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L_∞-ESTIMATES FOR SOLUTIONS OF NONLINEAR PARABOLIC SYSTEMS WITH GRADIENT LINEAR GROWTH

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Abstract. Existence of weak solutions and an L_∞-estimate are shown for nonlinear nondegenerate parabolic systems with linear growth conditions with respect to the gradient. The L_∞-estimate is proved for equations with coefficients continuous with respect to x and t in the general main part, and for diagonal systems with coefficients satisfying the Carathéodory condition.

1. Introduction. We consider the following initial boundary value problem for a nonlinear system of parabolic equations:

$$\begin{aligned}
&u_{it} - \sum_{j=1}^m \nabla\cdot(a_{ij}(x,t,u,\nabla u)\cdot\nabla u_j) = f_i(x,t,u,\nabla u) && \text{in } \Omega^T = \Omega\times(0,T),\\
&u_i|_{t=0} = u_{0i} && \text{in } \Omega,\\
&u_i = u_{bi} && \text{on } S^T = S\times(0,T),
\end{aligned}\tag{1.1}$$

where $i=1,\ldots,m$, $\Omega\subset\mathbb{R}^n$ is a bounded domain, $S=\partial\Omega$ and the dot denotes scalar product in $\mathbb{R}^n$. Strictly speaking the main term in $(1.1)_1$ takes the form

$$\sum_{j=1}^m \nabla\cdot(a_{ij}\cdot\nabla u_j) = \sum_{j=1}^m\sum_{r,s=1}^n \partial_{x_r}(a_{ij}^{rs}\partial_{x_s}u_j).$$

Moreover, $u=(u_1,\ldots,u_m)\in\mathbb{R}^m$, $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$.

Our aim is to prove existence of solutions to (1.1) and then to show regularity under appropriate assumptions on the coefficients of $(1.1)_1$.

To this end we assume the following structure conditions. First

$$a_{ij}:\Omega^T\times\mathbb{R}^n\times\mathbb{R}^{mn}\to\mathbb{R}^{n^2},\quad i,j=1,\ldots,m,$$

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satisfy the Carathéodory condition and

$$\alpha_1|\nabla u|^2 \le \sum_{i,j=1}^m a_{ij}(x,t,u,\nabla u)\cdot\nabla u_j\cdot\nabla u_i \le \alpha_2|\nabla u|^2, \tag{1.2}$$

where α_1,α_2 are positive constants and $|\ |$ denotes the norm in $\mathbb{R}^\alpha$.

Moreover, there exists a positive constant α_0 such that

$$\begin{aligned} &\alpha_0|\nabla u_1-\nabla u_2|^2 \\ &\quad\le \sum_{i,j=1}^m (a_{ij}(x,t,u,\nabla u_1)\cdot\nabla u_{1j} - a_{ij}(x,t,u,\nabla u_2)\cdot\nabla u_{2j})\cdot(\nabla u_{1i}-\nabla u_{2i}). \end{aligned} \tag{1.3}$$

Finally, the r.h.s. (right hand side) functions

$$f_i : \Omega^T\times\mathbb{R}^m\times\mathbb{R}^{mn}\to\mathbb{R}, \quad i=1,\dots,m,$$

satisfy the Carathéodory condition and there exist positive constants β_1, β_2, β_3 such that

$$|f_i(x,t,u,\nabla u)| \le \beta_1|\nabla u| + \beta_2|u| + \beta_3, \quad i=1,\dots,m. \tag{1.4}$$

Now, we introduce some definitions and auxiliary results. First we define the Steklov averages

$$v_h(x,t) = \begin{cases} \frac{1}{h}\int_{t-h}^t v(x,\tau)d\tau, & t\in(h,T], \\ 0, & t<h. \end{cases}$$

Next,

$$\overset{\circ}{W}{}^1_2(\Omega) = \{u\in W^1_2(\Omega) : u|_S = 0\}.$$

In this paper we prove existence of weak solutions to nonlinear parabolic systems with linear growth conditions with respect to ∇u for the right-hand side functions. Next an L_∞-estimate is shown in two cases. In the first case using the technique of Solonnikov (see [5]) an L_∞–estimate is shown for general parabolic systems with coefficients of the main part continuous with respect to x and t. In the case of coefficients which are measurable with respect to x and t the L_∞-estimate is shown by the method of Di Benedetto (see [3]) for diagonal systems only. Moreover, the diagonal elements are the same. In this paper the methods of [7] cannot be applied for general n.

