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# GLOBAL SOLUTIONS VIA PARTIAL INFORMATION AND THE CAHN–HILLIARD EQUATION

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**Abstract.** Global solutions of semilinear parabolic equations are studied in the case when some weak *a priori* estimate for solutions of the problem under consideration is already known. The focus is on the rapid growth of the nonlinear term for which existence of the semigroup and certain dynamic properties of the considered system can be justified. Examples including the famous Cahn–Hilliard equation are finally discussed.

**1. Introduction.** Global solvability and qualitative behaviour of solutions are usually a very important part of studies on parabolic equations. It is known that although in general local existence is merely a consequence of regularity of the coefficients and the (nonlinear) right side only ([2], [4], [8]), the global existence and all the more the dynamic behaviour of the system are much more delicate properties.

In the study of global solvability many partial results are known, e.g. *a priori* estimates [13], the method of *invariant regions* [16] or the comparison technique [3]. Each of these methods has its own interesting applications but one could hardly expect to find any general approach covering all interesting examples. However, it very often happens, especially in the case of equations describing physical or biological processes, that some introductory global in time estimate resulting from the phenomena described by the equation (e.g. consequences of mass conservation or properties of the energy functional) is initially given. With this partial introductory information the proof of the global existence becomes much simpler, and also suitable time independent estimate of the solutions (necessary for dissipativeness of the system) can be derived, very often enabling the construction of an absorbing set and attractor. It is also possible to study growth rates of

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the nonlinearities for which global solutions exist, defining in the autonomous case the semigroup $\{T(t)\}_{t\geq 0}$ by the formula $T(t)u_0 = u(t, u_0)$, $t \geq 0$.

Our model example in which such a situation can be observed is the Cahn–Hilliard equation, where the introductory information is a $H^1(\Omega)$ *a priori* bound of the solution (resulting from existence of the Lyapunov functional). This partial information allows further $H^{2+\mu}(\Omega)$, $\mu \in [0, 2)$ estimates (cf. [6], [7]) to be obtained, from which existence of the global solution and also certain dynamic properties of the system can then be deduced (cf. Example 1 of Section 4).

In this paper the ideas described above will be developed for a semilinear parabolic equation of the form

$$u_t = Au + f(t, x, d^{m_0}u), \qquad (t, x) \in R^+ \times \Omega, \tag{1}$$

where $-A = \sum_{|\xi|,|\zeta|\leq m}(-1)^{|\zeta|}D^{\zeta}(a_{\xi,\zeta}(x)D^{\xi})$ denotes a $2m$-th order uniformly strongly elliptic operator in a bounded domain $\Omega \subset R^n$, the function $f : R \times cl\ \Omega \times R^{d_0} \to R$ is locally Lipschitz continuous (here $d_0 = \frac{(n+m_0)!}{n!(m_0)!}$ is the number of all multi-indices $\beta$ with $|\beta| \leq m_0$) and $d^{m_0}u$, $m_0 \leq 2m - 1$, stands for the vector $\{D^{\beta}u\}_{|\beta|\leq m_0} = \{u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^{m_0} u}{\partial x_n^{m_0}}\}$ of the spatial partial derivatives of $u$ of order not exceeding $m_0$.

Together with (1) the following initial-boundary conditions are considered:

$$u(0, x) = u_0(x) \quad \text{ in } \Omega, \tag{2}$$

$$B_0 u = B_1 u = \ldots = B_{m-1}u = 0 \quad \text{ on } \partial\Omega. \tag{3}$$

In our studies we assume that:

A-I. The triple $(-A, \{B_j\}, \Omega)$ forms a "regular elliptic boundary value problem" in the sense of [8, p. 76] (i.e. the *root condition*, *smoothness condition*, *strong complementary condition* are satisfied, and moreover, the system $\{B_j\}$ is *normal*).

A-II. The condition

$$\int_{\Omega} (Av)w\,dx = -\int_{\Omega} a(v, w)\,dx$$

holds for all $v \in W^{2m,2}_{\{B_j\}}(\Omega) = \{\varphi \in W^{2m,2}(\Omega) : B_0\varphi = \ldots = B_{m-1}\varphi = 0\}$, $w \in W^{m,2}_{\{B_j\}}(\Omega)$, where the form $a(w, v) = \sum_{|\xi|,|\zeta|\leq m} a_{\xi,\zeta}(x)D^{\xi}vD^{\zeta}w$ is *symmetric* and *coercive* [12, p. 217], i.e. for some $\lambda_0 > 0$, $c > 0$:

$$\int_{\Omega} a(w, w)\,dx + \lambda_0\|w\|^2_{L^2(\Omega)} \geq c\|w\|^2_{W^{m,2}(\Omega)}, \qquad w \in W^{m,2}_{\{B_j\}}(\Omega). \tag{4}$$

The paper is organized as follows: Section 2 contains preliminaries, Section 3 is devoted to global existence and *a priori* estimates, while in Sections 4, 5 the Cahn–Hilliard equation and the Kuramoto-Sivashinsky equation are considered as illustrations of the ideas presented in the paper.

