\lambda_{\max}=\lambda_{\min}+(p_0/n)s$. Then (A) with

$\sigma = 0$ is equivalent to $\lambda_{\min} < \lambda < \lambda_{\max}$. It is easy to see that

$$1 + \frac{2p_0}{n} > \lambda_{\max} = 1 + \frac{2p_0}{n} - p_0\frac{s'-s}{n} > \max(\hat\lambda_0, \lambda_{\min}) \qquad \text{if } n > 1,$$
$$1 + \frac{2p_0}{n} > \lambda_{\max} = 1 + 2p_0 - p_0(1 + 1/p - s) > \max(\hat\lambda_0, \lambda_{\min}) \qquad \text{if } n = 1$$

provided s is sufficiently close to $1+1/p$. Moreover, $r(1+2p_0/n) < pn/(n-sp)$ if $n > sp$ and s is close to $1 + 1/p$ due to our assumption $p > p_0(n-1)/(p_0+n)$. Hence, it is sufficient to choose $\lambda_0 \in (\max(\hat\lambda_0, \lambda_{\min}), \lambda_{\max})$.

Now let $1 \le \hat\lambda_1 < 1 + \min(p_0/(n+p_0), 1/p)$.

If $n > p_0(p-1)$ (i.e. $p_0/(n+p_0) < 1/p$), put $R = r$,

$$\text{(1)} \qquad \Lambda_{\max} = 1 + p_0\frac{s-1+n(1/R-1/p)}{n+p_0}, \qquad \Lambda_{\min} = 1 + p_0(1/R - 1/p).$$

Then

$$\Lambda_{\max} = 1 + \frac{p_0}{n+p_0}(1-(s'-s)), \qquad \Lambda_{\min} = 1 + \frac{p_0}{n}(2-s') \qquad \text{if } n > 1,$$
$$\Lambda_{\max} = 1 + \frac{p_0}{n+p_0}\left(s - \frac{1}{p}\right), \qquad \Lambda_{\min} = 1 + p_0\left(1 - \frac{1}{p}\right) \qquad \text{if } n = 1.$$

In both cases, $\Lambda_{\max} > \max(\hat\lambda_1, \Lambda_{\min})$ if s is sufficiently close to $1+1/p$ so that we may choose λ_1 between these values to get (A) with $\sigma = 1$. Moreover, $R\lambda_1 < R\Lambda_{\max} < pn/(n-(s-1)p)$ if s is close to $1+1/p$ since $p > p_0(n-1)/(n+p_0)$.

If $n \le p_0(p-1)$ (i.e. $p_0/(n+p_0) \ge 1/p$), put $\tilde R = pp_0/(p_0+1)$. Then $\tilde R \in [r,p)$ so that we may choose $R \in (\tilde R, p)$. Define $\Lambda_{\min} = \Lambda_{\min}(R)$ and $\Lambda_{\max} = \Lambda_{\max}(R)$ by (1). Then $\Lambda_{\max}(R) > \Lambda_{\min}(R)$ if and only if $R > pp_0/(p_0 + p(s-1))$. Since $\Lambda_{\max}(\tilde R) = 1 + 1/p + (s-1-1/p)p_0/(n+p_0) > \hat\lambda_1$ for s sufficiently close to $1+1/p$, we have also $\Lambda_{\max}(R) > \hat\lambda_1$ for s close to $1+1/p$ and R close to $\tilde R$. Consequently, $\Lambda_{\max}(R) > \max(\hat\lambda_1, \Lambda_{\min}(R))$ if s is close to $1+1/p$, R is close to $\tilde R$, $R > pp_0/(p_0+p(s-1))$, so that we may choose λ_1 between these values to get (A) with $\sigma = 1$ (and r or λ replaced by R or λ_1, respectively). Moreover, $R\lambda_1 < R\Lambda_{\max}(R) < pn/(n-(s-1)p)$ if s is close to $1+1/p$ and R is close to $\tilde R$ since $p > p_0(n-1)/(n+p_0)$. ∎

P r o o f o f t h e T h e o r e m. Let us write $F = F_f + F_g$ where F_f or F_g represents the contribution of the function f or g, respectively, i.e.

$$F_f(t,u) = f(\cdot, t, u(\cdot), \nabla u(\cdot)),$$
$$F_g(t,u) = (\sigma + A)\mathcal{R}g(\cdot, t, u(\cdot)),$$

where $\sigma > 0$ and the operator $\mathcal{R}$ is described in [1, Section 11]. Denote by $\hat f$ and $\hat g$ the Nemytskiĭ operators defined by

$$\hat f(t,u,v) = f(\cdot, t, u(\cdot), v(\cdot)),$$
$$\hat g(t,u) = g(\cdot, t, u(\cdot)).$$