2. Existence of weak solutions. First we need

Definition 2.1. By a weak solution of problem (1.1) we mean solutions $u_i\in L_\infty(0,T;L_2(\Omega))\cap L_p(0,T;W^1_p(\Omega))$, $i=1,\dots,m$, of the integral identity

$$-\sum_{i=1}^m\int_{\Omega^T}(u_i-u_{0i})\phi_{it}\,dx\,dt + \sum_{i,j=1}^m\int_{\Omega^T}a_{ij}\cdot\nabla u_j\cdot\nabla\phi_i\,dx\,dt = \sum_{i,j=1}^m\int_{\Omega^T}f_i\phi_i\,dx\,dt, \tag{2.1}$$

which holds for any ϕ_i such that $\phi_i|_S=0$, $\phi_i|_{t=T}=0$, $\phi_{it}\in L_2(\Omega^T)$, $\phi_i\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))$, $i=1,\dots,m$.

To obtain necessary estimates we need the following identity with Steklov averages:

$$\sum_{i=1}^m\int_{\Omega\times(h,T)}\Big(\partial_t u_{hi}\phi_i + \sum_{j=1}^m(a_{ij}\cdot\nabla u_j)_h\cdot\nabla\phi_i - f_{ih}\phi_i\Big)dx\,dt = 0. \tag{2.2}$$

Hence, we have

LEMMA 2.2. *Let* $u_b \in L_\infty(0,T;L_2(\Omega)) \cap W_2^1(\Omega^T)$, $u_0 - u_b(0) \in L_2(\Omega)$. *Let* (1.2) *and* (1.4) *hold.*

Then there exist constants $c_1 = c_1(\alpha_1,\beta_1,\beta_2,\beta_3)$, $c_2 = c_2(\alpha_1,\alpha_2,\beta_2)$ *such that*

$$
\begin{aligned}
(2.3)\qquad \int_\Omega |u|^2dx + \alpha_1 \int_{\Omega^t} |\nabla u|^2 dx\,dt
&\le e^{c_1 t}\Big[c_2 \int_{\Omega^t} (|u_b|^2 + |u_{bt}|^2 + |\nabla u_b|^2)dx\,dt \\
&\quad + \beta_3|\Omega^t| + \operatorname{ess\,sup}_t \int_\Omega |u_b|^2dx + \int_\Omega |u_0 - u_b(0)|^2dx\Big], \qquad t \le T,
\end{aligned}
$$

where $|\Omega^t| = t \operatorname{vol}\Omega$.

Proof. Putting $\phi_i = u_{hi} - u_{bi}$ into (2.2), integrating with respect to time and passing with h to 0 we obtain

$$
\begin{aligned}
\frac12 \int_\Omega |u-u_b|^2dx + \alpha_1 \int_{\Omega^t} |\nabla u|^2dx\,dt
&\le \int_{\Omega^t} |u-u_b|\,|u_{bt}|\,dx\,dt + \alpha_2 \int_{\Omega^t} |\nabla u|\,|\nabla u_b|\,dx\,dt \\
&\quad + \int_{\Omega^t} (\beta_1|\nabla u| + \beta_2|u| + \beta_3)|u-u_b|\,dx\,dt + \frac12\int_\Omega |u_0 - u_b(0)|^2dx,
\end{aligned}
$$

where we have used (1.2) and (1.4).

In view of the Hölder and Young inequalities we have

$$
\begin{aligned}
\frac12 \int_\Omega |u-u_b|^2dx + \alpha_1 \int_{\Omega^t} |\nabla u|^2dx\,dt &\le \frac12 \int_{\Omega^t} (|u-u_b|^2 + |u_{bt}|^2)dx\,dt \\
&\quad + \varepsilon \int_{\Omega^t} |\nabla u|^2dx\,dt + \frac{\alpha_2^2}{2\varepsilon}\int_{\Omega^t} |\nabla u_b|^2dx\,dt \\
&\quad + \frac{\beta_1^2}{2\varepsilon}\int_{\Omega^t} |u-u_b|^2dx\,dt + \beta_2\int_{\Omega^t} |u-u_b|^2dx\,dt \\
&\quad + \beta_2\int_{\Omega^t} |u_b||u-u_b|\,dx\,dt + \beta_3 \int_{\Omega^t} |u-u_b|\,dx\,dt \\
&\quad + \frac12\int_\Omega (u_0 - u_b(0))^2dx.
\end{aligned}
$$

