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ANALYTIC HYPOELLIPTICITY AND LOCAL SOLVABILITY FOR A CLASS OF PSEUDO-DIFFERENTIAL OPERATORS WITH SYMPLECTIC CHARACTERISTICS

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1. Introduction. Let us consider a classical analytic pseudo-differential operator P of order μ on an open set Ω in $\mathbf{R}^N$ with the symbol

$$p(x,\xi) \sim p_\mu(x,\xi) + p_{\mu-1}(x,\xi) + \dots,$$

where $p_{\mu-j}(x,\xi)$ is positively homogeneous of degree $\mu - j$ with respect to ξ. We assume that the characteristic set $\Sigma = p_\mu^{-1}(0)$ of P is a symplectic real analytic submanifold of $T^*(\Omega)\backslash 0$ of codimension $2d$ and that p_μ vanishes exactly at the order m on Σ. As in Grušin [4], Sjöstrand [11] and Métivier [8], we also assume that $p_{\mu-j}$ vanishes at the order $m-2j$ on Σ for $j \le m/2$.

C^∞ and analytic hypoellipticity of this class of operators has been extensively studied by many mathematicians (see e.g., [1], [2], [4], [8], [9], [11], [13] and others). Among them Métivier [8] has proved analytic hypoellipticity of P by constructing a left parametrix when P is subelliptic with loss of $m/2$ derivatives.

In this note, we study hypoellipticity and local solvability of P at a point where the above subellipticity condition is not satisfied. We shall then construct a system of analytic pseudo-differential operators on $\mathbf{R}^{N-d}$ to which we can reduce the study of analytic hypoellipticity and local solvability of P.

Typical examples of the operators are

$$P = D_1^2 + x_1^2 D_2^2 - (1 + x_1^k) D_2, \quad \text{in } \mathbf{R}^2 \tag{1.1}$$

with $k \in \mathbf{N}$,

$$P = D_1^2 + x_1^2(D_2^2 + D_3^2) - (1 - x_2^2) D_3 - c, \quad \text{in } \mathbf{R}^3 \tag{1.2}$$

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with $c \in \mathbf{C}$. We can show that the operators (1.1) and (1.2) are analytic hypoelliptic and locally solvable for all k and all c respectively.

2. Notation and statement of the main result

2.1. *Notation.* Let Ω be an open set in $\mathbf{R}^N$. We denote by $x^* = (x, \xi)$ a point in $T^*(\Omega)\backslash 0$. For a distribution $u \in \mathcal{D}'(\Omega)$, $WF_A(u)$ is the analytic wave front set of u. We introduce the presheaf $\mathcal{C}^f_\Omega$ of micro-distributions on Ω as follows: With each open set $\omega \subset T^*(\Omega)\backslash 0$ we associate the space

$$\mathcal{C}^f_\Omega(\omega) = \mathcal{D}'(\Omega)/\{u \in \mathcal{D}'(\Omega); WF_A(u) \cap \omega = \emptyset\}.$$

We shall also use the notation:

$$\mathcal{A}_\Omega(\mathring{x}^*) = \{u \in \mathcal{D}'(\Omega); \mathring{x}^* \notin WF_A(u)\},$$
$$\mathcal{C}^f_\Omega(\mathring{x}^*) = \lim_{\substack{\rightarrow \\ \omega \ni \mathring{x}^*}} \mathcal{C}^f_\Omega(\omega) = \mathcal{D}'(\Omega)/\mathcal{A}_\Omega(\mathring{x}^*)$$

for $\mathring{x}^* \in T^*(\Omega)\backslash 0$, for the space of distributions on Ω which are micro-analytic at $\mathring{x}^*$ and for the space of germs at $\mathring{x}^*$ of micro-distributions on Ω respectively.

Let $\Omega \times \Gamma$ be a conic neighborhood of a point $(\mathring{x}, \mathring{\theta})$ in $\mathbf{R}^N \times (\mathbf{R}^n\backslash 0)$. Let $\mu \in \mathbf{R}$ and h be the reciprocal of a positive integer. A formal sum $\sum_{j=0}^\infty a_j(x, \theta)$ will be called a *polyhomogeneous analytic symbol* on $\Omega \times \Gamma$ of degree μ and step h if $a_j(x, \theta)$ is a holomorphic function on $\widetilde{\Omega} \times \widetilde{\Gamma}$, positively homogeneous of degree $\mu - jh$ with respect to θ and satisfying the estimate

$$|a_j(x, \theta)| \le C^{j+1}(j!)^h|\theta|^{m-jh}$$

for all $(x, \theta) \in \widetilde{\Omega} \times \widetilde{\Gamma}$ with C independent of j, where $\widetilde{\Omega}$ is a complex neighborhood of Ω in $\mathbf{C}^N$ and $\widetilde{\Gamma}$ is a conic complex neighborhood of Γ in $\mathbf{C}^n\backslash 0$. Then we shall write $\sum_{j=0}^\infty a_j(x, \theta) \in a\text{-}S^{\mu,h}_{\mathrm{phg}}(\Omega \times \Gamma)$.

Let us also recall the definition of analytic symbols of type (ρ, δ) introduced by Métivier [8]: For $\rho \in (0, 1]$, $\delta \in [0, 1)$ and a conic set $\Omega \times \Gamma \subset \mathbf{R}^N \times (\mathbf{R}^n\backslash 0)$, the space $a\text{-}S_{\rho,\delta}(\Omega \times \Gamma)$ of analytic symbols on $\Omega \times \Gamma$ of degree μ and type (ρ, δ) is the set of C^∞ functions $a(x, \theta)$ on $\Omega \times \Gamma$ for which there are $C > 0$ and $R > 0$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)| \le C^{|\alpha|+|\beta|+1}(1 + |\theta|)^\mu(|\alpha| + |\alpha|^{1-\delta}|\theta|^\delta)^{|\alpha|}(|\beta|/|\theta|)^{\rho|\beta|}$$

for all multi-indices α, β and all $(x, \theta) \in \Omega \times \Gamma$ such that $R|\beta| \le |\theta|$. Moreover, a symbol $a \in a\text{-}S^\mu_{\rho,\delta}(\Omega \times \Gamma)$ is said to be equivalent to 0 ($a \sim 0$) in $\Omega_0 \times \Gamma_0 \subset \Omega \times \Gamma$ if there is a constant $\varepsilon > 0$ such that

$$|\partial_x^\alpha a(x, \theta)| \le (1/\varepsilon)^{|\alpha|+1} e^{-\varepsilon|\theta|}$$

for all multi-indices α and all $(x, \theta) \in \Omega_0 \times \Gamma_0$.

Each polyhomogeneous symbol has a realization in $a\text{-}S^\mu_{1,0}(\Omega \times \Gamma)$ as follows: Let $\{\chi_j(\theta)\}_{j=0}^\infty$ be a sequence in $C^\infty(\mathbf{R}^n)$ such that $\chi_j(\theta) = 0$ for $|\theta| \le j$, $\chi_j(\theta) = 1$ for $|\theta| \ge 2j$ and there is a constant $C > 0$ for which we have $|\partial_\theta^\alpha \chi_j(\theta)| \le C^{|\alpha|}$ for all j, α such that $|\alpha| \le j$. If $\sum_{j=0}^\infty a_j \in a\text{-}S^{\mu,h}_{\mathrm{phg}}(\Omega \times \Gamma)$ then, for $\lambda > 0$ large enough,

$$a(x, \theta) = \sum_{j=0}^\infty \chi_{j+1}(\theta/\lambda) a_j(x, \theta) \tag{2.1}$$

is in $a\text{-}S^{\mu}_{1,0}(\Omega\times\Gamma)$. (See e.g. Treves [14, Chap. V] or Métivier [M, Section III].) Any symbol $a \in a\text{-}S^{\mu}_{\rho,\delta}(\Omega\times\Gamma)$ which is equivalent to the symbol (2.1) will be called a *realization* of $\sum_{j=0}^{\infty} a_j$ and we shall then write $a \sim \sum_{j=0}^{\infty} a_j$. Also, we let $\sigma_\mu(a)(x,\theta) = a_0(x,\theta)$ denote the principal symbol of a.

If $\mathring{x}^* = (\mathring{x},\mathring{\xi}) \in \Omega\times\Gamma \subset T^*(\mathbf{R}^N)\backslash 0$ and $a(x,\xi) \in a\text{-}S^{\mu}_{\rho,\delta}(\Omega\times\Gamma)$, then we define the operator

$$\mathrm{op}(a)_{\mathring{x}^*} : \mathcal{C}^f_\Omega(\mathring{x}^*) \to \mathcal{C}^f_\Omega(\mathring{x}^*)$$

via the distribution kernel

$$A_{\mathring{x}^*}(x,y) = \phi(x)\Bigl((2\pi)^{-N}\int_{\mathbf{R}^N} e^{i(x-y)\xi}a(x,\xi)g(\xi)d\xi\Bigr)\phi(y), \tag{2.2}$$

where $\phi \in C_0^\infty(\Omega)$, $\phi(x) = 1$ in a neighborhood of $\mathring{x}$ and $g(\xi) \in C^\infty(\mathbf{R}^N)$ is a cut-off function introduced in Lemma 3.1 of Métivier [8] such that $\mathrm{supp}(g) \subset \Gamma$, $g(\xi) = 1$ in a conic neighborhood of $\mathring{\xi}$ for $|\xi| \geq 2$ and there are $C > 0$, $\rho' \in (0,1)$ for which we have

$$|\partial_\xi^\alpha g(\xi)| \leq C^{|\alpha|+1}(|\alpha|/|\xi|)^{\rho'|\alpha|} \tag{2.3}$$

for all α, ξ such that $|\alpha| \leq |\xi|$.

The operator $\mathrm{op}(a)_{\mathring{x}^*}$ is well defined; that is, independent of the choice of the cut-off functions ϕ and g in (2.2). Moreover, when $a(x,\xi)$ is a realization of a formal symbol $\sum_{j=0}^{\infty} a_j(x,\xi)$, $\mathrm{op}(a)_{\mathring{x}^*}$ is also independent of the choice of the realization. Then $a(x,D_x) = \mathrm{op}(a)$ which stands for $\bigsqcup_{\mathring{x}^*\in\Omega\times\Gamma}\mathrm{op}(a)_{\mathring{x}^*}$ is called an analytic pseudo-differential operator on $\Omega\times\Gamma$ with the symbol $a(x,\xi)$ (or $\sum_{j=0}^{\infty} a_j(x,\xi)$).

2.2. *Statement of the result.* Let Σ be a symplectic submanifold of codimension $2d$ in a conic set $\omega \subset T^*(\mathbf{R}^N)\backslash 0$. We consider a classical analytic pseudo-differential operator P of order μ whose symbol $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{\mu-j}(x,\xi)$ defined on ω is such that $p_{\mu-j}$ is homogeneous of degree $\mu - j$, and vanishes to order $m - 2j$ on Σ for $j \leq m/2$.