**2. Preliminary notes.** The notation of monographs [8], [10] will be followed throughout the paper. In particular, we denote by $D^{\beta}u$, $\beta \in N^n$, the spatial partial derivative $\frac{\partial^{|\beta|}u}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}}$ of order $|\beta| = \beta_1 + \ldots + \beta_n$. Also the symbol $D^j u$ with $j \in N$ is used for the vector $\{D^\beta u, |\beta| = j\}$ and furthermore, as already mentioned in the introduction, $d^{m_0}u$ stands for $\{D^\beta u, |\beta| \le m_0\}$. Consequently, $|D^j u| = \sqrt{\sum_{|\beta|=j} (D^\beta u)^2}$, whereas $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

Since the triple $(-A, \{B_j\}, \Omega)$ is a "regular elliptic boundary value problem", $\mathcal{A} := -A + \lambda$ with $D(\mathcal{A}) = W^{2m,p}_{\{B_j\}}(\Omega)$, $1 < p < \infty$ is sectorial [8, p. 101] for sufficiently large $\lambda > \lambda_0$, $\lambda$ fixed from now on. In particular, the Sobolev Embedding Theorem can then be quoted in the form (cf. [10, Th. 1.6.1], [8, pp. 177–179]):

$$D(\mathcal{A}^\alpha) \hookrightarrow W^{k,q}(\Omega), \quad \text{with} \quad k - \frac{n}{q} < 2m\alpha - \frac{n}{p},\ p \le q,\ 0 \le \alpha \le 1, \tag{5}$$

whereas the Calderón-Zygmund estimate [8, Th. 19.2, p. 77] may be rewritten as:

$$\|v\|_{W^{2m,p}(\Omega)} \le C_1 \|\mathcal{A}v\|_{L^p(\Omega)}, \qquad \text{for } v \in D(\mathcal{A}),\ 1 < p < \infty. \tag{6}$$

For convenience the Nirenberg-Gagliardo inequality [8, Chap. I, Th.10.1] and the elementary Young inequality are also recalled:

$$\|D^j v\|_{L^q(\Omega)} \le C_2 \|v\|^\theta_{W^{k,p}(\Omega)} \|v\|^{1-\theta}_{L^r \Omega)}, \qquad \text{for } \theta \in [j/k, 1],\ 0 \le j < k, \tag{7}$$

if $\frac{1}{q} = \frac{j}{n} + \theta(\frac{1}{p} - \frac{k}{n}) + (1-\theta)\frac{1}{r}$ and $k - j - \frac{n}{p}$ is not a nonnegative integer,

$$ab \le \delta a^s + C_\delta b^{\frac{s}{s-1}},\ C_\delta = \frac{s-1}{s}(s\delta)^{\frac{1}{1-s}}, \qquad \text{for } a \ge 0,\ b \ge 0,\ \delta > 0,\ s > 1. \tag{8}$$

Furthermore, from the assumptions introduced in Section 1, $\mathcal{A}$ with domain $D(\mathcal{A}) = H^{2m}_{\{B_j\}}(\Omega)$ is symmetric and its range is the whole space $L^2(\Omega)$ [8, p. 77]. Hence, $\mathcal{A}$ is selfadjoint on $L^2(\Omega)$ [11, Chapt. IV, §1] and by the coercivity condition (4) its spectrum lies in the interval $[c, \infty)$ [11, Chapt. IV, §1]. Moreover, this last property is preserved also in the case when $\mathcal{A}$ is considered on the domain $D(\mathcal{A}) = W^{2m,p}_{\{B_j\}}(\Omega)$ with any $1 < p < \infty$ [18, §5.5.1], so that in particular:

$$Re(\sigma(\mathcal{A})) \ge c > 0 \quad \text{for any choice of } D(\mathcal{A}) = W^{2m,p}_{\{B_j\}}(\Omega),\ 1 < p < \infty. \tag{9}$$

Additionally, the resolvent of $\mathcal{A}$ is compact (cf. [18, Th. 5.5.1 (b)]).

In further considerations we shall treat (1)–(3) as an evolution problem

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u = F(t,u), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{10}$$

with $D(\mathcal{A}) = W^{2m,p}_{\{B_j\}}(\Omega)$ and $F(t,u) := f(t,x,d^{m_0}u) + \lambda u$. If for some $\alpha \in (\frac{m_0}{2m}, 1)$, $p \in (1,\infty)$ the function $F : R^+ \times D(\mathcal{A}^\alpha) \to L^p(\Omega)$ is Lipschitz continuous on bounded sets and $u_0 \in D(\mathcal{A}^\alpha)$, then (cf. [9, Th. 4.2.1, p. 73], [10, Th. 3.5.2, p. 71]):

PROPOSITION 1. *There is a unique solution of* (10) *on a maximal interval of existence* $[0, \tau_{u_0})$; *i.e. there exists a continuous function* $u : [0, \tau_{u_0}) \to D(\mathcal{A}^\alpha)$ *satisfying* (10), *such that* $\frac{du}{dt} : (0, \tau_{u_0}) \to D(\mathcal{A}^\alpha)$ *and* $F(\cdot, u(\cdot)) : [0, \tau_{u_0}) \to L^p(\Omega)$ *are continuous and* $u(t)$ *belongs to* $D(A)$ *for* $t \in (0, \tau_{u_0})$. *Moreover, if* $\tau_{u_0} < \infty$, *then* $\|u(t_n)\|_{D(\mathcal{A}^\alpha)} \to \infty$ *for some sequence* $t_n \to \tau^-_{u_0}$; *i.e.* $u$ *"blows up" in a finite time.*

Remark 1. Lipschitz continuity of $F$ from $R^+ \times D(\mathcal{A}^\alpha)$ to $L^p(\Omega)$ on bounded sets follows easily when $\alpha \in (\frac{m_0}{2m}, 1)$ and $2m\alpha - m_0 > \frac{n}{p}$. By Sobolev Embedding [10, Th.