Choosing $\varepsilon = \frac{\alpha_1}{2}$ and using again the Hölder and Young inequalities implies

$$
\frac12 \int_\Omega (u-u_b)^2dx + \frac{\alpha_1}{2}\int_{\Omega^t} |\nabla u|^2dx\,dt \le \left(\frac12 + \frac{\beta_1^2}{\alpha_1} + \frac{3\beta_2}{2} + \frac{\beta_3}{2}\right)\int_{\Omega^t} (u-u_b)^2dx\,dt
$$

$$+\frac{\alpha_2^2}{\alpha_1}\int\limits_{\Omega^t}|\nabla u_b|^2dx\,dt+\frac{1}{2}\int\limits_{\Omega^t}|u_{bt}|^2dx\,dt$$
$$+\frac{\beta_2}{2}\int\limits_{\Omega^t}|u_b|^2dx\,dt+\frac{\beta_3}{2}|\Omega^t|+\frac{1}{2}\int\limits_{\Omega}(u_0-u_b(0))^2dx,$$

where $|\Omega^t|=t|\Omega|$ and $|\Omega|=\operatorname{vol}\Omega$.

In view of the Gronwall inequality we have

$$\int\limits_{\Omega}|u-u_b|^2dx+\alpha_1\int\limits_{\Omega^t}|\nabla u|^2dx\,dt$$
$$\le e^{(1+\frac{2\beta_1^2}{\alpha_1}+3\beta_2+\beta_3)t}\bigg[\bigg(1+\frac{2\alpha_2^2}{\alpha_1}+\beta_2\bigg)\int\limits_{\Omega^t}(|\nabla u_b|^2+|u_{bt}|^2+|u_b|^2)dx\,dt$$
$$+\beta_3|\Omega^t|+\int\limits_{\Omega}(u_0-u_b(0))^2dx\bigg].$$

Using $\int_\Omega|u|^2dx\le\int_\Omega|u-u_b|^2dx+\int_\Omega|u_b|^2dx$ in the above inequality gives (2.3). This concludes the proof.

Now, we prove existence of solutions to (1.1).

LEMMA 2.3. *Let the assumptions of Lemma 2.2 hold. Let* (1.3) *hold and let S be Lipschitz continuous. Then there exists a weak solution to problem* (1.1) *such that*

$$u_i\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;W_2^1(\Omega)),\qquad i=1,\dots,m,$$

and the estimate (2.3) *holds.*

Proof. To prove existence of solution to problem (1.1) we replace ∂_tu by the backward difference quotient

$$\partial_t^{-h}u=\frac{1}{h}[u(t)-u(t-h)].$$

Hence, to prove existence of solutions to (1.1) we approximate (1.1) using time and space discretization. Successively, on time levels we solve approximated (projected on finite-dimensional space) elliptic equations.

Then, we prove estimates for approximate solutions. Finally, we pass to the limit to show existence.

Let $e_i(x)$, $i=1,\dots,\lambda$, be linearly independent smooth functions in $\overset{\circ}{W}{}_2^1(\Omega)$ such that their linear combinations are dense in $\overset{\circ}{W}{}_2^1(\Omega)$. Then we are looking for an approximate solution of (1.1) in the form

$$u_\alpha(x,t)=u_{bh}+\sum_{i=1}^{\lambda}d_{\alpha,i}(t)\,e_i(x),\qquad (x,t)\in\Omega^T,\tag{2.4}$$

where $\alpha=(h,\lambda^{-1})$, $d_{\alpha,i}(t)\in L_\infty(0,T)$ are constant on the subintervals $I_k=(t_{k-1},t_k)$, $t_k=kh$, $k=1,\dots,s$, $h=\frac{T}{s}$, $s\in\mathbf{N}$. The values of d_α on I_k are determined successively for $k = 1, \dots, \frac{T}{h}$ by solving the elliptic problems