Let s, λ, r be from Lemma 2. Denoting by Tr and i the trace operator and the imbedding, respectively, the operator F_g can be written in the form (cf. [1, p. 258])

$$\begin{aligned} F_g(t,\cdot): W_{\mathcal{B}}^{s/2} &= W_p^s(\Omega,\mathbb{R}^N) \xrightarrow{\mathrm{Tr}} W_p^{s-1/p}(\partial\Omega,\mathbb{R}^N) \xrightarrow{i} L_{r\lambda}(\partial\Omega,\mathbb{R}^N) \\ &\xrightarrow{\hat{g}(t,\cdot)} L_r(\partial\Omega,\mathbb{R}^N) \xrightarrow{i} W_p^{s'-1-1/p}(\partial\Omega,\mathbb{R}^N) \xrightarrow{(\sigma+A)\mathcal{R}} W_{\mathcal{B}}^{s'/2-1} \end{aligned}$$

for some $s' \in (s, 1+1/p)$ (the imbeddings are guaranteed by the inequalities in Lemma 2). Hence, using Lemmas 1 and 2 we can estimate

$$\begin{aligned} \|F_g(t,u)\|_{W_{\mathcal{B}}^{s'/2-1}} &\le C\|\hat{g}(t,u)\|_{L_r(\partial\Omega,\mathbb{R}^N)} \le C(1+\|u\|^{\lambda}_{L_{r\lambda}(\partial\Omega,\mathbb{R}^N)}) \\ &\le C(1+\|u\|^{\lambda}_{1/r\lambda,r\lambda}) \le C(1+\|u\|^{\lambda-\varepsilon}_{p_0}\|u\|^{\varepsilon}_{s,p}) \\ &\le Cc(t)^{\lambda-\varepsilon}(1+\|u\|^{\varepsilon}_{W_{\mathcal{B}}^{s/2}}). \end{aligned}$$

Similarly, if $s, \lambda_0, \lambda_1, r, R$ are the constants from Lemma 3 then the operator F_f can be written as

$$\begin{aligned} F_f(t,\cdot): W_{\mathcal{B}}^{s/2} &= W_p^s(\Omega,\mathbb{R}^N) \xrightarrow{i\times\nabla} W_p^s(\Omega,\mathbb{R}^N) \times (W_p^{s-1}(\Omega,\mathbb{R}^N))^n \\ &\xrightarrow{i} L_{r\lambda_0}(\Omega,\mathbb{R}^N) \times (L_{R\lambda_1}(\Omega,\mathbb{R}^N))^n \xrightarrow{\hat{f}(t,\cdot,\cdot)} L_r(\Omega,\mathbb{R}^N) \xrightarrow{i} W_{\mathcal{B}}^{s'/2-1} \end{aligned}$$

together with the corresponding estimate

$$\|F_g(t,u)\|_{W_{\mathcal{B}}^{s'/2-1}} \le C(t)(1+\|u\|^{\varepsilon}_{W_{\mathcal{B}}^{s/2}}). \ \blacksquare$$

Remark 2. If we assume $f \equiv 0$ and $\sup_{t\in[0,T)} \|u(t)\|_{L_{p_0}(\partial\Omega,\mathbb{R}^N)} \le c(t)$ instead of $\sup_{t\in[0,T)} \|u(t)\|_{L_{p_0}(\Omega,\mathbb{R}^N)} \le c(t)$ in our Theorem then we may repeat the considerations above with the corresponding estimate

$$\begin{aligned} \|F_g(t,u)\|_{W_{\mathcal{B}}^{s'/2-1}} &\le C(1+\|u\|^{\lambda}_{L_{r\lambda}(\partial\Omega,\mathbb{R}^N)}) \\ &\le C(1+\|u\|^{\lambda-\varepsilon}_{L_{p_0}(\partial\Omega,\mathbb{R}^N)}\|u\|^{\varepsilon}_{W_p^{s-1/p}(\partial\Omega,\mathbb{R}^N)}) \\ &\le Cc(t)^{\lambda-\varepsilon}(1+\|u\|^{\varepsilon}_{W_{\mathcal{B}}^{s/2}}) \end{aligned}$$

under the following hypothesis on p, p_0 and $\hat{\lambda}$:

$$p > \max\left(1, \frac{p_0(n-1)}{p_0+n-1}\right), \qquad \hat{\lambda} < 1 + \frac{p_0}{n-1}.$$

This corresponds to the results of J. Filo in [4].

Example. Let $n = N = 1$, $\Omega = (-1,1)$, $f \equiv 0$, $g(x,t,u) = u^{\lambda}$, $\lambda > 1$, let $u_0 : [-1,1] \to \mathbb{R}^+$ be a smooth function, $u_0(-x) = u_0(x)$ for $x \in [-1,1]$, $u_0'(1) = u_0^{\lambda}(1) > 0$ and let the first four derivatives of u_0 restricted to the interval $[0,1]$ be non-negative. Then [3] implies that the solution u is non-negative, it blows up in a finite time $T = T(u_0)$ and choosing $p_0 \geq 1$ we get

$$\begin{aligned}\frac{1}{p_0}\frac{d}{dt}\int_0^1 u^{p_0}(x,t)\,dx &= \int_0^1 u^{p_0-1}u_t\,dx = \int_0^1 u^{p_0-1}u_{xx}\,dx \\ &= -\int_0^1 (p_0-1)u^{p_0-2}u_x^2\,dx + u^{p_0-1}u_x\Big|_{x=0}^1 \\ &\leq u^{p_0+\lambda-1}(1,t) \leq \left(\frac{\lambda-1}{T-t}\right)^{\frac{p_0+\lambda-1}{2(\lambda-1)}} = C(T-t)^{-\frac{p_0+\lambda-1}{2(\lambda-1)}}\end{aligned}$$

where we have used the estimate (2.1) from [3]. Hence $\|u(t)\|_{p_0}$ stays bounded if $\frac{p_0+\lambda-1}{2(\lambda-1)} < 1$, i.e. if $\lambda > 1+p_0$. This shows that the condition $\hat{\lambda} < 1+p_0/n$ in our Theorem is (except for the equality sign) optimal if $n = 1$.

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