1.6.1] we then have $D(\mathcal{A}^\alpha) \hookrightarrow W^{m_0,\infty}(\Omega)$ and, since $f$ is locally Lipschitz continuous, for any bounded subset $I \times \mathcal{U}$ of $R^+ \times D(\mathcal{A}^\alpha)$ we obtain:

$$\begin{aligned}\forall_{\substack{v_1,v_2\in\mathcal{U}\\ t_1,t_2\in I}} \quad & \|F(t_1,v_1)-F(t_2,v_2)\|_{L^p(\Omega)} \\ & \le \|F(t_1,v_1)-F(t_2,v_1)\|_{L^p(\Omega)} + \|F(t_2,v_1)-F(t_2,v_2)\|_{L^p(\Omega)} \\ & \le L_I|t_1-t_2| + \|f(t_2,\cdot,d^{m_0}v_1) - f(t_2,\cdot,d^{m_0}v_2) + \lambda(v_1-v_2)\|_{L^p(\Omega)} \\ & \le L_I|t_1-t_2| + \sum_{|\beta|\le m_0} L_{\mathcal{U},\beta}\|D^\beta(v_1-v_2)\|_{L^p(\Omega)} + \lambda\|v_1-v_2\|_{L^p(\Omega)} \\ & \le C_{I\times\mathcal{U}}(|t_1-t_2| + \|v_1-v_2\|_{D(\mathcal{A}^\alpha)}),\end{aligned}$$

where the constant $C_{I\times\mathcal{U}}$ depends on $I$ and $\mathcal{U}$ (note that since $\mathcal{U}$ is bounded in $W^{m_0,\infty}(\Omega)$ the range of the arguments of $f$ is then restricted to a compact subset of $R^{1+n+d_0}$).

Our task can now be introduced as follows:

*Knowing for some* $0 \le l \le m_0$ *the following a priori estimate for the solution* $u$ *of the problem* (1):

(11) $$\|D^l u(t)\|_{L^r(\Omega)} \le \rho(t), \qquad t > 0,$$

*with a function* $\rho \in C^0\big([0,\infty)\big)$, *find the growth condition for the nonlinear term* $f$ *in* (1) *for which the global solution* $u$ *of* (10) *exists defining* (*when* $f$ *is time independent*) *the semigroup* $\{T(t)\}_{t\ge 0}$ *by the formula* $T(t)u_0 = u(t,u_0)$. *We are further interested in finding time independent estimates of solutions suitable for the study of the dynamics of the considered system.*

Remark 2. When $l > 0$, the estimate (11) is not sufficient to control the derivatives $D^j u$ with $0 \le j < l$. Using (11) and the boundary conditions (3), we can often estimate these lower derivatives basing on the *Generalized Poincaré Inequality* [17, p. 50]:

(12) $$\|w\|_{H^{l-1}(\Omega)} \le c\{\|D^l w\|_{L^2(\Omega)} + p(w)\}, \qquad \text{for } w \in H^l(\Omega),$$

where $p$ is a continuous seminorm on $H^l(\Omega)$ which is a norm on the space $\mathcal{P}^{l-1}$ of polynomials of degree not exceeding $l-1$. Clearly such an estimate is true when (3) are Dirichlet boundary conditions and $p(w) = \sqrt{\sum_{j=0}^{l-1}\int_\Gamma |D^j w|^2 d\sigma}$ ($\Gamma \subset \partial\Omega$, $|\Gamma| > 0$, $l \le m$). Inequality (12) then guarantees, in particular, that $\|D^l w\|_{L^2(\Omega)}$ is the norm on $H^l_{\{B_j\}}(\Omega)$ equivalent to the standard $H^l(\Omega)$ norm.

Thus, if $l > 0$ in (11) we shall assume that for the solution $u$ of (10):

(13) the full $W^{l,r}(\Omega)$ norm of $u$ is estimated *a priori* for $t > 0$ by $\rho(t)$.

Remark 3. Nevertheless, in order to ensure (13) in the case when (11) is known, it merely suffices to obtain some weak estimate of $L^s(\Omega)$ norm or even seminorm of $u$. Such a situation takes place, for instance, in Examples 1, 2 of Section 4.

**3. Global solutions.** We assume throughout this section that the conditions A-I, A-II of Section 1 are satisfied, an *a priori* estimate (11) holds and if $l > 0$ in (11) then also (13) is valid. Additionally we require that the nonlinear term $f$ in (1) satisfies the following growth condition:

$$|f(\cdot,\cdot,d^{m_0}u)+\lambda u| \le C_3\Big(1+\sum_{j=0}^{m_0}|D^j u|^{\gamma_j}\Big), \qquad m_0 \le 2m-1, \tag{14}$$

where each exponent $\gamma_j$ is restricted by the conditions:

RESTRICTION 1. $\gamma_j \ge 1$ for $j=0,\dots,m_0$ and

1a. $\gamma_j \le 1+\dfrac{2m-1-j}{j-l+n/r}$ if $r(l-j)<n$,

1b. $\gamma_j$ arbitrarily large if $n \le r(l-j)$.

We shall then prove that:

LEMMA 1. *If $D(\mathcal{A})=W^{2m,p}_{\{B_j\}}(\Omega)$ with $p>\frac{nr}{n+(2m-1-l)r}$ and $u(t)$ is a solution of* (10) *on* $[0,\tau]$ ($\tau$ *arbitrarily large*), *then for each $\alpha\in(\frac{2m-1}{2m},1)$*

$$\|F(t,u)\|_{L^p(\Omega)} \le k(t)\big(1+\|\mathcal{A}^\alpha u\|_{L^p(\Omega)}\big), \qquad 0\le t\le\tau, \tag{15}$$

*where $k$ is the continuous function defined in* (25).

Proof. The proof rests on the Nirenberg-Gagliardo inequality (7).