$$S_\alpha(u_\alpha, \phi) := \sum_{i=1}^{m} \int_{\Omega} \Big[\partial_t^{-h} u_{\alpha i}(t)\,\phi_i + \sum_{j=1}^{m} a_{ijh} \cdot \nabla u_{\alpha j} \cdot \nabla \phi_i - f_{ih}\phi_i\Big]\,dx = 0, \tag{2.5}$$

which hold for any $\phi_i \in V_\lambda = \operatorname{span}\{e_1, \dots, e_\lambda\}$,

$$a_{ijh} = \frac{1}{h} \int_{(k-1)h}^{kh} a_{ij}(s, x, u_\alpha(t), \nabla u_\alpha(t))\,ds,$$

$$f_{ih} = \frac{1}{h} \int_{(k-1)h}^{kh} f_i(s, x, u_\alpha(t), \nabla u_\alpha(t))\,ds, \qquad t \in ((k-1)h, kh).$$

We take the initial data

$$u_\alpha(t) := u_{0h}(t) \quad \text{ for } -h < t \le 0, \tag{2.6}$$

and

$$u_{0h} := \min\left(1, \frac{1}{h|u_0|}\right) u_0, \tag{2.7}$$

and the boundary conditions

$$u_{bh}(x,t) := \frac{1}{h} \int_{(k-1)h}^{kh} u_b(x,s)\,ds, \qquad t \in ((k-1)h, kh), \tag{2.8}$$

where u_{bh} is time independent also in each interval $((k-1)h, kh)$.

The choice of u_{0h} implies that we can determine $u_\alpha(t)$ inductively for $t \in ((k-1)h, kh)$ as a solution of an elliptic problem. In fact if $u_\alpha(t-h)$ is known the l.h.s. of (2.5) defines a continuous mapping $\Phi_\alpha : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$, where the λ parameters are the unknown coefficients of $u_\alpha(t)$.

To prove the existence of $u_\alpha(t)$ for $t \in (0, kh)$ we assume that $u_\alpha(t)$ is already known in $(0, (k-1)h)$. Therefore, we have to determine $\{d_{\alpha,i}\}_{i=1,\dots,\lambda}$ for $t \in (0, kh)$. Consider a continuous mapping $\Phi_\alpha : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ such that

$$\Phi_{\alpha i}(d_\alpha) := S_\alpha(u_\alpha, e_i), \qquad i = 1, \dots, \lambda,$$

where $d_\alpha = u_\alpha - u_{bh}$, $d_\alpha = \sum_{i=1}^{\lambda} d_{\alpha,i}(t) e_i(x)$. Then using (2.5) we obtain

$$\begin{aligned}
\Phi_\alpha(d_\alpha) \cdot d_\alpha &= \sum_{i=1}^{\lambda} \Phi_{\alpha i}(d_\alpha)\,d_{\alpha,i} = \sum_{i=1}^{\lambda} S_\alpha(u_\alpha, e_i)\,d_{\alpha,i} \\
&= \sum_{i=1}^{\lambda} S_\alpha(u_\alpha, u_\alpha - u_{bh}) \\
&= \sum_{i=1}^{m} \int_{\Omega} \frac{1}{h}(u_{\alpha i}(t) - u_{\alpha i}(t-h))(u_{\alpha i}(t) - u_{bhi})dx \\
&\quad + \sum_{i=1}^{m} \int_{\Omega} \Big[\sum_{j=1}^{m} a_{ijh} \cdot \nabla u_{\alpha j}(t) \cdot \nabla(u_{\alpha i} - u_{bhi}) - f_{ih}(u_{\alpha i} - u_{bhi})\Big]dx.
\end{aligned} \tag{2.9}$$