After transforming P by a suitable elliptic Fourier integral operator, we may suppose Σ is given by the equation

$$x_1 = \ldots = x_d = 0; \qquad \xi_1 = \ldots = \xi_d = 0.$$

Henceforth, we write $t_i = x_i$, $\tau_i = \xi_i$ for $i = 1,\ldots,d$ and $y_i = x_{d+i}$, $\eta_i = \xi_{d+i}$ for $i = 1,\ldots,n(= N-d)$ and set

$$\iota : T^*(\mathbf{R}^n)\backslash 0 \ni (y,\eta) \mapsto (0,y,0,\eta) \in T^*(\mathbf{R}^N)\backslash 0.$$

In this coordinate, Σ can be identified with $\iota(T^*(\mathbf{R}^n)\backslash 0)$ in ω and P has the form

$$P = \sum_{|\alpha|+|\beta|\leq m} t^\alpha c_{\alpha\beta}(x,D_x)D_t^\beta, \qquad c_{\alpha\beta} \in a\text{-}S^{\mu-m/2+|\alpha|/2-|\beta|/2,1/2}_{\mathrm{phg}}(\omega). \tag{2.4}$$

For $\mathring{x}^* = \iota(\mathring{y}^*) = (0,\mathring{y},0,\mathring{\eta}) \in \Sigma\cap\omega$, we set

$$\sigma^0_{\Sigma}(P)_{\mathring{x}^*}(t,\tau) = \sum_{|\alpha|+|\beta|=m} \sigma_{\mu-m/2+|\alpha|/2-|\beta|/2}(c_{\alpha\beta})(\mathring{x}^*)t^\alpha\tau^\beta,$$

$$\widehat{\sigma}_{\Sigma}(P)_{\mathring{x}^*}(t,D_t) = \sum_{|\alpha|+|\beta|\leq m} \sigma_{\mu-m/2+|\alpha|/2-|\beta|/2}(c_{\alpha\beta})(\mathring{x}^*)t^\alpha D_t^\beta$$

and assume

$$\exists C > 0 \quad \text{such that} \quad |\sigma^0_\Sigma(P)_{\mathring{x}^*}(t,\tau)| \geq C(|t| + |\tau|)^m. \tag{2.5}$$

With this assumption $\widehat{\sigma}_\Sigma(P)_{\mathring{x}^*}$ becomes a Fredholm operator from $\mathcal{S}'$ to $\mathcal{S}'$, and if $\mathrm{Ker}(\widehat{\sigma}_\Sigma(P)_{\mathring{x}^*}) \cap \mathcal{S} = \{0\}$ (resp. $\mathrm{Coker}(\widehat{\sigma}_\Sigma(P)_{\mathring{x}^*}) \cap \mathcal{S} = \{0\}$) then P (resp. P^*) is subelliptic with loss of $m/2$ derivatives. Our interest is now focusing at a point where this subellipticity condition of P or P^* is not satisfied. So we set

$$k_+ = \dim(\mathrm{Ker}(\widehat{\sigma}_\Sigma(P)_{\mathring{x}^*}) \cap \mathcal{S}), \qquad k_- = \dim(\mathrm{Coker}(\widehat{\sigma}_\Sigma(P)_{\mathring{x}^*}) \cap \mathcal{S}).$$

The main theorem of this note is

THEOREM 2.1. *Let P be an operator of the form* (2.4) *satisfying* (2.5). *Then there exist a $k_- \times k_+$-matrix of pseudo-differential operators*

$$M(y, D_y) : (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_+} \to (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_-}$$

and two operators

$$H^+ : \ (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_+} \to \mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*) \quad \textit{and} \quad H^{-*} : \ \mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*) \to (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_-}$$

for which we have the isomorphisms:

$$\begin{aligned} H^+ &: \ \mathrm{Ker}(M : (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_+} \to (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_-}) \\ &\qquad \xrightarrow{\sim} \ \mathrm{Ker}(P : \mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*) \to \mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*)) \\ H^{-*} &: \ \mathrm{Coker}(P : \mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*) \to \mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*)) \\ &\qquad \xrightarrow{\sim} \ \mathrm{Coker}(M : (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_+} \to (\mathcal{C}^f_{\mathbf{R}^n}(\mathring{y}^*))^{k_-}). \end{aligned}$$

Remark. Grigis-Rothschild [3] have treated the case $c_{\alpha\beta} = c_{\alpha\beta}(D_y)$ and obtained the same result as above. See also Kashiwara-Kawai-Oshima [7] and Stein [12].

3. Operator valued symbols

3.1. *Symbol spaces.* Let $\mathring{y}^* = (\mathring{y}, \mathring{\eta}) \in \mathbf{R}^n \times (\mathbf{R}^n \backslash 0)$ $(|\mathring{\eta}| = 1)$. For $\rho > 0$, we consider a complex neighborhood of $\mathring{y}^*$ of the form

$$\omega_\rho = \{(y, \eta) \in \mathbf{C}^n \times (\mathbf{C}^n \backslash 0); |y - \mathring{y}| < \rho, |\eta - \mathring{\eta}| < \rho\}$$

and let $\widetilde{\omega}_\rho$ denote the cone generated by ω_ρ; that is,

$$\widetilde{\omega}_\rho = \{(y, \eta) \in \mathbf{C}^n \times (\mathbf{C}^n \backslash 0); \ |y - \mathring{y}| < \rho, |\eta/|\eta| - \mathring{\eta}| < \rho\}.$$

Let $B = B(\lambda)$ be some Banach space whose norm may depend on λ.

DEFINITION 3.1. Let $\mu \in \mathbf{R}$. The space $\mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; B)$ of *B-valued homogeneous symbols* (also denoted by $B^{(\mu)}_\rho$ for short) and the space $S^{\mu,h}_{\mathrm{phg}}(\widetilde{\omega}_\rho; B)$ of *B-valued polyhomogeneous symbols* are defined by:

(1) $p(y, \eta) \in \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; B)$ if and only if $p(y, \eta)$ is a holomorphic function defined on $\widetilde{\omega}_\rho$ with values in $B(|\eta|)$ which satisfies

$$\|p(y, \lambda\eta)\|_{B(\lambda)} = \lambda^\mu \|p(y, \eta)\|_{B(1)} \qquad \text{for } (y, \eta) \in \omega_\rho$$

and

$$\|p\|_{B^{(\mu)}_\rho} \overset{\text{def}}{=} \sup_{(y,\eta) \in \omega_\rho} \|p(y, \eta)\|_{B(1)} < +\infty.$$

(2) $\sum_{j=0}^{\infty} p_j(y,\eta) \in S^{\mu,h}_{\mathrm{phg}}(\widetilde{\omega}_\rho; B)$ if and only if $p_j(y,\eta) \in \mathcal{O}^{(\mu-jh)}(\widetilde{\omega}_\rho; B)$ and there exists a $C > 0$ such that

$$\|p_j\|_{B_\rho^{(\mu-jh)}} \le C^{j+1}(j!)^h.$$

3.2. *Banach spaces and estimates.* Let us now introduce several Banach spaces following Métivier [8] and quote some of their properties from [8].

Definition 3.2. $\mathcal{A}^m(\lambda)$ denotes the space of differential operators on $\mathbf{R}^d$ of the form

$$A(t, D_t) = \sum_{|\alpha|+|\beta| \le m} C_{\alpha\beta} t^\alpha {D_t}^\beta, \qquad C_{\alpha\beta} \in \mathbf{C},$$

with the norm $\|A\|_{\mathcal{A}^m(\lambda)} = \sum_{\alpha,\beta} |C_{\alpha\beta}| \lambda^{(|\beta|-|\alpha|)/2}$.

Definition 3.3. $\mathcal{M}^{\pm}$ denotes the space of $k_- \times k_+$-matrices $M = (m_{ij}) \in L(\mathbf{C}^{k_+}, \mathbf{C}^{k_-})$ with the norm $\|M\|_{\mathcal{M}^{\pm}(\lambda)} = (\sum |m_{ij}|^2)^{1/2}$ independent of λ.

Let t denote a point in $\mathbf{R}^d$. We consider the operators

$$T_j = T_j(\lambda) = \lambda^{-\frac{1}{2}} \frac{\partial}{\partial t_j}, \quad T_{-j} = T_{-j}(\lambda) = i\lambda^{1/2} t_j, \qquad j = 1, \dots, d.$$

For a sequence $I = (j_1, \dots, j_k) \in \{\pm 1, \dots, \pm d\}^k$ we write $|I| = k$ and $T_I = T_{j_1}, \dots, T_{j_k}$. If L is an operator acting from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ we write

$$(\operatorname{ad} T_j)(L) = [T_j, L] = T_j L - L T_j$$

and because the $\operatorname{ad} T_j$'s commute, we write for a multi-index $\alpha = (\alpha_j)_{j=\pm 1,\dots,\pm d} \in \mathbf{N}^{2d}$,

$$(\operatorname{ad} T)^\alpha = \prod_j (\operatorname{ad} T_j)^{\alpha_j}.$$

Also we write $\|L\|_0$ for the operator-norm of L from $L^2(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$.

Definition 3.4. Let m be a non-negative integer. For a real $R > 0$, $\mathcal{L}^m_R(\lambda)$ denotes the space of the operators for which there is a constant C such that for all multi-indices $\alpha \in \mathbf{N}^{2d}$ and for all I, J with $|I| + |J| \le |\alpha| + m$,

$$\|T_I (\operatorname{ad} T_j)^\alpha (L) T_J\|_0 \le C|\alpha|! R^{|\alpha|}.$$

Clearly $\mathcal{L}^m_R(\lambda)$ becomes a Banach space and there exists $C > 0$ such that

$$\|AL\|_{\mathcal{L}^0_R(\lambda)} \le C\|A\|_{\mathcal{A}^m(\lambda)} \|L\|_{\mathcal{L}^m_R(\lambda)} \tag{3.1}$$

for all $A \in \mathcal{A}^m(\lambda)$ and $L \in \mathcal{L}^m_R(\lambda)$.

For an operator K from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ we write $K(t,s)$ for its distribution kernel. We also introduce the operator $\widetilde{K}$ induced from K via the Fourier transform; that is,

$$\widetilde{K}\widehat{u} = \widehat{Ku}.$$

Definition 3.5. For $\varepsilon > 0$, $\mathcal{B}_\varepsilon(\lambda)$ is the space of Hilbert-Schmidt operators K such that for all $j = 1, \dots, d$,

$$\|e^{\varepsilon\lambda\phi_j(t,s)} K(t,s)\|_{L^2(\mathbf{R}^d \times \mathbf{R}^d)} < +\infty, \tag{3.2}$$

$$\|e^{\varepsilon\phi_j(\tau,\sigma)/\lambda} \widetilde{K}(\tau,\sigma)\|_{L^2(\mathbf{R}^d \times \mathbf{R}^d)} < +\infty, \tag{3.3}$$

where $\phi_j(t,s) = |t_j|t_j| - s_j|s_j||$. The norm of $\mathcal{B}_\varepsilon(\lambda)$ is the maximum for $j = 1, \ldots, d$ of the norms in (3.2) and (3.3).

The space $\mathcal{B}_\varepsilon(\lambda)$ plays an important role in the construction of a relative parametrix. The crucial points are

LEMMA 3.6 (Métivier [8], Proposition 2.8). *If $m > d$ then for all $R > 0$ there exist $\varepsilon > 0$ and C such that*

$$\|K\|_{\mathcal{B}_\varepsilon(\lambda)} \leq C\|K\|_{\mathcal{L}^m_R(\lambda)}$$

for all $K \in \mathcal{L}^m_R(\lambda)$.

LEMMA 3.7 (loc. cit., Proposition 2.9). *For all $R > 0$, there exist $\varepsilon_0 > 0$ and C such that for all $\varepsilon \in (0, \varepsilon_0]$,*

$$\|LK\|_{\mathcal{B}_\varepsilon(\lambda)} \leq C\|L\|_{\mathcal{L}^0_R(\lambda)}\|K\|_{\mathcal{B}_\varepsilon(\lambda)}$$

for all $L \in \mathcal{L}^0_R(\lambda)$ and all $K \in \mathcal{B}_\varepsilon(\lambda)$.