From the growth condition (14) we obtain:

$$\|F(t,u)\|_{L^p(\Omega)} \le C_3|\Omega|^{\frac{1}{p}} + C_3\sum_{j=0}^{m_0}\|D^j u\|^{\gamma_j}_{L^{p\gamma_j}(\Omega)}. \tag{16}$$

Whenever $n\le r(l-j)$, the Sobolev Embedding $W^{l-j,r}(\Omega)\hookrightarrow L^q(\Omega)$, $q\ge 1$ (cf. [1, Th. 5.4]), and a priori estimate (13) give immediately:

$$\|D^j u\|^{\gamma_j}_{L^{p\gamma_j}(\Omega)} \le C_4\rho^{\gamma_j}(t), \qquad \text{for arbitrarily large } p\gamma_j. \tag{17}$$

If $n>r(l-j)$ and $j\ne 2m-1$, then the Nirenberg-Gagliardo inequality (7) together with the Sobolev Embedding (5) give ($\varepsilon=0$ when $2m-1-j-\frac{n}{p}$ is not a nonnegative integer, otherwise $\varepsilon>0$ and sufficiently small):

$$\begin{aligned}\|D^j u\|^{\gamma_j}_{L^{p\gamma_j}(\Omega)} &\le C_5\|u\|^{\theta_j\gamma_j}_{W^{2m-1,p+\varepsilon}(\Omega)}\|D^{l_j}u\|^{(1-\theta_j)\gamma_j}_{L^{r_j}(\Omega)}\\ &\le C_5'\|\mathcal{A}^\alpha u\|^{\theta_j\gamma_j}_{L^p(\Omega)}\|D^{l_j}u\|^{(1-\theta_j)\gamma_j}_{L^{r_j}(\Omega)},\end{aligned} \tag{18}$$

with parameters

$$l_j=\begin{cases} j, & 0\le j<l,\\ l, & l\le j\le m_0\end{cases} \qquad\text{and}\qquad r_j=\begin{cases}\frac{nr}{n-(l-j)r}, & 0\le j<l,\\ r, & l\le j\le m_0,\end{cases} \tag{19}$$

provided that the following requirements are satisfied:

$$\begin{cases}\frac{1}{\gamma_j p}=\frac{j-l_j}{n}+\theta_j\big(\frac{1}{p+\varepsilon}-\frac{2m-1-l_j}{n}\big)+(1-\theta_j)\frac{1}{r_j},\\ \theta_j\in\big[\frac{j-l_j}{2m-1-l_j},1\big].\end{cases} \tag{20}$$

Additionally we shall require that:

$$\gamma_j\theta_j \le 1. \tag{21}$$

Taking in (20) $p > \frac{nr}{n+(2m-1-l)r}$ (note that $\frac{nr}{n+(2m-1-l)r} = \frac{nr_j}{n+(2m-1-l_j)r_j}$ by (19)) we get:

$$\gamma_j\theta_j(\gamma_j) = \frac{\gamma_j(\frac{j-l_j}{n} + \frac{1}{r_j}) - \frac{1}{p}}{\frac{2m-1-l_j}{n} + \frac{1}{r_j} - \frac{1}{p+\varepsilon}}, \tag{22}$$

which shows that $\gamma_j\theta_j(\gamma_j)$ (and also $\theta_j(\gamma_j)$) is increasing with respect to $\gamma_j$ and clearly, since $j < 2m-1$, $\gamma_j\theta_j(\gamma_j)$ must reach 1 at some $\gamma_{j\,max} \geq 1$. Hence, we have

$$\gamma_{j\,max}\theta_j(\gamma_{j\,max}) = 1, \tag{23}$$

and next, considering (22) and (19), we obtain immediately

$$\gamma_{j\,max} = 1 + \frac{2m-1-j+n(\frac{1}{p} - \frac{1}{p+\varepsilon})}{j-l+\frac{n}{r}}. \tag{24}$$

Moreover, analyzing the dependence between $\theta_j$ and $\gamma_j$ it is easy to see that the value $\theta_j(\gamma_{j\,max})$ satisfying condition (23) is always attained in the interior of $[\frac{j-l_j}{2m-1-l_j}, 1]$ admissible for $\theta_j$ (if $\varepsilon \geq 0$ is sufficiently small).

Inserting estimates (17), (18) in the right side of (16) and applying conditions (11), (13) (note that $\|D^{l_j}v\|_{L^{r_j}(\Omega)} \leq \mathrm{const}\|v\|_{W^{l,r}(\Omega)}$ for $0 \leq j < l$) we obtain finally:

$$\begin{aligned} \|F(t,u)\|_{L^p(\Omega)} &\leq \Big(C_3|\Omega|^{\frac{1}{p}} + C_3C_4 \sum_{\{j;n\leq r(l-j)\}} \rho^{\gamma_j}(t) \\ &\quad + C_3C_5' \sum_{\{j;n>r(l-j)\}} \rho^{\gamma_j-1}(t)\Big)(1 + \|\mathcal{A}^\alpha u(t)\|_{L^p(\Omega)}) \\ &=: k(t)(1 + \|\mathcal{A}^\alpha u(t)\|_{L^p(\Omega)}). \end{aligned} \tag{25}$$

The proof of Lemma 1 is completed. ■

From Lemma 1 and Proposition 1 with $D(\mathcal{A}) = W^{2m,p}_{\{B_j\}}(\Omega)$, we obtain directly:

THEOREM 1. *For each* $\alpha \in (\frac{2m-1}{2m}, 1)$ *and* $p > \frac{nr}{n+(2m-1-l)r}$ *such that* $F : R^+ \times D(\mathcal{A}^\alpha) \to D(\mathcal{A}^\alpha)$ *is Lipschitz continuous on bounded sets*, *the solution* $u$ *of the problem* (10) *with* $u_0 \in D(\mathcal{A}^\alpha)$ *exists globally for* $t \geq 0$ *and*, *when* $F$ *is time independent*, $T(t)u_0 = u(t,u_0)$, $t \geq 0$, *defines a strongly continuous semigroup of operators* $T(t) : D(\mathcal{A}^\alpha) \to D(\mathcal{A}^\alpha)$, $t \geq 0$.