In view of the Hölder and Young inequalities we have

$$
\begin{aligned}
(2.10)\quad \Phi_\alpha(d_\alpha)\cdot d_\alpha \geq \int_\Omega \frac{1}{h}\bigg(&|u_\alpha(t)|^2-\varepsilon_1|u_\alpha(t)|^2-\frac{1}{2\varepsilon_1}|u_{bh}|^2\\
&-\frac{1}{2\varepsilon_1}|u_\alpha(t-h)|^2-|u_\alpha(t-h)||u_{bh}|\bigg)dx\\
&+\int_\Omega \alpha_1|\nabla u_\alpha|^2dx-\varepsilon_2\int_\Omega|\nabla u_\alpha|^2dx-\frac{\alpha_2^2}{2\varepsilon_2}\int_\Omega|\nabla u_{bh}|^2dx\\
&-\frac{\beta_1^2}{2\varepsilon_2}\int_\Omega|u_\alpha-u_{bh}|^2dx-\beta_2\int_\Omega|u_\alpha|^2dx\\
&-\frac{\beta_2}{2}\int_\Omega(|u_\alpha|^2+|u_{bh}|^2)dx-\beta_3\int_\Omega(|u_\alpha|+|u_{bh}|)dx.
\end{aligned}
$$

Choosing $\varepsilon_1=\frac{1}{2}$ and $\varepsilon_2=\frac{\alpha_1}{2}$ we obtain

$$
\begin{aligned}
(2.11)\quad \Phi_\alpha(d_\alpha)\cdot d_\alpha \geq &\left(\frac{1}{2h}-\frac{2\beta_1^2}{\alpha_1}-\frac{3\beta_2}{2}-\beta_3\right)\int_\Omega|u_\alpha|^2\,dx\\
&+\frac{\alpha_1}{2}\int_\Omega|\nabla u_\alpha|^2\,dx-\frac{1}{h}\int_\Omega(|u_{bh}|^2\,dx+|u_\alpha(t-h)|^2\,dx\\
&+|u_\alpha(t-h)||u_{bh}|)dx-\frac{\alpha_2^2}{\alpha_1}\int_\Omega|\nabla u_{bh}|^2\,dx-\frac{2\beta_1^2}{\alpha_1}\int_\Omega|u_{bh}|^2\,dx\\
&-\frac{\beta_2}{2}\int_\Omega|u_{bh}|^2\,dx-\beta_3\int_\Omega|u_{bh}|\,dx-\frac{\beta_3}{4}|\Omega|.
\end{aligned}
$$

Therefore, for sufficiently large $|d_\alpha(t)|$ and sufficiently small h we have $\Phi_\alpha(d_\alpha)\cdot d_\alpha>0$, so there exists $d_{\alpha_0}(t)$ such that $\Phi_{\alpha_0}(d_{\alpha_0})=0$, that is, $u_\alpha(t)$ exists.

Now, we obtain an estimate for solutions of (2.5). We put $\phi=u_\alpha(t)-u_b$ into (2.5) and integrate the result over t from 0 to t_{i+1}, where $t_i=ih$, $i\leq\frac{T}{h}$. Then we obtain

$$
\begin{aligned}
(2.12)\quad &\int_0^{t_{i+1}}\frac{1}{h}\int_\Omega(u_\alpha(t)-u_\alpha(t-h))(u_\alpha(t)-u_{bh}(t))dx\,dt\\
&+\int_0^{t_{i+1}}\sum_{k,l=1}^m\int_\Omega a_{lkh}\cdot\nabla u_{\alpha k}\cdot\nabla(u_{\alpha l}-u_{bhl})\,dx\,dt-\sum_{i=1}^m\int_0^{t_{i+1}}\int_\Omega f_{lh}(u_{\alpha l}-u_{bhl})\,dx\,dt.
\end{aligned}
$$

Using the formula in line 6 on page 316 of [1] and the structure conditions (1.2) and (1.4) we get

$$
\begin{aligned}
(2.13)\quad &\frac{1}{h}\int_{t_i}^{t_{i+1}}\int_\Omega u_\alpha^2(t)\,dx\,dt-\int_\Omega u_{0h}^2\,dx+\alpha_1\int_0^{t_{i+1}}\int_\Omega|\nabla u_\alpha|^2\,dx\,dt\\
&\leq-\int_0^{t_i}\int_\Omega(u_\alpha-u_{0h})\partial_t^h u_{bh}\,dx\,dt+\frac{1}{h}\int_{t_i}^{t_{i+1}}\int_\Omega(u_\alpha-u_{0h})u_{bh}\,dx\,dt
\end{aligned}
$$

$$+ \alpha_2 \int_0^{t_{i+1}} \int_\Omega |\nabla u_\alpha||\nabla u_{bh}|\,dx\,dt$$
$$+ \int_0^{t_{i+1}} \int_\Omega (\beta_1|\nabla u_\alpha| + \beta_2|u_\alpha| + \beta_3)|u_\alpha - u_{bh}|\,dx\,dt.$$