LEMMA 3.8 (loc. cit., Proposition 2.10). *There exists a constant M_0 such that for all $0 < \varepsilon' < \varepsilon \leq 1$ and all $j = \pm 1, \ldots, \pm d$,*

$$\|(\operatorname{ad} T_j)(K)\|_{\mathcal{B}_{\varepsilon'}(\lambda)} \leq \left(\frac{M_0}{\varepsilon - \varepsilon'}\right)^{1/2} \|K\|_{\mathcal{B}_\varepsilon(\lambda)}$$

for all $K \in \mathcal{B}_\varepsilon(\lambda)$.

For the operator K of kernel $K(t,s)$, we define its symbol $k = \sigma(K)$ by

$$k(t,\tau) = \int_{\mathbf{R}^d} K(t, t-s)e^{-is\tau} ds.$$

Then

$$Ku(t) = k(t, D_t)u(t) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{it\tau} k(t,\tau)\widehat{u}(\tau)\, d\tau.$$

LEMMA 3.9. *For all $\varepsilon > 0$, there exists a $C > 0$ such that for all $(\alpha, \beta) \in \mathbf{R}^d \times \mathbf{R}^d$,*

$$\sup_{(t,\tau)\in\mathbf{R}^{2d}} |\partial_t^\alpha \partial_\tau^\beta \sigma(K)(t,\tau)| \leq C^{j+1}(|\alpha| + |\beta|)^{(|\alpha|+|\beta|)/2} \lambda^{(|\alpha|-|\beta|)/2} \|K\|_{\mathcal{B}_\varepsilon(\lambda)}$$

for all $K \in \mathcal{B}_\varepsilon(\lambda)$.

We also introduce the space of *Hermite operators*. First we define its symbol space.

DEFINITION 3.10. For $\varepsilon > 0$, $\mathcal{H}_\varepsilon(\lambda)$ is the space of functions $h(t) \in \mathcal{S}(\mathbf{R}^d)$ such that for all $j = 1, \ldots, d$,

$$\|e^{\lambda\varepsilon t_j^2} h(t)\|_{L^2(\mathbf{R}^d)} < +\infty, \tag{3.4}$$

$$\|e^{\varepsilon\tau_j^2/\lambda} \widehat{h}(\tau)\|_{L^2(\mathbf{R}^d)} < +\infty. \tag{3.5}$$

The norm of $\mathcal{H}_\varepsilon(\lambda)$ is the maximum for $j = 1, \ldots, d$ of the norms in (3.4) and (3.5).

For $H = (h_1, \ldots, h_k) \in (\mathcal{H}_\varepsilon(\lambda))^k$, define the operators H and H^* by

$$H : \mathbf{C}^k \ni (z_l)_{l=1}^k \mapsto \sum\nolimits_{l=1}^k z_l h_l(t) \in \mathcal{S}(\mathbf{R}^d), \tag{3.6}$$

$$H^* : \mathcal{S}'(\mathbf{R}^d) \ni u(t) \mapsto \Big(\int_{\mathbf{R}^d} \overline{h_l(t)} u(t) dt \Big)_{l=1}^k \in \mathbf{C}^k, \tag{3.7}$$

where $\overline{h_l(t)}$ is the complex conjugate of $h_l(t)$. We denote by $\mathcal{H}_\varepsilon^k(\lambda)$ and $\mathcal{H}_\varepsilon^{k*}(\lambda)$ the spaces of operators of the form (3.6) and (3.7) respectively. The norm in them is defined by

$$\|H\|_{\mathcal{H}_\varepsilon^k(\lambda)} = \|H^*\|_{\mathcal{H}_\varepsilon^{k*}(\lambda)} = \big(\textstyle\sum_{l=1}^k \|h_l\|^2_{\mathcal{H}_\varepsilon(\lambda)}\big)^{1/2}$$

and we write $\sigma(H) = \sigma(H^*) = (h_1, \dots, h_k)$.

By definition, we have

Lemma 3.11. *Let $k, k' \in \mathbf{N}$ and $\varepsilon > 0$. If $K \in \mathcal{B}_\varepsilon(\lambda)$, $H_1, H_2 \in \mathcal{H}_\varepsilon^k(\lambda)$ and $H_3 \in \mathcal{H}_\varepsilon^{k'}(\lambda)$ then $KH_1 \in \mathcal{B}_\varepsilon(\lambda)$, $H_2H_1^* \in \mathcal{B}_\varepsilon(\lambda)$ and $H_1^*H_3 \in L(\mathbf{C}^{k'}, \mathbf{C}^k)$. Moreover,*

$$\begin{aligned}
&\|KH_1\|_{\mathcal{H}_\varepsilon^k(\lambda)} \le \|K\|_{\mathcal{B}_\varepsilon(\lambda)} \|H_1\|_{\mathcal{H}_\varepsilon^k(\lambda)},\\
&\|H_2H_1^*\|_{\mathcal{B}_\varepsilon(\lambda)} \le \|H_2\|_{\mathcal{H}_\varepsilon^k(\lambda)} \|H_1\|_{\mathcal{H}_\varepsilon^k(\lambda)},\\
&\|H_1^*H_3\|_{L(\mathbf{C}^{k'}, \mathbf{C}^k)} \le \|H_1\|_{\mathcal{H}_\varepsilon^k(\lambda)} \|H_3\|_{\mathcal{H}_\varepsilon^{k'}(\lambda)}.
\end{aligned}$$

Also, the following lemma has been proved in Métivier [8, Lemma A.3].

Lemma 3.12. *There exists a constant M_0 such that for all $0 < \varepsilon' < \varepsilon \le 1$ and all $j = \pm 1, \dots, \pm d$,*

$$\|T_j(h)\|_{H_{\varepsilon'}(\lambda)} \le \Big(\frac{M_0}{\varepsilon - \varepsilon'}\Big)^{1/2} \|h\|_{H_\varepsilon(\lambda)}$$

for all $h \in H_\varepsilon(\lambda)$.

Finally, we set $\mathcal{H}_\varepsilon^{\pm}(\lambda) = \mathcal{H}_\varepsilon^{k_\pm}(\lambda)$ and $\mathcal{H}_\varepsilon^{\pm *}(\lambda) = \mathcal{H}_\varepsilon^{k_\pm *}(\lambda)$.

4. Construction of parametrix

4.1. *The case $c_{\alpha\beta} = c_{\alpha\beta}(y, D_y)$.* Let $P = \sum_{|\alpha|+|\beta|\le m} t^\alpha c_{\alpha\beta}(x, D_x) D_t^\beta$ be an operator of the form (2.4) satisfying (2.5). Multiplying P by an elliptic factor we may assume $\mu = m/2$. Also we suppose $m \ge d+1$ in the construction of a parametrix. Otherwise we replace P by $P(P^*P+1)^k$ for some integer k. Because $(P^*P+1)^k$ is isomorphic on $\mathcal{C}^f_{\mathbf{R}^N}(\mathring{x}^*)$, this does not affect the conclusion of Theorem 2.1. Moreover, we assume in this section

$$c_{\alpha\beta}(x, \xi) = c_{\alpha\beta}(y, \eta) \qquad \text{independent of } t, \tau. \tag{4.1}$$

Then $c_{\alpha\beta}(y^*) = \sum_{j=0}^\infty c_{\alpha\beta,j}(y^*) \in S_{\mathrm{phg}}^{(|\alpha|-|\beta|)/2, 1/2}(\widetilde{\omega}_\rho)$, where $\widetilde{\omega}_\rho$ is a conic complex neighborhood of $\mathring{y}^* = (\mathring{y}, \mathring{\eta}) = \iota^{-1}(\mathring{x}^*)$ generated by

$$\omega_\rho = \{(y, \eta) \in \mathbf{C}^n \times (\mathbf{C}^n \backslash 0);\ |y - \mathring{y}| < \rho, |\eta - \mathring{\eta}| < \rho\}$$

and $c_{\alpha\beta,j}$ is positively homogeneous of degree $(|\alpha| - |\beta| - j)/2$.

Now, we set

$$P_j(y^*) = \sum_{|\alpha|+|\beta|\le m} c_{\alpha\beta,j}(y^*) t^\alpha D_t{}^\beta.$$

Then $P_j \in \mathcal{O}^{(-j/2)}(\widetilde{\omega}_\rho; \mathcal{A}^m)$ and

$$P(y^*) \left(\overset{\text{def}}{=} \sum_{|\alpha|+|\beta| \le m} c_{\alpha\beta}(y^*) t^\alpha D_t{}^\beta\right) = \sum_{j=0}^{\infty} P_j(y^*) \in S^{0,1/2}_{\text{phg}}(\widetilde{\omega}_\rho; \mathcal{A}^m).$$

For $y^* \in \widetilde{\omega}_\rho$, we let $P_0^*(y^*) = (P_0(\overline{y}^*))^*$ and write $P_0^* P_0(y^*) = P_0^*(y^*) P_0(y^*)$ and $P_0 P_0^*(y^*) = P_0(y^*) P_0^*(y^*)$. By the assumption (2.4), $P_0^* P_0(\mathring{y}^*)$ and $P_0 P_0^*(\mathring{y}^*)$ are Fredholm operators from $\mathcal{S}'(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ together with $P_0(\mathring{y}^*)$. (Note that $P_0(\mathring{y}^*) = \widehat{\sigma}_\Sigma(P)_{\mathring{y}^*}$.)

Let $\gamma \subset \mathbf{C}$ be a positively oriented closed curve enclosing only the 0-eigenvalue of $P_0^* P_0(\mathring{y}^*)$ and $P_0 P_0^*(\mathring{y}^*)$. If $\rho > 0$ is sufficiently small then for all $y^* \in \widetilde{\omega}_\rho$ and all $\zeta \in \gamma$, $P_0^* P_0(y^*) - \zeta$ and $P_0 P_0^*(y^*) - \zeta$ are invertible. So we set for $y^* \in \widetilde{\omega}_\rho$,

$$\begin{aligned}
Q_0(y^*) &= \frac{1}{2\pi i}\Big(\int_\gamma \zeta^{-1}(P_0^* P_0(y^*) - \zeta)^{-1} d\zeta\Big) P_0^*(y^*),\\
\Pi_0^+(y^*) &= \frac{-1}{2\pi i}\int_\gamma (P_0^* P_0(y^*) - \zeta)^{-1} d\zeta,\\
\Pi_0^-(y^*) &= \frac{-1}{2\pi i}\int_\gamma (P_0 P_0^*(y^*) - \zeta)^{-1} d\zeta,\\
\mathrm{E}_0^\pm(y^*) &= \Pi_0^\pm(\mathcal{S}'(\mathbf{R}^d)).
\end{aligned}$$

Note that $\Pi_0^+(\mathring{y}^*)$ (resp. $\Pi_0^-(\mathring{y}^*)$) are the projections onto $\mathrm{Ker}(P_0(\mathring{y}^*))$ (resp. $\mathrm{Ker}(P_0^*(\mathring{y}^*)) \simeq \mathrm{Coker}(P_0(\mathring{y}^*))$). Also, from the choice of ρ, $\dim(E_0^\pm(y^*))$ is constant for $y^* \in \widetilde{\omega}_\rho$, hence equal to $k_\pm$.

Then we have

PROPOSITION 4.1 (Métivier [8], Proposition 2.3). *There exist $\rho_0 > 0$ and $R_0 > 0$ such that*

$$Q_0(y^*) \in \mathcal{O}^{(0)}(\widetilde{\omega}_{\rho_0}; \mathcal{L}^m_{R_0}).$$

PROPOSITION 4.2. *We can choose bases $\{h^+_{0,l}(t;y^*)\}_{l=1}^{k_+}$ (resp. $\{h^-_{0,l}(t;y^*)\}_{l=1}^{k_-}$) of $\mathrm{E}_0^+(y^*)$ (resp. $\mathrm{E}_0^-(y^*)$) in $L^2(\mathbf{R}^d)$ which are orthonormal if y^* is real and such that*

$$h^\pm_{0,l}(t;y^*) \in \mathcal{O}^{(0)}(\widetilde{\omega}_{\rho_0}; H_{\varepsilon_0}), \qquad l = 1, \dots, k_\pm,$$

for some $\rho_0 > 0$ and $\varepsilon_0 > 0$.