Proof. According to Proposition 1 it suffices to show that $u$ cannot "blow up" in a finite time. Although the proof, which rests on consideration of the integral equation

$$u(t) = e^{-\mathcal{A}t}u_0 + \int_0^t e^{-\mathcal{A}(t-s)}F\big(s,u(s)\big)ds, \tag{26}$$

is standard (cf. [8, Th. 16.7, p. 176], [10, Corollary 3.3.5]), we insert it for completeness.

From (26) and (15), we obtain

$$\|\mathcal{A}^\alpha u(t)\|_{L^p(\Omega)} \leq \|\mathcal{A}^\alpha e^{-\mathcal{A}t}u_0\|_{L^p(\Omega)} + \int_0^t k(s)\|\mathcal{A}^\alpha e^{-\mathcal{A}(t-s)}\|ds \tag{27}$$

$$+\int_0^t \|\mathcal{A}^\alpha e^{-\mathcal{A}(t-s)}\| \|\mathcal{A}^\alpha u(s)\|_{L^p(\Omega)} ds$$
$$=: \mathcal{I}(t) + \int_0^t \mathcal{J}(t,s) \|\mathcal{A}^\alpha u(s)\|_{L^p(\Omega)} ds.$$

Moreover from (9), according to the results of [10, Th. 1.4.3, Ex. 4, §1.4]:

$$\begin{aligned}
\mathcal{I}(t) &\le \|e^{-\mathcal{A}t}\mathcal{A}^\alpha u_0\|_{L^p(\Omega)} + \sup_{s\in[0,t]}\{k(s)\} C_6 \int_0^\infty \frac{e^{-c(t-s)}}{(t-s)^\alpha} ds \\
&\le C_7 e^{-ct}\|\mathcal{A}^\alpha u_0\|_{L^p(\Omega)} + \sup_{s\in[0,t]}\{k(s)\} C_6 \frac{\Gamma(1-\alpha)}{c^{1-\alpha}} \\
&\le C_7 + C_8 \sup_{s\in[0,t]}\{k(s)\},
\end{aligned} \tag{28}$$

and also,

$$\mathcal{J}(t,s) \le C_6 \frac{e^{-c(t-s)}}{(t-s)^\alpha} \le \frac{C_6}{(t-s)^\alpha}. \tag{29}$$

Making use of (28) and (29) we get from (27) the Volterra type integral inequality:

$$\|\mathcal{A}^\alpha u(t)\|_{L^p(\Omega)} \le C_7 + C_8 \sup_{s\in[0,t]}\{k(s)\} + \int_0^t \frac{C_6}{(t-s)^\alpha} \|\mathcal{A}^\alpha u(s)\|_{L^p(\Omega)} ds, \tag{30}$$

which, by [10, Lem. 7.1.1], gives an estimate of $\|\mathcal{A}^\alpha u(t)\|_{L^p(\Omega)}$ for $t \ge 0$. The proof of Theorem 1 is completed. ∎

Consider further the special case $m_0 \le m$ when the nonlinear term $f$ in (1) is time independent and contains the derivatives of order not exceeding half the order of $\mathcal{A}$. Let $r \ge 2$ in (11) (or (13), respectively) and also $F : D(\mathcal{A}^{\frac{1}{2}}) \to D(\mathcal{A}^{\frac{1}{2}})$ (where $D(\mathcal{A}) = H^{2m}_{\{B_j\}}(\Omega)$) is Lipschitz continuous on bounded sets. Then:

THEOREM 2. *If the introductory a priori estimate* (11) (*or* (13), *respectively*) *is time independent then*

$$\|\mathcal{A}^{\frac{1}{2}} u(t)\|_{L^2(\Omega)} \le \max\{\|\mathcal{A}^{\frac{1}{2}} u_0\|_{L^2(\Omega)}, (C_9)^{\frac{1}{2}}\}, \qquad t > 0, \tag{31}$$

*and Theorem* 1 *holds with* $\alpha = \frac{1}{2}$. *Moreover*, $T(t) : D(\mathcal{A}^{\frac{1}{2}}) \to D(\mathcal{A}^{\frac{1}{2}})$ *takes bounded sets into bounded sets and is compact on* $D(\mathcal{A}^{\frac{1}{2}})$ *for each* $t > 0$.

Proof. Since the resolvent of $\mathcal{A}$ is compact, according to Proposition 1 and [9, Th. 4.2.2], it suffices only to prove that uniform estimate (31) is valid. Multiplying (10) by $\mathcal{A}u$ we get:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{\frac{1}{2}} u\|^2_{L^2(\Omega)} &= -\|\mathcal{A}u\|^2_{L^2(\Omega)} - (F(u), \mathcal{A}u)_{L^2(\Omega)} \\
&\le -\|\mathcal{A}u\|^2_{L^2(\Omega)} + \|F(u)\|_{L^2(\Omega)} \|\mathcal{A}u\|_{L^2(\Omega)}.
\end{aligned} \tag{32}$$

Since (11) and (13) are time independent then $k(t) \equiv k$ in (15). Inserting (15) with $k(t) \equiv k$ in the right side of (32) and using Young and standard Interpolation Inequality

[10, Th. 1.4.4], for arbitrarily chosen $\alpha \in (\frac{2m-1}{2m}, 1)$ we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L^2(\Omega)} &\le -\|\mathcal{A}u\|^2_{L^2(\Omega)} + k(1+\|\mathcal{A}^{\alpha}u\|_{L^2(\Omega)})\|\mathcal{A}u\|_{L^2(\Omega)} \\ &\le -\frac{1}{2}\|\mathcal{A}u\|^2_{L^2(\Omega)} + \frac{1}{2}k^2 + k\|\mathcal{A}u\|^{1+\alpha}_{L^2(\Omega)}\|u\|^{1-\alpha}_{L^2(\Omega)}.\end{aligned} \tag{33}$$