Since u_α and u_{bh} are constants in the intervals (t_i, t_{i+1}), $i = 0, \ldots, \frac{T}{h} - 1$, we have

$$\begin{aligned}\int_\Omega u_\alpha^2(t_{i+1})\,dx + \alpha_1 \int_0^{t_{i+1}} dt \int_\Omega |\nabla u_\alpha|^2 dx \\ \le c_1 \int_0^{t_{i+1}} dt \int_\Omega u_\alpha^2(t)\,dx + c_2 \int_\Omega (u_{bh}^2(t_{i+1}) + u_{0h}^2)dx \\ + c_3 \int_0^{t_{i+1}} dt \int_\Omega (|u_{0h}|^2 + |\partial_t^h u_{bh}|^2 + |u_{bh}|^2 + |\nabla u_{bh}|^2)dx + c_4.\end{aligned} \tag{2.14}$$

Hence, in view of the Gronwall lemma we obtain

$$\int_\Omega u_\alpha^2(t_{i+1})\,dx + \alpha_1 \int_0^{t_{i+1}} dt \int_\Omega |\nabla u_\alpha|^2\,dx \le c, \tag{2.15}$$

so (2.15) holds for any $t \in (0, T)$.

From (2.15) we can choose a subsequence of $\{u_\alpha\}$ still denoted by $\{u_\alpha\}$ such that $u_\alpha \to u$ weakly in $L_2(0, T; \mathring{W}_2^1(\Omega))$, and $u_\alpha \to u$ weak star in $L_\infty(0, T; L_2(\Omega))$, as $\alpha \to 0$.

Now, we shall show almost everywhere convergence of $u_\alpha \to u$ in Ω^T. Changing the time variable in (2.5) from t to $t + h$ and integrating the result over t from 0 to $T - h$ we obtain

$$\begin{aligned}\sum_{i=1}^m \bigg(\frac{1}{h} \int_0^{T-h} \int_\Omega (u_{\alpha i}(t+h) - u_{\alpha i}(t)\bigg) \cdot \phi_i\,dx\,dt \\ + \int_0^{T-h} \int_\Omega \Big(\sum_{j=1}^m a_{ijh} \nabla u_{\alpha j} \nabla \phi_i - f_{ih}\phi_i\Big) dx\,dt\bigg) = 0.\end{aligned} \tag{2.16}$$

Putting $\phi = u_\alpha(t+h) - u_\alpha(t) - (u_{bh}(t+h) - u_{bh}(t))$ we get

$$\int_0^{T-h} dt \int_\Omega (u_\alpha(t+h) - u_\alpha(t))^2\,dx \le ch. \tag{2.17}$$

Hence, in view of Lemma 1.9 from [1] $u_\alpha \to u$ strongly in $L_1(\Omega^T)$, so

$$u_\alpha \to u \quad \text{a.e. in } \Omega^T. \tag{2.18}$$

Now, from Lemma 6.3, Ch. 5, Sect. 6 of [4] we see that $u_\alpha \to u$ strongly in $L_r(\Omega^T)$, where $r < q = p\frac{n+2}{n}$.

Finally, we prove strong convergence of ∇u_α to ∇u. To show this we put $\phi = u_\alpha - v_\alpha =: w_\alpha$ into (2.5), where $v_\alpha \in L_2(0, T; V_\lambda)$ are approximations of u in $L_2(0, T; \mathring{W}_2^1(\Omega))$, which are time independent in each interval $((k-1)h, kh)$, so

$$v_\alpha \to u \quad \text{strongly in } L_2(0,T;W_2^1(\Omega)). \tag{2.19}$$

From (2.5) we have

$$\sum_{i=1}^m \int_0^t \int_\Omega \partial_t^{-h} u_{\alpha i} w_{\alpha i}\,dx\,dt + \sum_{i,j=1}^m \int_0^t \int_\Omega a_{ijh} \nabla u_{\alpha j} \nabla w_{\alpha i}\,dx\,dt = \sum_{i=1}^m \int_0^t \int_\Omega f_{ih} \cdot w_{\alpha i}\,dx\,dt. \tag{2.20}$$