Proof. It follows from Theorem 3.9 in Chap. VII of Kato [6] that we can choose bases $\{h^+_{0,l}(t;y^*)\}_{l=1}^{k_+}$ (resp. $\{h^-_{0,l}(t;y^*)\}_{l=1}^{k_-}$) of $\mathrm{E}_0^+(y^*)$ (resp. $\mathrm{E}_0^-(y^*)$), depending holomorphically on $y^* \in \widetilde{\omega}_{\rho_0}$, orthonormal for real y^*. Then, for each fixed y^*, $h^\pm_{0,l}(t;y^*)$ are in $\mathcal{H}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. (See e.g. Melin [9, Lemma A.1].) ∎

Let $\{h^\pm_{0,l}(t;y^*)\}_{l=1}^{k_-}$ be chosen as above and define the operators $H_0^\pm \in \mathcal{H}^\pm_{\varepsilon_0}$ and $H_0^{\pm *} \in \mathcal{H}^{\pm *}_{\varepsilon_0}$ by

$$\begin{aligned}
H_0^\pm &: \ \mathbf{C}^k \ni (z_l)_{l=1}^{k_\pm} \mapsto \sum\nolimits_{l=1}^{k_\pm} z_l h^\pm_{0,l}(t;y^*) \in \mathcal{S}(\mathbf{R}^d),\\
H_0^{\pm *} &: \ \mathcal{S}'(\mathbf{R}^d) \ni u(t) \mapsto \Big(\int_{\mathbf{R}^d} \overline{h^\pm_{0,l}(t,\overline{y}^*)} u(t) dt\Big)_{l=1}^{k_\pm} \in \mathbf{C}^{k_\pm}.
\end{aligned}$$

Then we have

$$\Pi_0^{\pm}(y^*) = H_0^{\pm}(y^*)H_0^{\pm *}(y^*).$$

Let us also introduce a matrix

$$M_0(y^*) = -H_0^{-*}(y^*)P_0(y^*)H_0^{+}(y^*).$$

Then, by Lemma 3.11 and Lemma 3.12,

$$M_0(y^*) \in \mathcal{O}^{(0)}(\widetilde{\omega}_{\rho_0}; \mathcal{M}^{\pm}) \tag{4.2}$$

and we have

PROPOSITION 4.3. *There is a $\rho_0 > 0$ such that for all $y^* \in \widetilde{\omega}_{\rho_0}$,*

$$\begin{pmatrix} P_0(y^*) & H_0^-(y^*) \\ H_0^{+*}(y^*) & 0 \end{pmatrix} \begin{pmatrix} Q_0(y^*) & H_0^+(y^*) \\ H_0^{-*}(y^*) & M_0(y^*) \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{\mathcal{S}'(\mathbf{R}^d)} & 0 \\ 0 & \mathrm{Id}_{\mathbf{C}^{k_+}} \end{pmatrix},$$
$$\begin{pmatrix} Q_0(y^*) & H_0^+(y^*) \\ H_0^{-*}(y^*) & M_0(y^*) \end{pmatrix} \begin{pmatrix} P_0(y^*) & H_0^-(y^*) \\ H_0^{+*}(y^*) & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{\mathcal{S}'(\mathbf{R}^d)} & 0 \\ 0 & \mathrm{Id}_{\mathbf{C}^{k_-}} \end{pmatrix}.$$

P r o o f. This is an easy consequence of the resolvent equation. (See e.g. Kato [6, I-§5.3].)

We write

$$L(y^*) = \begin{pmatrix} P(y^*) & H_0^-(y^*) \\ H_0^{+*}(y^*) & 0 \end{pmatrix} = \sum_{j=0}^{\infty} L_j(y^*),$$

where

$$L_0(y^*) = \begin{pmatrix} P_0(y^*) & H_0^-(y^*) \\ H_0^{+*}(y^*) & 0 \end{pmatrix}, \quad L_j(y^*) = \begin{pmatrix} P_j(y^*) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } j \geq 1$$

and construct a right parametrix $E(y^*) = \sum_{j=0}^{\infty} E_j(y^*)$ of $L(y^*)$ so that

$$L\#E = \sum_{l=0}^{\infty} \sum_{i+j+2|\alpha|=l} \frac{1}{\alpha!}(\partial_\eta^\alpha L_i)(D_y^\alpha E_j) = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix}, \tag{4.3}$$

where $\#$ denotes the pseudo-differential composition of symbols in (y, η).

By Proposition 4.3 we can take

$$E_0(y^*) = \begin{pmatrix} Q_0(y^*) & H_0^+(y^*) \\ H_0^{-*}(y^*) & M_0(y^*) \end{pmatrix}.$$

Then, for $j \geq 1$, E_l's are determined recurrently by

$$E_l(y^*) = -\sum_{\substack{i+j+2|\alpha|=l \\ j \leq l-1}} \frac{1}{\alpha!} E_0(y^*)(\partial_\eta^\alpha L_i(y^*))(D_y^\alpha E_j(y^*)). \tag{4.4}$$

We want to show $\sum_{j=0}^{\infty} E_j$ has a meaning as a formal sum of operator valued *analytic* pseudo-differential operators. For this purpose we introduce a norm for E_j as follows:

DEFINITION 4.4. For $\varepsilon > 0$ and $\rho > 0$, $\mathcal{E}_{\varepsilon,\rho}^{(\mu)}$ denotes the space of operator valued symbols on $\widetilde{\omega}_\rho$ of the form

$$E(y^*) = \begin{pmatrix} Q(y^*) & H^+(y^*) \\ H^{-*}(y^*) & M(y^*) \end{pmatrix} \in \begin{pmatrix} \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{B}_\varepsilon) & \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{H}_\varepsilon^+) \\ \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{H}_\varepsilon^{-*}) & \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{M}^{\pm}) \end{pmatrix}.$$

The norm of $\mathcal{E}^{(\mu)}_{\varepsilon,\rho}$ is defined by

$$\|E\|_{\mathcal{E}^{(\mu)}_{\varepsilon,\rho}} = \max\{\|Q\|_{\mathcal{B}^{(\mu)}_{\varepsilon,\rho}},\ \|H^+\|_{\mathcal{H}^{+,(\mu)}_{\varepsilon,\rho}},\ \|H^{-*}\|_{\mathcal{H}^{-*,(\mu)}_{\varepsilon,\rho}},\ \|M\|_{\mathcal{M}^{\pm,(\mu)}_{\rho}}\}.$$

We have

Lemma 4.5. *Suppose $m \geq d+1$. Then there exist ε_0, ρ_0 and C such that for all $0<\rho<\rho_0$,*

$$\|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}} \leq C\left(\frac{Cj}{\rho_0-\rho}\right)^{j/2} \tag{4.5}$$

for $j=0,1,2,\dots$

Proof. By Proposition 4.1, Q_0 is in $\mathcal{O}^{(0)}(\widetilde{\omega}_{\rho_0};\mathcal{L}^m_{R_0})$ for some $\rho_0>0, R_0>0$. Then by Lemma 3.6 there is a ε_0 for which we have $Q_0 \in \mathcal{O}^{(0)}(\widetilde{\omega}_{\rho_0};\mathcal{B}_{\varepsilon_0})$. Hence, together with Proposition 4.2 and (4.2), E_0 is in $\mathcal{E}^{(0)}_{\varepsilon_0,\rho_0}$ by decreasing ε_0 if necessary. Here, for later convenience, we suppose ε_0 is so chosen that Lemma 3.7 holds. Also we can assume the following estimates are satisfied for a constant C_0:

$$\|\partial^\alpha_\eta P_i\|_{\mathcal{A}^{m,(-|\alpha|-i/2)}_{\rho_0}} \leq C_0^{|\alpha|+i/2+1}\alpha!(i!)^{1/2}, \tag{4.6}$$

$$\|\partial^\alpha_\eta H_0^\pm\|_{\mathcal{H}^{\pm,(-|\alpha|)}_{\varepsilon_0,\rho_0}} \leq C_0^{|\alpha|+1}\alpha!, \tag{4.7}$$

$$\|Q_0\|_{\mathcal{L}^{m,(0)}_{R_0,\rho_0}} \leq C_0, \qquad \|Q_0\|_{\mathcal{B}^{(0)}_{\varepsilon_0,\rho_0}} \leq C_0, \tag{4.8}$$

$$\|M_0\|_{\mathcal{M}^{\pm,(0)}_{\rho_0}} \leq C_0. \tag{4.9}$$

For $j\geq 1$, we shall prove (4.5) by induction. First we note that if $E_j \in \mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}$ then, by Cauchy's inequality, there is an M_0 which depends only on d such that for all $0<\rho'<\rho<\rho_0$,

$$\|D^\alpha_y E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho'}} \leq \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|}\|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}}. \tag{4.10}$$

We write (4.4) as

$$E_l = -\sum_{k=1}^{l}\mathcal{M}_k(E_{l-k}),$$

where

$$\mathcal{M}_k(E_j) = \sum_{2|\alpha|+i=k}\frac{1}{\alpha!}E_0\big(\partial^\alpha_\eta L_i\big)\big(D^\alpha_y E_j\big).$$

Then we have

$$\begin{aligned}
\mathcal{M}^{11}_k(E_j) = &\sum_{2|\alpha|+i=k}\frac{1}{\alpha!}Q_0\partial^\alpha_\eta P_i D^\alpha_y Q_j\\
&+\sum_{2|\alpha|=k}\frac{1}{\alpha!}(H_0^+\partial^\alpha_\eta H_0^{+*}D^\alpha_y Q_j + Q_0\partial^\alpha_\eta H_0^- D^\alpha_y H_j^{-*}),\\
\mathcal{M}^{12}_k(E_j) = &\sum_{2|\alpha|+i=k}\frac{1}{\alpha!}Q_0\partial^\alpha_\eta P_i D^\alpha_y H_j^+\\
&+\sum_{2|\alpha|=k}\frac{1}{\alpha!}(H_0^+\partial^\alpha_\eta H_0^{+*}D^\alpha_y H_j^+ + Q_0\partial^\alpha_\eta H_0^- D^\alpha_y M_j),
\end{aligned}$$

$$\mathcal{M}_k^{21}(E_j) = \sum_{2|\alpha|+i=k} \frac{1}{\alpha!} H_0^{-*} \partial_\eta^\alpha P_i D_y^\alpha Q_j + \sum_{2|\alpha|=k} \frac{1}{\alpha!} (M_0 \partial_\eta^\alpha H_0^{+*} D_y^\alpha Q_j + H_0^{-*} \partial_\eta^\alpha H_0^- D_y^\alpha H_j^{-*}),$$

$$\mathcal{M}_k^{22}(E_j) = \sum_{2|\alpha|+i=k} \frac{1}{\alpha!} H_0^{-*} \partial_\eta^\alpha P_i D_y^\alpha H_j^+ + \sum_{2|\alpha|=k} \frac{1}{\alpha!} (M_0 \partial_\eta^\alpha H_0^{+*} D_y H_j^+ + H_0^{-*} \partial_\eta^\alpha H_0^- D_y^\alpha M_j).$$

We shall show that there exists an M such that for all $0 < \rho' < \rho < \rho_0$,

$$\|\mathcal{M}_k(E_j)\|_{\mathcal{E}^{(-j/2-k/2)}_{\varepsilon_0,\rho'}} \le M\left(\frac{Mk}{\rho-\rho'}\right)^{k/2} \|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}}. \tag{4.11}$$