For $r \ge 2$ and $\rho(t) \equiv \rho$ in (11) (or in (13) respectively), (33) may be rewritten as

$$\frac{d}{dt}\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L^2(\Omega)} \le -\|\mathcal{A}u\|^2_{L^2(\Omega)} + k^2 + 2k(\rho|\Omega|^{\frac{r-2}{2r}})^{1-\alpha}\|\mathcal{A}u\|^{1+\alpha}_{L^2(\Omega)}. \tag{34}$$

Then combining (34) with an obvious inequality

$$\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L^2(\Omega)} \le C\|\mathcal{A}u\|^2_{L^2(\Omega)}, \tag{35}$$

it is easy to see that for some $C_9 > 0$ (cf. [7, Lem. 5]):

$$\|\mathcal{A}^{\frac{1}{2}}u\|^2_{L^2(\Omega)} \le \max\{\|\mathcal{A}^{\frac{1}{2}}u_0\|^2_{L^2(\Omega)}, C_9\}, \tag{36}$$

(more precisely, $C_9 := Cz_0$, where $z_0$ is the positive root of an algebraic equation: $-z + k^2 + 2k(\rho|\Omega|^{\frac{r-2}{2r}})^{1-\alpha}z^{\frac{1+\alpha}{2}} = 0$ and $C$ appears in (35)). The proof is completed. ■

## 4. Examples

Example 1. As the first example we shall consider the Cahn–Hilliard equation ([17], [7]):

$$\begin{cases} u_t = -\varepsilon^2\Delta^2 u + \Delta(g(u)), & (t,x) \in R^+ \times \Omega,\ n \le 3, \\ u(0,x) = u_0(x) & \text{for } x \in \partial\Omega, \\ \frac{\partial u}{\partial N} = \frac{\partial(\Delta u)}{\partial N} = 0 & \text{on } \partial\Omega, \end{cases} \tag{37}$$

where $g$ is a polynomial of order $2l-1$,

$$g(s) = \sum_{j=1}^{2l-1} a_j s^j, \qquad l \in N,\ l \ge 2 \text{ and } l = 2 \text{ if } n = 3,$$

with $a_{2l-1} > 0$. From (37) it is easy to deduce (cf. [7]) that

$$\text{the average } \overline{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(t,x)dx \text{ of } u(t) \text{ is preserved}, \tag{38}$$

and moreover, that

$$\mathcal{L}(\phi) = \frac{\varepsilon^2}{2}\|\nabla\phi\|^2_{L^2(\Omega)} + \int_\Omega \Big(\int_0^\phi g(s)ds\Big)dx \tag{39}$$

is the Lyapunov functional for (37). We have

$$\Delta(g(u)) = g''(u)|\nabla u|^2 + g'(u)\Delta u, \tag{40}$$

and for $n = 3$, with the prescribed growth of $g$:

$$|\Delta(g(u))| \le \text{const}\,((1+|u|)|\nabla u|^2 + (1+|u|^2)|\Delta u|). \tag{41}$$

Making use of (38) and (39) our introductory *a priori* estimate (11) now reads (cf. Remark 2):

$$\|u(t,\cdot)\|_{H^1(\Omega)} \le \text{const}'(\|u_0\|_{H^1(\Omega)}, |\overline{u}_0|), \tag{42}$$

i.e. (13) holds with $l = 1$ and $r = 2$. Evidently, $A = -\varepsilon^2 \Delta^2$ and $D(A) = \{\phi \in H^4(\Omega) : \frac{\partial \phi}{\partial N} = \frac{\partial(\Delta\phi)}{\partial N} = 0 \text{ on } \partial\Omega\}$.

Restriction 1a. with $(l = 1,\ r = 2,\ m = 2,\ n = 3,\ j)_{j=0,1,2}$ allows for

$$\gamma_0 \le 7, \quad \gamma_1 \le \frac{7}{3}, \quad \gamma_2 \le \frac{7}{5},$$

in (14), so that the maximal growth for $f$ given by $\Delta \circ g$ can be:

$$|f(\cdot, d^2u)| \le C_3(1 + |u|^7 + |\nabla u|^{\frac{7}{3}} + |\Delta u|^{\frac{7}{5}}). \tag{43}$$

By simple application of the Young inequality to the components of the right side in (41) we find that:

$$|u|\ |\nabla u|^2 \le \frac{6}{7}|\nabla u|^{\frac{7}{3}} + \frac{1}{7}|u|^7,$$
$$|u|^2\ |\Delta u| \le \frac{5}{7}|\Delta u|^{\frac{7}{5}} + \frac{2}{7}|u|^7,$$

which shows that the restriction (43) is satisfied by the Cahn–Hilliard equation when $n = 3$. For dimensions $n = 1, 2$ we have no restrictions on $l$ in the definition of $g$ and also Restriction 1b. allows for arbitrarily large $\gamma_0 > 1$ so that the results of Section 3 are applicable to (37) for all $n \le 3$. Since also $F : D(\mathcal{A}^{\frac{1}{2}}) \to D(\mathcal{A}^{\frac{1}{2}})$ (here $D(\mathcal{A}) = H^{2m}_{\{B_j\}}(\Omega)$) is Lipschitz continuous on bounded sets, whereas the introductory estimate (42) is time independent, then Theorem 2 ensures that for $D(\mathcal{A}) = H^{2m}_{\{B_j\}}(\Omega)$ the solution $u$ of the problem (10) with $u_0 \in D(\mathcal{A}^{\frac{1}{2}})$ exists globally for $t \ge 0$ and $T(t)u_0 = u(t, u_0)$ $(t \ge 0)$ defines a strongly continuous semigroup of operators $T(t) : D(\mathcal{A}^{\frac{1}{2}}) \to D(\mathcal{A}^{\frac{1}{2}})$, $t \ge 0$. Moreover, $T(t)$ takes bounded sets into bounded sets (see (31)) and is compact on $D(\mathcal{A}^{\frac{1}{2}})$ for $t > 0$. Additionally, since the estimate (31) of $\|\mathcal{A}^{\frac{1}{2}}u\|_{L^2(\Omega)}$ is time independent then also $\|F(u)\|_{L^2(\Omega)}$ is globally bounded for $t \in [0, \infty)$. Hence, in the presence of [10, Th. 3.3.6], we obtain global boundedness of $\|\mathcal{A}^\alpha u\|_{L^2(\Omega)}$ with any $\alpha \in [\frac{1}{2}, 1)$ and in consequence for each $\alpha \in [\frac{1}{2}, 1)$ we get a compact semigroup $T(t) : D(\mathcal{A}^\alpha) \to D(\mathcal{A}^\alpha)$, $t \ge 0$.

According to [9, Th. 4.2.4] and the results of Section 3, only point dissipativeness needs to be additionally checked in order to show that the global attractor for the Cahn–Hilliard problem (37) exists on $D(\mathcal{A}^\alpha)$ for each $\alpha \in [\frac{1}{2}, 1)$. We leave this part of studies until Section 5 where existence of an absorbing set will be briefly justified.

Example 2. Our second example will be the Kuramoto-Sivashinsky equation in dimension $n \le 3$. Following [14], we shall treat the problem of the form:

$$\begin{cases} u_t + \varepsilon^2 \Delta^2 u + \Delta u + \frac{1}{2}|\nabla u|^2 = 0, & (t, x) \in R^+ \times \Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega \subset R^n, & \\ \frac{\partial u}{\partial N} = \frac{\partial(\Delta u)}{\partial N} = 0 \quad \text{on } \partial\Omega, & \end{cases} \tag{44}$$

although usually space periodic boundary conditions in (44) are considered (cf. [17], [14]). The conditional result, that the $L^2(\Omega)$ global in time boundedness of $|\nabla u|$ implies $H^k(\Omega)$ global boundedness of $u$, is formulated in [14]. Let us then assume that for $u$ solving (44):

$$\|\nabla u(t, \cdot)\|_{L^2(\Omega)} \le M, \qquad t > 0, \tag{45}$$

which is true e.g. for even solutions when $\Omega \subset R$; see [14]. Integrating the first equation in (44) over $\Omega$ we get:

$$\frac{d}{dt}\overline{u}(t) \equiv \frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(t,x)dx\right) = -\frac{1}{2|\Omega|}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)},$$

so that, in the presence of (45), the $H^1(\Omega)$ norm of $u$ (by Remark 2; $\|u\|_{H^1(\Omega)} = (\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + |\overline{u}(t)|^2)^{\frac{1}{2}}$) is estimated for $t > 0$. The operator $A$ and the domain $D(A)$ are clearly the same as for the Cahn–Hilliard equation, while the nonlinear term is:

$$f(\cdot, d^2u) = -\Delta u - \frac{1}{2}|\nabla u|^2. \tag{46}$$

The growth condition (43) is then admissible for validity of the results of Section 3 and in this example it is satisfied by $f$ directly from (46). Thus, assuming (45) we have the semigroup $T(t) : D(\mathcal{A}^\alpha) \to D(\mathcal{A}^\alpha)$, $t \geq 0$ (with any $\alpha \in [\frac{1}{2}, 1)$) defined for the problem (44) by $T(t)u_0 = u(t, u_0)$. Moreover, $T(t)$ takes bounded sets into bounded sets and is compact for $t > 0$.

**5. Dissipativeness of the Cahn–Hilliard equation.** We shall develop here the results of Example 1 showing existence of the global attractor for the Cahn–Hilliard equation (37).

Note that for any $\alpha \in [\frac{1}{2}, 1)$ an element $u_0 \in D(\mathcal{A}^\alpha)$ is an equilibrium point of the semigroup $T(t) : D(\mathcal{A}^\alpha) \to D(\mathcal{A}^\alpha)$, $t \geq 0$, if and only if $u(t, u_0) \equiv u_0$ $(t \geq 0)$ is a stationary solution of (37) constructed in Proposition 1. Hence the set $\mathcal{S}$ of all equilibrium points of the semigroup generated by (37) on $D(\mathcal{A}^\alpha)$ does not depend on $\alpha \in [\frac{1}{2}, 1)$ and in particular $\mathcal{S} \subset D(\mathcal{A})$. Furthermore, using the identity $(u_0 \in \mathcal{S})$:

$$0 = \mathcal{L}(T(t)u_0) - \mathcal{L}(u_0) = -\int_0^t \|\nabla[-\varepsilon^2\Delta u_0 + g(u_0)]\|^2_{L^2(\Omega)},$$

following from (37) and (39), and elliptic regularity theory (note that by the assumption A-I of Section 1 the boundary $\partial\Omega$ appearing in (37) is of the class $C^4$), it is easy to see that the elements of $\mathcal{S}$ coincide with $H^2(\Omega)$ solutions of the elliptic boundary value problem:

$$\begin{cases} -\varepsilon^2\Delta v + g(v) = \frac{1}{|\Omega|}\int_\Omega g(v)dx, & x \in \Omega, \\ \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{47}$$

(cf. [5, Lem. 2] for detailed proof).