By Lemmas 3.7 and 3.11, $\mathcal{M}_k^{11}(E_j)$ is in $\mathcal{O}^{(-j/2-k/2)}(\widetilde{\omega}_{\rho'}; \mathcal{B}_{\varepsilon_0})$ and we have

$$\begin{aligned}
&\|\mathcal{M}_k^{11}(E_j)\|_{\mathcal{B}^{(-j/2-k/2)}_{\varepsilon_0,\rho'}} \\
&\quad\le \sum_{2|\alpha|+i=k} \frac{C_1C_2}{\alpha!} \|Q_0\|_{\mathcal{L}^{m,(0)}_{R_0,\rho_0}} \|\partial_\eta^\alpha P_i\|_{\mathcal{A}^{m,(-|\alpha|-i/2)}_{\rho_0}} \|D_y^\alpha Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho'}} \\
&\qquad + \sum_{2|\alpha|=k} \frac{1}{\alpha!} (\|H_0^+\|_{\mathcal{H}^{+,(0)}_{\varepsilon_0,\rho_0}} \|\partial_\eta^\alpha H_0^{+*}\|_{\mathcal{H}^{+*,(-|\alpha|)}_{\varepsilon_0,\rho_0}} \|D_y^\alpha Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho'}} \\
&\qquad + \|Q_0\|_{\mathcal{B}^{(0)}_{\varepsilon_0,\rho_0}} \|\partial_\eta^\alpha H_0^-\|_{\mathcal{H}^{-,(-|\alpha|)}_{\varepsilon_0,\rho_0}} \|D_y^\alpha H_j^{-*}\|_{\mathcal{H}^{-*,(-1/2)}_{\varepsilon_0,\rho'}}) \\
&\quad\le \sum_{2|\alpha|+i=k} C_1C_2C_0^2 C_0^{|\alpha|+i/2} (i!)^{1/2} \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|} \|Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho}} \\
&\qquad + \sum_{2|\alpha|=k} C_0^2 C_0^{|\alpha|} \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|} \|Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho}} \\
&\qquad + \sum_{2|\alpha|=k} C_0^2 C_0^{|\alpha|} \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|} \|H_j^{-*}\|_{\mathcal{H}^{-*,(-j/2)}_{\varepsilon_0,\rho}} \\
&\quad\le \left(C_1C_2C_0^2\left(\frac{C_0M_0(n+1)k}{\rho-\rho'}\right)^{k/2} + 2C_0^2\left(\frac{C_0M_0nk}{\rho-\rho'}\right)^{k/2}\right) \|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}} \\
&\quad\le M\left(\frac{Mk}{\rho-\rho'}\right)^{k/2} \|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}},
\end{aligned}$$

provided $M \ge \max\{(C_1C_2+2)C_0^2, C_0M_0(n+1)\}$, where C_1 is a constant appearing in (3.1) and C_2 is a constant appearing in Lemma 3.7.

$\mathcal{M}_k^{12}(E_j)$ can be estimated in the same way by using Lemma 3.12 instead of Lemma 3.7.

To estimate $\mathcal{M}_k^{21}(E_j)$ we suppose further that

$$\|H_0^{-*}\|_{\mathcal{H}^{-*,(0)}_{2\varepsilon_0,\rho_0}} \le C_0. \tag{4.12}$$

(We need only replace ε_0 by $\varepsilon_0/2$.) Then by Lemma 3.13 we have, for $A \in \mathcal{A}_{\rho_0}^{m,(\mu)}$,

$$\|H_0^{-*}A\|_{\mathcal{H}^{-*,(\mu)}_{\varepsilon_0,\rho_0}} \le \left(\frac{M_0 m}{\varepsilon_0}\right)^{m/2} \|H_0^{-*}\|_{\mathcal{H}^{(0)}_{2\varepsilon_0}} \|A\|_{\mathcal{A}^{m,(\mu)}_{\rho}} \le C_3 C_0 \|A\|_{\mathcal{A}^{m,(\mu)}_{\rho}}.$$

Here we set $C_3 = (M_0 m/\varepsilon_0)^{m/2}$. We have

$$\begin{aligned}
&\|\mathcal{M}_k^{21}(E_j)\|_{\mathcal{H}^{*,(-j/2-k/2)}_{\varepsilon_0,\rho'}} \\
&\qquad \le \sum_{2|\alpha|+i=k} \frac{1}{\alpha!} \|H_0^{-*}\partial_\eta^\alpha P_i\|_{\mathcal{H}^{-*,(-|\alpha|-i/2)}_{\varepsilon_0,\rho_0}} \|D_y^\alpha Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho'}} \\
&\qquad\quad + \sum_{2|\alpha|=k} \frac{1}{\alpha!} (\|M_0\|_{\mathcal{M}^{\pm,(0)}_{\varepsilon_0,\rho_0}} \|\partial_\eta^\alpha H_0^{+*}\|_{\mathcal{H}^{+*,(-|\alpha|)}_{\varepsilon_0,\rho_0}} \|D_y^\alpha Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho'}} \\
&\qquad\quad + \|H_0^{-*}\|_{\mathcal{H}^{-*,(j)}_{\varepsilon_0,\rho_0}} \|\partial_\eta^\alpha H_0^{-}\|_{\mathcal{H}^{-,(-|\alpha|)}_{\varepsilon_0,\rho_0}} \|D_y^\alpha H_j^{-*}\|_{\mathcal{H}^{-*,(-j/2)}_{\varepsilon_0,\rho'}}) \\
&\qquad \le \sum_{2|\alpha|+i=k} C_3 C_0^2 C_0^{|\alpha|+i/2} (i!)^{1/2} \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|} \|Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho}} \\
&\qquad\quad + \sum_{2|\alpha|=k} C_0^2 C_0^{|\alpha|} \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|} \|Q_j\|_{\mathcal{B}^{(-j/2)}_{\varepsilon_0,\rho}} \\
&\qquad\quad + \sum_{2|\alpha|=k} C_0^2 C_0^{|\alpha|} \left(\frac{M_0|\alpha|}{\rho-\rho'}\right)^{|\alpha|} \|H_j^{-*}\|_{\mathcal{H}^{-*,(-j/2)}_{\varepsilon_0,\rho}} \\
&\qquad \le \left(C_3 C_0^2 \left(\frac{C_0 M_0 (n+1) k}{\rho-\rho'}\right)^{k/2} + 2C_0^2 \left(\frac{C_0 M_0 n k}{\rho-\rho'}\right)^{k/2}\right) \|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}} \\
&\qquad \le M \left(\frac{Mk}{\rho-\rho'}\right)^{k/2} \|E_j\|_{\mathcal{E}^{(-j/2)}_{\varepsilon_0,\rho}},
\end{aligned}$$

provided $M \ge \max\{(C_3+2)C_0^2, C_0 M_0(n+1)\}$.

$\mathcal{M}_k^{22}(E_j)$ can be estimated in the same way and we have proved (4.11).

Now assume that (4.5) has been proved up to order $j = l-1$. Using (4.11) with $\rho = \rho' + (k/l)(\rho_0 - \rho')$ we obtain

$$\begin{aligned}
\|\mathcal{M}_k(E_{l-k})\|_{\mathcal{E}^{(-l/2)}_{\varepsilon_0,\rho'}} &\le M \left(\frac{Mk}{\rho-\rho'}\right)^{k/2} \|E_{l-k}\|_{\mathcal{E}^{(-l/2+k/2)}_{\varepsilon_0,\rho}} \\
&\le M \left(\frac{Mk}{\rho-\rho'}\right)^{k/2} C \left(\frac{C(l-k)}{\rho_0-\rho}\right)^{(l-k)/2} \\
&\le C \left(\frac{Cl}{\rho_0-\rho'}\right)^{l/2} M \left(\frac{M}{C}\right)^{k/2}.
\end{aligned}$$

Therefore, $E_l = -\sum_{k=1}^{l} \mathcal{M}_k(E_{l-k})$ satisfies

$$\|E_l\|_{\mathcal{E}^{(-l/2)}_{\varepsilon_0,\rho'}} \le C \left(\frac{Cl}{\rho_0-\rho'}\right)^{l/2} M \sum_{k=1}^{l} \left(\frac{M}{C}\right)^{k/2},$$

which implies (4.5) at order $j = l$, if C is large enough ($C \ge \max\{4M, 4M^3\}$).

In the same way we can construct a left parametrix of L and find that the above E is a two-side parametrix of L.

4.2. *General case.* In this section we remove the assumption (4.1) and describe needed modifications in the construction of a relative parametrix.

Let

$$P = \sum_{|\alpha|+|\beta|\le m} t^{\alpha} c_{\alpha\beta}(x, D_x) D_t^{\beta}$$

be an operator of order $\mu = m/2$ of the form (2.4) satisfying (2.5), where $c_{\alpha\beta}(x,\xi) = \sum_{j=0}^{\infty} c_{\alpha\beta,j}(x,\xi)$ is in a-$S_{\mathrm{phg}}^{(|\alpha|-|\beta|)/2,1/2}$ in a conic neighborhood of $\mathring{x}^* = (0,\mathring{y},0,\mathring{\eta})$. As in Section 4.1 we assume $m \ge d+1$ from the beginning.

After taking Taylor expansion of $c_{\alpha\beta,j}$ in (t,τ) we set

$$P_j(y^*) = \sum_{i+|\gamma|=j} \sum_{|\alpha|+|\beta|\le m} \partial_t^{\gamma_-}\partial_\tau^{\gamma_+} c_{\alpha\beta,i}(0,y,0,\eta) t^{\alpha+\gamma_-} D_t^{\beta+\gamma_+}.$$

Interchanging the order of t^{γ_-} and D_t's we can write P_j in the form

$$P_j(y^*) = \sum_{|\gamma|\le j} P_{j,\gamma}(y^*) D_t^{\gamma_+} t^{\gamma_-}$$

with

$$P_{j,\gamma}(y^*) \in \mathcal{O}^{(-j/2-|\gamma_+|/2+|\gamma_-|/2)}(\widetilde{\omega}_{\rho_0}; \mathcal{A}^m).$$

Then $P_{j,\gamma}$ satisfies

$$\|P_{j,\gamma}\|_{\mathcal{A}_{\rho_0}^{m,(-j/2-|\gamma_+|/2+|\gamma_-|/2)}} \le C_0 C_0^j \sqrt{(j-|\gamma|)!} \tag{4.13}$$

for all j and $\gamma = (\gamma_+,\gamma_-)$.

Proceeding just as in Section 4.1, we arrive at the construction of a parametrix $E = \sum_{j=0}^{\infty} E_j$ of

$$L = \sum_{j=0}^{\infty} L_j = \begin{pmatrix} P_0 & H_0^- \\ H_0^{+*} & 0 \end{pmatrix} + \sum_{j=1}^{\infty} \begin{pmatrix} P_j & 0 \\ 0 & 0 \end{pmatrix}$$

so that (4.3) is satisfied. Then E_j's must be given by (4.4). It only remains to prove the estimate like Lemma 4.5 so that we can realize $\sum_{j=0}^{\infty} E_j$ as an analytic micro-local operator. For this purpose we define $\mathcal{E}_\rho^{(\mu)}$ as follows: For $\rho > 0$ we write in this section $\mathcal{B}_\rho^{(\mu)} = \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{B}_\rho)$, $\mathcal{H}_\rho^{\pm,(\mu)} = \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{H}_\rho^{\pm})$ and $\mathcal{H}_\rho^{\pm*,(\mu)} = \mathcal{O}^{(\mu)}(\widetilde{\omega}_\rho; \mathcal{H}_\rho^{\pm*})$. We let $\mathcal{B}_\rho^{(\mu)} \otimes \mathcal{A}^l$ (resp. $\mathcal{H}_\rho^{-*,(\mu)} \otimes \mathcal{A}^l$) denote the space of operator valued symbols for which we can write

$$Q(y^*) = \sum\nolimits_{|\gamma|\le l} Q_\gamma(y^*) D_t^{\gamma_+} t^{\gamma_-} \quad \Big(\text{resp. } H(y^*) = \sum\nolimits_{|\gamma|\le l} H_\gamma(y^*) D_t^{\gamma_+} t^{\gamma_-}\Big)$$

with $Q_\gamma \in \mathcal{B}_\rho^{(\mu-|\gamma_+|/2+|\gamma_-|/2)}$ (resp. $H_\gamma \in \mathcal{H}_\rho^{-*,(\mu-|\gamma_+|/2+|\gamma_-|/2)}$).