It is also clear that $\mathcal{S}$ contains all constant functions $v \equiv \text{const}$ and that the compact global attractor (if it exists) has to contain $\mathcal{S}$. To allow existence of the global attractor it is thus necessary to restrict further the semigroup $T(t)$ from the whole $D(\mathcal{A}^\alpha)$ $(\alpha \in [\frac{1}{2}, 1))$ to its positively invariant (with respect to $T(t)$; cf. property (38)) metric subspace:

$$\mathcal{H}^\alpha_b = \{\phi \in D(\mathcal{A}^\alpha); |\overline{\phi}| \leq b\}, \qquad b > 0, \tag{48}$$

i.e. consider for each $\alpha \in [\frac{1}{2}, 1)$, $b > 0$ the semigroup

$$T(t) : \mathcal{H}^\alpha_b \to \mathcal{H}^\alpha_b, \qquad t \geq 0.$$

We shall then recall the following (cf. [7, Lem. 1]):

LEMMA 2. *The set $\mathcal{S} \cap \mathcal{H}^\alpha_b$ is bounded in $D(\mathcal{A})$ with the bound depending on* $\varepsilon$, $b$, $\Omega$ *and constants characterizing the nonlinear term* $g$.

Proof. From [17, p. 152] we have immediately:

$$\exists_{C>0}\ \forall_{s\in R} \quad -g(s)s \le -\frac{1}{2}a_{2l-1}s^{2l} + C, \tag{49}$$

$$\forall_{\nu>0}\ \exists_{C_\nu>0}\ \forall_{s\in R} \quad |g(s)| \le \nu a_{2l-1}s^{2l} + C_\nu. \tag{50}$$

We also recall that elements of $\mathcal{S}$ are $H^2(\Omega)$ solutions of the elliptic boundary value problem (47). Then considering (47), we find that (cf. [7, Lem. 1] for the direct calculations):

$$\varepsilon^2\|\nabla v\|^2_{L^2(\Omega)} \le (C + C_\nu|\overline{v}|)|\Omega|, \tag{51}$$

$$\varepsilon^2\|\Delta v\|^2_{L^2(\Omega)} \le \sup_{s\in R}\{-g'(s)\}\|\nabla v\|^2_{L^2(\Omega)} = C_{10}\|\nabla v\|^2_{L^2(\Omega)} \tag{52}$$

(note that $(-g')$ is bounded from above). Next, taking the Laplacian of both sides of the equation in (47) and multiplying the result by $\Delta^2 v$, we get the equality:

$$-\varepsilon^2\|\Delta^2 v\|^2_{L^2(\Omega)} + \int_\Omega (g''(v)|\nabla v|^2 + g'(v)\Delta v)\Delta^2 v\,dx = 0, \tag{53}$$

which, in the presence of the Young inequality, leads to the estimate:

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\Delta^2 v\|^2_{L^2(\Omega)} \le \frac{1}{\varepsilon^2}(&\|g''(v)\|^2_{L^\infty(\Omega)}\||\nabla v|^2\|^2_{L^2(\Omega)} \\ &+ \|g'(v)\|^2_{L^\infty(\Omega)}\|\Delta v\|^2_{L^2(\Omega)}).\end{aligned} \tag{54}$$

Since for $n \le 3$ the Sobolev Embeddings $D(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow L^\infty(\Omega)$, $D(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow W^{1,4}(\Omega)$ are valid, it follows that (based on Remark 2 with $p(w) = |\overline{w}|$) the estimates (51), (52) are sufficient to bound the right side of (54) and to obtain the required $H^4(\Omega)$ boundedness of $\mathcal{S} \cap \mathcal{H}^\alpha_b$. Lemma 2 is thus proved. ■

We can now justify that:

LEMMA 3. *For arbitrary $\alpha \in [\frac{1}{2}, 1)$, $b > 0$ the semigroup generated by the Cahn–Hilliard equation on the metric space $\mathcal{H}^\alpha_b$ has a global attractor.*

Proof. Based on the results contained in Example 1 it suffices to show that for any $\alpha \in [\frac{1}{2}, 1)$, $b > 0$ the semigroup $T(t) : \mathcal{H}^\alpha_b \to \mathcal{H}^\alpha_b$, $t \ge 0$ generated by (37) is point dissipative (cf. [9, Th. 4.2.4]).

Let us define the set

$$\mathcal{B}^\alpha_b = \{\phi \in \mathcal{H}^\alpha_b; \|\mathcal{A}^\alpha\phi\|_{L^2(\Omega)} \le \sup_{v\in\mathcal{S}\cap\mathcal{H}^\alpha_b}\|\mathcal{A}^\alpha v\|_{L^2(\Omega)}\}$$

and note that by Lemma 2, $\mathcal{B}^\alpha_b$ is bounded in $D(\mathcal{A}^\alpha)$ ($\alpha \in [\frac{1}{2}, 1)$, $b > 0$). Choosing next some $u_0 \in \mathcal{H}^\alpha_b$ we obtain immediately, from the results of Example 1, that the $\omega$-limit set $\omega(u_0) \subset \mathcal{H}^\alpha_b$ attracts $u_0$ in $\mathcal{H}^\alpha_b$ (cf. [9, Lem. 3.2.1]). Moreover, by the proof of [10, Th. 4.3.4] (based on considerations of the Lyapunov functional), the set $\omega(u_0)$ consists only of the elements of $\mathcal{S}$. Hence $\omega(u_0)$ is a subset of $\mathcal{B}^\alpha_b$, which proves simultaneously that $\mathcal{B}^\alpha_b$ attracts each point of $\mathcal{H}^\alpha_b$. Our considerations are completed. ■

Additionally we have reported in our paper [5] certain natural generalizations of the results concerning the single Cahn–Hilliard equation (37) to the system case (i.e. when multicomponent alloys and, consequently, systems of equations similar to (37) are considered).

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