Definition 4.6. For $\mu \le 0$ and $\rho > 0$, $\mathcal{E}_\rho^{(\mu)}$ denotes the space of operator valued symbol of the form

$$E = \begin{pmatrix} Q & H^+ \\ H^{-*} & M \end{pmatrix} \in \begin{pmatrix} \mathcal{B}_\rho^{(\mu)} \otimes \mathcal{A}^{2|\mu|} & \mathcal{H}_\rho^{+,(\mu)} \\ \mathcal{H}_\rho^{-*,(\mu)} \otimes \mathcal{A}^{2|\mu|} & \mathcal{M}_\rho^{\pm,(\mu)} \end{pmatrix}.$$

Then we can prove the following lemma for the estimate of E_j's.

LEMMA 4.7. *There exist* $\rho_0 > 0$ *and* $C > 0$ *such that for all* $0 < \rho < \rho_0$,

$$E_j = \begin{pmatrix} Q_j & H_j^+ \\ H_j^{-*} & M_j \end{pmatrix} \in \mathcal{E}_\rho^{(-j/2)},$$

and such that

$$\|Q_{j,\gamma}\|_{\mathcal{B}_\rho^{(-j/2-|\gamma_+|/2+|\gamma_-|/2)}} \le C\left(\frac{C(j-|\gamma|)}{\rho_0-\rho}\right)^{(j-|\gamma|)/2}\left(\frac{1}{\rho}\right)^{|\gamma|}, \tag{4.14}$$

$$\|H_{j,\gamma}^{-*}\|_{\mathcal{H}_\rho^{-*,(-j/2-|\gamma_+|/2+|\gamma_-|/2)}} \le C\left(\frac{C(j-|\gamma|)}{\rho_0-\rho}\right)^{(j-|\gamma|)/2}\left(\frac{1}{\rho}\right)^{|\gamma|}, \tag{4.15}$$

$$\|H_j^+\|_{\mathcal{H}_\rho^{+,(-j/2)}} \le C\left(\frac{Cj}{\rho_0-\rho}\right)^{j/2}, \tag{4.16}$$

$$\|M_j\|_{\mathcal{M}_\rho^{\pm,(-j/2)}} \le C\left(\frac{Cj}{\rho_0-\rho}\right)^{j/2}, \tag{4.17}$$

where $Q_j = \sum_{|\gamma|\le j} Q_{j,\gamma} D_t^{\gamma_+} t^{\gamma_-}$ *and* $H_j^{-*} = \sum_{|\gamma|\le j} H_{j,\gamma}^{-*} D_t^{\gamma_+} t^{\gamma_-}$.

The proof of this lemma is straightforward but very long and tedious. So we only describe here how the induction works for Q_j.

First we note that there is a constant M_0 such that for all $0 < \rho' < \rho < 1/2$ we have

$$\|(\mathrm{ad}\, D_t)^{\beta_+}(\mathrm{ad}\, t)^{\beta_-} D_y^\alpha Q\|_{\mathcal{B}_{\rho'}^{(\mu-|\beta_+|/2+|\beta_-|/2)}} \le \left(\frac{M_0(2|\alpha|+|\beta|)}{\rho-\rho'}\right)^{(2|\alpha|+|\beta|)/2}\|Q\|_{\mathcal{B}_\rho^{(\mu)}},$$

$$\|D_t^{\beta_+} t^{\beta_-} D_y^\alpha H^\pm\|_{\mathcal{H}_{\rho'}^{\pm,(\mu-|\beta_+|/2+|\beta_-|/2)}} \le \left(\frac{M_0(2|\alpha|+|\beta|)}{\rho-\rho'}\right)^{(2|\alpha|+|\beta|)/2}\|H\|_{\mathcal{H}_\rho^{\pm,(\mu)}}.$$

This follows from Lemma 3.8, 3.12 and Cauchy's inequality. We also assume (4.7) through (4.9) in Section 4.1 and (4.13) are satisfied for a constant $C_0 \ge 1$.

We write

$$Q_l = Q_l^{\mathrm{I}} + Q_l^{\mathrm{II}} + Q_l^{\mathrm{III}},$$

where

$$Q_l^{\mathrm{I}} = \sum_{k=1}^{l}\sum_{2|\alpha|+i=k}\frac{1}{\alpha!}Q_0\partial_\eta^\alpha P_i D_y^\alpha Q_{l-k},$$

$$Q_l^{\mathrm{II}} = \sum_{k=1}^{l}\sum_{2|\alpha|=k}\frac{1}{\alpha!}H_0^+\partial_\eta^\alpha H_0^{+*} D_y^\alpha Q_{l-k},$$

$$Q_l^{\mathrm{III}} = \sum_{k=1}^{l}\sum_{2|\alpha|=k}\frac{1}{\alpha!}Q_0\partial_\eta^\alpha H_0^- D_y^\alpha H_{l-k}^{-*}.$$

For Q_l^{I}, we have

$$Q_l^{\mathrm{I}} = \sum_{k=1}^{l}\sum_{2|\alpha|+i=k}\frac{1}{\alpha!}Q_0\Big(\sum_{|\beta|\le i}\partial_\eta^\alpha P_{i,\beta} D_t^{\beta_+} t^{\beta_-}\Big)\Big(\sum_{|\gamma|\le l-k} D_y^\alpha Q_{l-k,\gamma} D_t^{\gamma_+} t^{\gamma_-}\Big).$$

Interchanging the order of $D_t^{\beta_+} t^{\beta_-}$ and $D_y^\alpha Q_{l-k,\gamma} D_t^{\gamma_+}$, we know that the coefficient of $D_t^{\gamma'_+} t^{\gamma'_-}$ consists of at most $\{3\sqrt{n+1}(2d+1)\}^k$ terms of the form

$$\sum_{k=1}^{l} \frac{1}{\alpha!} Q_0 \partial_\eta^\alpha P_{i,\beta}((\operatorname{ad} D_t)^{\beta_1^+} (\operatorname{ad} t)^{\beta_1^-} D_y^\alpha Q_{l-k,\gamma}) \frac{\gamma_+!}{(\gamma_+ - \beta_2^-)!}, \tag{4.18}$$

where $2|\alpha| + i = k$, $|\beta| \le i$, $\beta_1^+ + \beta_2^+ = \beta_+$, $\beta_1^- + \beta_2^- + \beta_3^- = \beta_-$, $\gamma_+ = \gamma'_+ - \beta_2^+ + \beta_2^-$ and $\gamma_- = \gamma'_- - \beta_3^-$.

Now assume (4.14) through (4.17) have been proved up to order $j = l-1$. Then the $\mathcal{B}_{\rho'}^{((-l-|\gamma'_+|+|\gamma'_-|)/2)}$-norm of each term in (4.18) can be estimated by, for a $\rho \in [\rho', \rho_0)$,

$$\begin{aligned}
&\frac{1}{\alpha!}\|Q_0\|_{\mathcal{B}_{\rho'}^{(0)}} \|\partial_\eta^\alpha P_{i,\beta}\|_{\mathcal{A}_{\rho'}^{((-i-|\beta_+|+|\beta_-|-2|\alpha|)/2)}} \\
&\quad \times \|(\operatorname{ad} D_t)^{\beta_1^+} (\operatorname{ad} t)^{\beta_1^-} D_y^\alpha Q_{l-k,\gamma}\|_{\mathcal{B}_{\rho'}^{((-l+k-|\gamma_+|+|\gamma_-|+|\beta_1^+|-|\beta_1^-|)/2)}} \left(\frac{\gamma_+!}{(\gamma_+ - \beta_2^-)!}\right) \\
&\le C_0^2 C_0^{|\alpha|+\frac{i}{2}} \sqrt{(i-|\beta|)!} \left(\frac{M_0(2|\alpha| + |\beta_1^+| + |\beta_1^-|)}{\rho - \rho'}\right)^{(2|\alpha|+|\beta_1^+|+|\beta_1^-|)/2} \\
&\quad \times C\left(\frac{C(l-k-|\gamma|)}{\rho_0 - \rho}\right)^{(l-k-|\gamma|)/2} \left(\frac{1}{\rho}\right)^{|\gamma|} \frac{\gamma_+!}{(\gamma_+ - \beta_2^-)!} \\
&\le C_0^2 C_0^{k/2} \left(\frac{M_0(k - |\beta_2^+| - |\beta_2^-| - |\beta_3^-|)}{\rho - \rho'}\right)^{(k-|\beta_2^+|-|\beta_2^-|-|\beta_3^-|)/2} \\
&\quad \times C\left(\frac{C(l-k-|\gamma'| + |\beta_2^+| - |\beta_2^-| + |\beta_3^-|)}{\rho_0 - \rho}\right)^{(l-k-|\gamma'|+|\beta_2^+|-|\beta_2^-|+|\beta_3^-|)/2} \\
&\quad \times \left(\frac{|\beta|_2^-}{\rho - \rho'}\right)^{|\beta_2^-|} \left(\frac{1}{\rho'}\right)^{|\gamma'|-|\beta_2^+|-|\beta_3^-|} \\
&\le C_0^2 C_0^{k/2} \left(\frac{M_0(k - |\beta_2^+| + |\beta_2^-| - |\beta_3^-|)}{\rho - \rho'}\right)^{(k-|\beta_2^+|+|\beta_2^-|-|\beta_3^-|)/2} \\
&\quad \times C\left(\frac{C(l-k-|\gamma'|+|\beta_2^+|-|\beta_2^-|+|\beta_3^-|)}{\rho_0 - \rho}\right)^{(l-k-|\gamma'|+|\beta_2^+|-|\beta_2^-|+|\beta_3^-|)/2} \left(\frac{1}{\rho'}\right)^{|\gamma'|-|\beta_2^+|-|\beta_3^-|}.
\end{aligned}$$

Here we have used the inequality

$$\frac{\gamma_+!}{(\gamma_+ - \beta_2^-)!}\left(\frac{1}{\rho}\right)^{|\gamma_+|} \le \left(\frac{|\beta_2^-|}{\rho - \rho'}\right)^{|\beta_2^-|} \left(\frac{1}{\rho'}\right)^{|\gamma_+|-|\beta_2^-|} \qquad \text{for } \rho' < \rho.$$

Taking ρ to satisfy

$$\frac{l - |\gamma'|}{\rho_0 - \rho'} = \frac{l - |\gamma'| - k + |\beta_2^+| - |\beta_2^-| + |\beta_3^-|}{\rho_0 - \rho},$$

this can be estimated by

$$C_0^2 C\left(\frac{C(l-|\gamma'|)}{\rho_0 - \rho'}\right)^{(l-|\gamma'|)/2} \left(\frac{1}{\rho'}\right)^{|\gamma'|} \left(\frac{C_0 M_0}{C}\right)^{(k-|\beta_2^+|+|\beta_2^-|-|\beta_3^-|)/2} \rho_0^{|\beta_2^+|+|\beta_3^-|}.$$

(Note that if $|\beta_2^+|+|\beta_3^-|=k$ then we can take $\rho=\rho'$ from the beginning.) If ρ_0 and C have been chosen as

$$\rho_0=\sqrt{\frac{C_0M_0}{C}}\le\frac{1}{18C_0^2\sqrt{n+1}(2d+1)}$$

then the sum (4.18) brings to Q_l^{I} the term $Q_{l.\gamma'}^{\mathrm{I}}D_t^{\gamma'_+}t^{\gamma'_-}$ such that

$$\|Q_{l.\gamma'}^{\mathrm{I}}\|_{\mathcal{B}_{\rho'}^{((-l-|\gamma'_+|+|\gamma'_-|)/2)}}\le\frac{1}{3}C\left(\frac{C(l-|\gamma'|)}{\rho_0-\rho'}\right)^{(l-|\gamma'|)/2}\left(\frac{1}{\rho'}\right)^{|\gamma'|}.$$

For Q_l^{II}, we have

$$Q_l^{\mathrm{II}}=\sum_{|\gamma|\le l-1}\sum_{k=1}^{l-|\gamma|}\sum_{2|\alpha|=k}\frac{1}{\alpha!}H_0^+\partial_\eta^\alpha H_0^{+*}D_y^\alpha Q_{l-k,\gamma}D_t^{\gamma_+}t^{\gamma_-}.$$

Hence, for $\rho_k\in(\rho',\rho_0)$,

$$\begin{aligned}
&\|Q_{l,\gamma}^{\mathrm{II}}\|_{\mathcal{B}_{\rho'}^{((-l-|\gamma_+|+|\gamma_-|)/2)}}\\
&\quad\le\sum_{k=1}^{l-|\gamma|}\sum_{2|\alpha|=k}\frac{1}{\alpha!}\|H_0^+\|_{\mathcal{H}_{\rho'}^{+,(0)}}\|\partial_\eta^\alpha H_0^{+*}\|_{\mathcal{H}_{\rho'}^{-*,(-|\alpha|)}}\|D_y^\alpha Q_{l-k,\gamma}\|_{\mathcal{B}_{\rho'}^{((-l+k-|\gamma_+|+|\gamma_-|)/2)}}\\
&\quad\le\sum_{k=1}^{l-|\gamma|}(n+1)^{\frac{k}{2}}C_0^2C_0^{\frac{k}{2}}\left(\frac{M_0k}{\rho_k-\rho'}\right)^{k/2}C\left(\frac{C(l-k-|\gamma|)}{\rho_0-\rho_k}\right)^{(l-k-|\gamma|)/2}\left(\frac{1}{\rho_k}\right)^{|\gamma|}\\
&\quad\le\sum_{k=1}^{l-|\gamma|}C_0^2C\left(\frac{(n+1)C_0M_0k}{\rho_k-\rho'}\right)^{k/2}\left(\frac{C(l-k-|\gamma|)}{\rho_0-\rho_k}\right)^{(l-k-|\gamma|)/2}\left(\frac{1}{\rho_k}\right)^{|\gamma|}.
\end{aligned}$$

If we choose ρ_k to satisfy

$$\frac{l-k-|\gamma|}{\rho_0-\rho_k}=\frac{l-|\gamma|}{\rho_0-\rho'}$$

then the sum can be estimated by

$$\begin{aligned}
&C_0^2C\left(\frac{C(l-|\gamma|)}{\rho_0-\rho'}\right)^{(l-|\gamma|)/2}\left(\frac{1}{\rho'}\right)^{|\gamma|}\sum_{k=1}^{l-|\gamma|}\left(\frac{(n+1)C_0M_0}{C}\right)^{k/2}\\
&\qquad\qquad\le\frac{1}{3}C\left(\frac{C(l-|\gamma|)}{\rho_0-\rho'}\right)^{(l-|\gamma|)/2}\left(\frac{1}{\rho'}\right)^{|\gamma|}
\end{aligned}$$

provided $C\ge 36(n+1)C_0^5M_0$.

The sum in Q_l^{III} can be estimated in the same way as Q_l^{II} and we obtain (4.14) at order $j=l$.

Now we suppose that (4.14)–(4.17) have been established for all j. Then we can realize $E=\sum_{j=0}^\infty E_j$ as follows: For a $\rho<\rho_0$ we set

$$\begin{aligned}
&V_\rho=\{(t,y,\tau,\eta)\in T^*(\mathbf{R}^N);|t|<\rho,\ |y-\mathring{y}|<\rho,\ |\tau/|\eta||<\rho,\ |\eta/|\eta|-\mathring{\eta}|<\rho\},\\
&W_\rho=\iota^{-1}(V_\rho)=\{(y,\eta)\in T^*(\mathbf{R}^n);|y-\mathring{y}|<\rho,\ |\eta/|\eta|-\mathring{\eta}|<\rho\}.
\end{aligned}$$

By Lemma 3.9, $Q_{l,\gamma} \in \mathcal{B}_\rho^{(-l/2-|\gamma_+|/2+|\gamma_-|/2)}$ has the symbol $b_{l,\gamma}(t,\tau,y,\eta)$ of order $(-l-|\gamma_+|+|\gamma_-|)/2$ of type $(1/2,1/2)$. The estimate (4.14) implies that

$$\sum_{l-|\gamma|=k} b_{l,\gamma}(t,y,\tau,\eta)\tau^{\gamma_+}s^{\gamma_-}$$

converges for $(t,s,y,\tau,\eta) \in \mathbf{R}^d \times V_\rho$ and that if we set

$$q(t,s,y,\tau,\eta) = \sum_{k=0}^{\infty} \chi_{k+1}(\eta/\lambda)\Big(\sum_{l-|\gamma|=k} b_{l,\gamma}(t,y,\tau,\eta)\tau^{\gamma_+}s^{\gamma_-}\Big)$$

then, for a sufficiently large λ, we have $q \in a\text{-}S^0_{1/2,1/2}(\mathbf{R}^d \times V_\rho)$, where χ is the function introduced in Section 2.1. The operator

$$Q = \mathrm{op}(q)_{\mathring{x}^*} : \mathcal{C}^f_{\mathbf{R}^N}(x^*) \to \mathcal{C}^f_{\mathbf{R}^N}(x^*)$$

is well defined through the kernel

$$Q(t,y,s,z) = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{i(t-s)\tau+i(y-z)\eta} q(t,s,y,\tau,\eta)g(\tau,\eta)d\tau d\eta,$$

where g is a suitable cut-off function of Métivier (see Section 2.1).

If $H^{-*}_{j,\gamma}$'s satisfy (4.15) then we have $\sum_{j=0}^{\infty} H_j(y^*) \in S^{0,1/2}_{\mathrm{phg}}(\widetilde{\omega}_\rho;\mathcal{H}^{-*}_\rho)$. In fact, by Lemma 3.12, (4.15) implies $H^{-*}_{j,\gamma}(y^*)D_t^{\gamma_+}t^{\gamma_-}$ is in $\mathcal{O}^{(-j/2)}(\widetilde{\omega}_\rho;\mathcal{H}^{-*}_\rho)$ and, taking $\rho_\gamma = \rho + (|\gamma|/j)(\rho_0-\rho)$, we have

$$\begin{aligned}\|H_j^{-*}\|_{\mathcal{H}_\rho^{-*,(-j/2)}} &\le \sum_{|\gamma|\le j}\Big(\frac{M_0|\gamma|}{\rho_\gamma-\rho}\Big)^{|\gamma|/2} C\Big(\frac{C(j-|\gamma|)}{\rho_0-\rho_\gamma}\Big)^{(j-|\gamma|)/2}\Big(\frac{1}{\rho_\gamma}\Big)^{|\gamma|}\\ &\le C\Big(\frac{Cj}{\rho_0-\rho}\Big)^{j/2}\sum_{|\gamma|\le j}\Big(\frac{M_0}{C\rho^2}\Big)^{|\gamma|/2}\\ &\le C_\rho\Big(\frac{C_\rho j}{\rho_0-\rho}\Big)^{j/2}.\end{aligned}$$

Let $(h^-_{j,1},\dots,h^-_{j,k_-})$ be a symbol of H_j^{-*}. Then $h^-_{j,l}$ satisfies

$$|\partial_t^\alpha h(t,y,\eta)| \le C_\rho e^{-\rho t^2|\eta|/2}(C_\rho(j+|\alpha|)/|\eta|)^{(j+|\alpha|)/2}$$

for $(y,\eta) \in \widetilde{\omega}_\rho$ with another constant C_ρ. This implies, for a sufficiently large λ,

$$h^-_l(t,y,\eta) = \sum_{j=0}^{\infty}\chi_{j+1}(\eta/\lambda)h^-_{j,l}(t,y,\eta)$$

is in $S^0_{1/2,1/2}(\mathbf{R}^d \times W_\rho)$ and bounded by $C_\rho e^{-\rho t^2|\eta|/2}$. The operator $H^{-*}_{(l)} = \mathrm{op}(h_l^{-*})$ is now defined through the kernel

$$H^{-*}_{(l)}(y;t,z) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{i(y-z)\eta}\overline{h^-_l(t,y,\eta)}g(\eta)d\eta$$

and we have

$$WF_A\big(H^{-*}_{(l)}\big) \subset \{(y,0,y,\eta,0,-\eta) \in T^*(R^n_y \times \mathbf{R}^N_{t,z});\eta \in \mathrm{supp}(g)\}.$$

Hence, $H^{-*} = \bigoplus_l H_{(l)}^{-*}$ is analytic micro-local with respect to ι^{-1} as desired.

In the same way we can realize H^+ to be analytic micro-local with respect to ι with symbol in $(a\text{-}S^0_{1/2,1/2}(\mathbf{R}^d \times W_\rho))^{k_+}$. Because $\widetilde{\omega}_\rho$ is a complex neighborhood of W_ρ we write $H^{-*} \in \mathrm{op}\left(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{H}_\rho^{-*})\right)$ and $H^+ \in \mathrm{op}\left(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{H}_\rho^{+})\right)$ for the above realizations of $\sum_{j=0}^\infty H_j^{-*}$ and $\sum_{j=0}^\infty H_j^+$. Clearly, we can realize M as a matrix of usual pseudo-differential operators on W_ρ.

Therefore we obtain

THEOREM 4.17. *Let*

$$P = \sum_{|\alpha|+|\beta|<m} t^\alpha c_{\alpha\beta}(x, D_x) D_t^\beta$$

be an operator of order $m/2$ *of the form* (2.4) *satisfying* (2.5). *If* $m \geq d+1$ *then there are* $\rho > 0$ *and operators* $H_0^- \in \mathrm{op}(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{H}_\rho^-))$, $H_0^{+*} \in \mathrm{op}(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{H}_\rho^{+*}))$, $Q \in \mathrm{op}(a\text{-}S^0_{1/2,1/2}(\mathbf{R}^d \times V_\rho))$, $H^+ \in \mathrm{op}(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{H}_\rho^+))$, $H^{-*} \in \mathrm{op}(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{H}_\rho^{-*}))$ *and* $M \in \mathrm{op}(a\text{-}S^{0,1/2}_{\mathrm{phg}}(W_\rho; \mathcal{M}^\pm))$ *such that*

$$\begin{pmatrix} P & H_0^- \\ H_0^{+*} & 0 \end{pmatrix} \begin{pmatrix} Q & H^+ \\ H^{-*} & M \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{\mathcal{C}^f_{\mathbf{R}^N}(\overset{\circ}{x}{}^*)} & 0 \\ 0 & \mathrm{Id}_{\left(\mathcal{C}^f_{\mathbf{R}^n}(\overset{\circ}{y}{}^*)\right)^{k_+}} \end{pmatrix},$$

$$\begin{pmatrix} Q & H^+ \\ H^{-*} & M \end{pmatrix} \begin{pmatrix} P & H_0^- \\ H_0^{+*} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{\mathcal{C}^f_{\mathbf{R}^N}(\overset{\circ}{x}{}^*)} & 0 \\ 0 & \mathrm{Id}_{\left(\mathcal{C}^f_{\mathbf{R}^n}(\overset{\circ}{y}{}^*)\right)^{k_-}} \end{pmatrix}.$$

Now Theorem 2.1 is an immediate consequence of this theorem.

5. Examples. In this section we shall present a few simple examples and illustrate how our results are applied to them.

Recall that the parametrix has been constructed through (4.4). Thus we have the following formulas for M_j:

$$\begin{aligned}
M_0(y^*) &= -H_0^{-*} P_0 H_0^+ \\
M_1(y^*) &= -H_0^{-*} P_1 H_0^+ \\
M_2(y^*) &= -H_0^{-*} P_2 H_0^+ + H_0^{-*} P_1 Q_0 P_1 H_0^+ \\
&\quad + i(H_0^{-*}\langle \nabla_\eta P_0, \nabla_y H_0^+\rangle + H_0^{-*}\langle \nabla_\eta H_0^-, \nabla_y M_0\rangle + M_0\langle \nabla_\eta H_0^{-*}, \nabla_y H_0^+\rangle) \\
&\ \ \vdots \\
M_j(y^*) &= -H_0^{-*} P_j H_0^+ + F(P_0, \ldots, P_{j-1}, Q_0, H_0^\pm, H_0^{\pm*}, M_0).
\end{aligned}$$

Our results are well applicable to operators with double characteristics because in that case the principal operator $\widehat{\sigma}_\Sigma(P)$ is transformed into a sum of harmonic oscillators, for which we can get a complete eigenexpansion by means of Hermite functions. So we set for $k = 0, 1, 2, \ldots$,

$$\psi_k(t) = (2^k k! \sqrt{\pi})^{-1/2} (t - d/dt)^k e^{-t^2/2}$$

so that

$$(-d^2/dt^2 + t^2)\psi_k(t) = (2k+1)\psi_k(t), \quad (\psi_k, \psi_l)_{L^2(\mathbf{R})} = \delta_{kl}.$$

Example 1. For a non-negative integer l, consider the operator

$$P = D_t^2 + t^2 D_y^2 - (2l+1)D_y$$

at $\mathring{x}^* = (0; dy) \in T^*(\mathbf{R}^2)$. Then

$$Q(y^*) = \sum_{k \neq l} \frac{1}{2(k-l)|\eta|} h_k(t,\eta) h_k^*(t,\eta),$$

$H_0^{\pm}(y^*) = H^{\pm}(y^*) = h_l(t,\eta)$ and $M(y^*) = 0$, where $h_k(t,\eta) = |\eta|^{1/4}\psi_k(t|\eta|^{1/2})$ for $k = 1, 2, \ldots$ Hence

$$H : u(y) \mapsto (2\pi)^{-1} \int e^{iy\eta} h_l(t,\eta)\widehat{u}(\eta)d\eta$$

gives the isomorphisms

$$\begin{aligned} \mathcal{C}_{\mathbf{R}}^f(\mathring{y}^*) &\xrightarrow{\sim} \mathrm{Ker}(P : \mathcal{C}_{\mathbf{R}^2}^f(\mathring{x}^*) \to \mathcal{C}_{\mathbf{R}^2}^f(\mathring{x}^*)), \\ \mathcal{C}_{\mathbf{R}}^f(\mathring{y}^*) &\xrightarrow{\sim} \mathrm{Coker}(P : \mathcal{C}_{\mathbf{R}^2}^f(\mathring{x}^*) \to \mathcal{C}_{\mathbf{R}^2}^f(\mathring{x}^*)). \end{aligned}$$

In particular, $f(t,y) \in \mathcal{D}'(\mathbf{R}^2)$ is in the range of P in a neighborhood of the origin if and only if $H^*f(y)$ is micro-analytic at $(0, dy) \in T^*(\mathbf{R})$.

Remark. If $l = 0$ then $P = (D_t + it_y)(D_t - itD_y)$ and the range of P is equal to the range of $D_t + itD_y$. In this case the characterization of the range of P obtained here is equivalent to that of $D_t + itD_y$ given by Sato-Kawai-Kashiwara [10]. To see this we note that H^*H is the identity on $\mathcal{C}_{\mathbf{R}}^f(\mathring{y}^*)$. Thus H^*f is micro-analytic at $\mathring{y}^*$ if and only if HH^*f is micro-analytic at $\mathring{x}^*$. Now HH^* has the kernel

$$(2\pi)^{-1} \int_0^\infty e^{i(y-z)-(t^2+s^2)|\eta|/2}|\eta|^{1/2}d\eta = \text{const.}\left(y - z + \frac{i}{2}(t^2+s^2) + i0\right)^{-3/2},$$

which is precisely the one appearing in Sato-Kawai-Kashiwara [10, Chap. III, Lemma 2.3.5].

Example 2. Consider

$$P = D_t^2 + t^2 D_y^2 - (1+t^k)D_y$$

at $\mathring{x}^* = (0; dy) \in T^*(\mathbf{R}^2)$. Then $Q_0(y^*) = \sum_{l\neq 0} \frac{1}{2l|\eta|} h_l(t,\eta)h_l^*(t,\eta)$, $H_0^+(y^*) = H_0^-(y^*) = h_0(t,\eta)$ and we have

$$\begin{aligned} &M_0(y^*) = \ldots = M_{k-1}(y^*) = 0, \\ &M_k(y^*) = -h_0^*(t,\eta)t^k\eta h_0(t,\eta), \\ &M_{k+1}(y^*) = \ldots = M_{2k-1}(y^*) = 0, \\ &M_{2k}(y^*) = -\textstyle\sum_{l\neq 0} \frac{1}{2l|\eta|} \left|h_l^*(t,\eta)t^k\eta h_0(t,\eta)\right|^2, \end{aligned}$$

where $h_l(t,\eta)$ are the same as in Example 1.

Hence, $M_k \neq 0$ if k is even, $M_k = 0$ but $M_{2k} \neq 0$ if k is odd. In both cases we know $\sum_{j=0}^\infty M_j$ is elliptic at $\mathring{y}^*$; therefore P is isomorphic on $\mathcal{C}_{\mathbf{R}^2}^f(\mathring{x}^*)$.

Example 3. Consider

$$P = D_t^2 + t^2(D_{y_1}^2 + D_{y_2}^2) - (1-y_1^2)D_{y_2} - c$$

at $\mathring{x}^* = (0; dy_2) \in T^*(\mathbf{R}^3)$ for a $c \in \mathbf{C}$. Then $Q_0(y^*) = \sum_{l\neq 0} \frac{1}{2l|\eta|} h_l(t,\eta) h_l^*(t,\eta)$, $H_0^+(y^*) = H_0^-(y^*) = h_0(t,\eta)$, where $h_l(t,\eta) = |\eta|^{1/2}\psi_l(t|\eta|^{1/2})$, $l = 1, 2, \ldots$, and we have

$$\begin{aligned} M_0(y^*) &= -[(\sqrt{\eta_1^2+\eta_2^2} - \eta_2) + y_1^2\eta_2], \\ M_1(y^*) &= 0, \\ M_2(y^*) &= c. \end{aligned}$$

We note that $\sqrt{\eta_1^2+\eta_2^2} - \eta_2 = |\eta_2|(\eta_1^2/(2\eta_2^2) + O(|\eta_1/\eta_2|^4))$ near $\mathring{y}^*$. Hence if $c \neq \frac{1}{\sqrt{2}}(2k+1)$ then $M(y, D_y)$ is isomorphic on $\mathcal{C}^f_{\mathbf{R}^2}(\mathring{y}^*)$ from the results of Métivier [8]. Otherwise, we can apply Theorem 4.17 once more to $M(y, D_y)$ and find that the reduced operator $\widetilde{M}(D_{y_2})$ is elliptic at $(0; dy_2) \in T^*(\mathbf{R})$. Therefore, for all $c \in \mathbf{C}$, we conclude that P is an isomorphism on $\mathcal{C}^f_{\mathbf{R}^2}(\mathring{x}^*)$.

Example 4. Consider

$$P = \sum_{i=1}^{d} (D_{t_i}^2 + t_i^2 D_y^2 - D_y) + \sum a_{ij}(y) t_i t_j D_y \ (= P_0 + P_2)$$

at $\mathring{x}^* = (0; dy) \in T^*(\mathbf{R}^{d+1})$. We set $A = D_y^{-1} P (D_y^{-1} P_0 + 1)^{[d/2]}$ and apply Theorem 4.17 to A. Then $H_0^+(y^*) = H_0^-(y^*) = |\eta|^{d/4} \Pi_{i=1}^d \psi_0(t|\eta|^{1/2})$ and we have

$$\begin{aligned} M_0(y^*) &= M_1(y^*) = 0, \\ M_2(y^*) &= -H_0^{-*}(y^*)(D_y^{-1} P_2 (D_y^{-1} P_0 + 1)^{[d/2]}) H_0^+(y^*) \\ &= -H_0^{-*}(y^*)\Big(\sum a_{ij}(y) t_i t_j\Big) H_0^+(y^*) \\ &= -\frac{1}{2}\sum_{i=1}^{d} a_{ii}(y). \end{aligned}$$

Here we have used the fact that $P_0 H_0^+(y^*) = 0$. Hence if $\sum_{i=1}^d a_{ii}(0) \neq 0$ then P is isomorphic on $\mathcal{C}^f_{\mathbf{R}^{d+1}}(\mathring{x}^*)$.

References

[1] L. Boutet de Monvel, *Hypoelliptic operators with double characteristics and related pseudodifferential operators*, Comm. Pure Appl. Math. 27 (1974), 585–639.

[2] L. Boutet de Monvel, A. Grigis et B. Helffer, *Paramétrixes d'opérateurs pseudo-différentiels à caractéristiques multiples*, Astérisque 34-35 (1976), 93–121.

[3] A. Grigis and L. P. Rothschild, *A criterion for analytic hypoellipticity of a class of differential operators with polynomial coefficients*, Ann. Math. 118 (1983), 443–460.

[4] V. V. Grushin, *On a class of elliptic pseudodifferential operators degenerate on a submanifold*, Math. USSR-Sb. 13 (1971), 155–185.

[5] B. Helffer, *Sur l'hypoellipticité des opérateurs pseudodifférentiels à caractéristiques multiples (perte de 3/2 dérivées)*, Bull. Soc. Math. France 51-52 (1977), 13–61.

[6] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1980.

[7] M. Kashiwara, T. Kawai and T. Oshima, *Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics I*, Proc. Japan Acad. 50 (1974), 420–425.

[8] G. Métivier, *Analytic hypoellipticity for operators with multiple characteristics*, Comm. Partial Differential Equations, 6 (1981), 1–90.

[9] A. Melin, *Parametrix constructions for some right invariant operators on the Heisenberg group*, ibid., 1363–1405.

[10] M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and Pseudo-Differential Operators*, Lecture Notes in Math. 287, Springer, 1973, 265–529.

[11] J. Sjöstrand, *Parametrix for pseudodifferential operators with multiple characteristics*, Ark. Mat. 12 (1974), 85–130.

[12] E. M. Stein, *An example on the Heisenberg group related to the Lewy operator*, Invent. Math. 69 (1982), 209–216.

[13] F. Treves, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications*, Comm. Partial Differential Equations 3 (1978), 475–642.

[14] F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators*, Vols. I, II, Plenum Press, New York and London, 1981.