From [1] we know that $\Phi = \frac{1}{2}(u_1^2 + \ldots + u_m^2)$, $b = (u_1, \ldots, u_m) = \nabla \Phi$, $B(u) = \frac{1}{2}(u_1^2 + \ldots + u_m^2)$, so

$$\sum_{i=1}^m \int_0^t \int_\Omega \partial_t^{-h} u_{\alpha i} w_{\alpha i}\,dx\,dt \ge \frac{1}{h} \int_{t-h}^t \int_\Omega B(u_\alpha(t))\,dx\,dt - \int_\Omega B(u(t))\,dx + o(\alpha), \tag{2.21}$$

where $o(\alpha) \to 0$ as $\alpha \to 0$.

The second term in (2.20) takes the form

$$\begin{aligned}
&\sum_{i,j=1}^m \int_0^t \int_\Omega a_{ijh} \nabla u_{\alpha j} \nabla w_{\alpha i}\,dx\,dt \\
&\quad = \sum_{i,j=1}^m \int_0^t \int_\Omega a_{ijh} (\nabla w_{\alpha j} \nabla w_{\alpha i} + \nabla (v_{\alpha j} - u_j) \nabla w_{\alpha i} + \nabla u_j \nabla w_{\alpha i}) dx\,dt \\
&\quad \equiv I_1 + I_2 + I_3,
\end{aligned}$$

where I_2 converges to zero because of strong convergence of $v_\alpha \to u$ in $L_2(0,T;\mathring{W}_2^1(\Omega))$.

Finally, $I_3 \to 0$ because w_α converges weakly to 0 in $L_2(0,T;\mathring{W}_2^1(\Omega))$.

Finally, we examine the last term in (2.20). Hence, we consider

$$\Big| \sum_{i=1}^m \int_0^t \int_\Omega f_{ih} \cdot w_{\alpha i}\,dx\,dt \Big| \le c(\nabla u_\alpha L_2(\Omega^t) + \|\nabla u_\alpha)\|_{L_2(\Omega^t)} + 1) \|w_\alpha\|_{L_2(\Omega^t)},$$

which converges to zero because $w_\alpha \to 0$ strongly in $L_2(\Omega^T)$.

Summarizing the above results we get

$$\frac{1}{h} \int_{t-h}^t \int_\Omega B(u_\alpha(t))\,dx\,dt - \int_\Omega B(u(t))\,dx + \int_{\Omega^t} |\nabla w_\alpha|^2\,dx\,dt \le o(\alpha),$$

where in view of the Fatou lemma

$$\liminf_{\alpha \to 0} \int_\Omega \frac{1}{h} \int_{t-h}^t B(u_\alpha(t))\,dx\,dt - \int_\Omega B(u(t))\,dx \ge 0.$$

Hence,

$$\nabla u_\alpha \to \nabla u \quad \text{strongly in } L_2(\Omega^T). \tag{2.22}$$

Finally, we pass to the limit in the integral identity

$$(2.23)\qquad \sum_{i=1}^{m}\int_0^T\int_\Omega \partial_t^{-h}u_{\alpha i}\phi_i\,dx\,dt+\sum_{i,j=1}^{m}\int_0^T\int_\Omega a_{ijh}\nabla u_{\alpha j}\nabla\phi_i\,dx\,dt = \sum_{i=1}^{m}\int_0^T\int_\Omega f_{ih}\cdot\phi_i\,dx\,dt.$$

In the first term we use the integration by parts formula and we can pass to the limit since $\phi\in H^1(\Omega^T)$. In the other two terms we can pass to the limit because of (2.18), (2.20) and Theorem 2, Ch. 1, Sect. 4 of [2]. Hence (2.1) follows. This concludes the proof.

3. Regularity of solutions. First we have

THEOREM 3.1. *Let $S\in C^2$, $a_{ij}=a_{ij}(x,t)\in C(\Omega^T;\mathbb{R}^{n^2})$, $i,j=1,\dots,m$. Let the assumptions of Lemma 2.2 hold. Then the weak solution belongs to $W_p^{2,1}(\Omega^T)$, $p>1$.*

Proof. Since $u_i\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;W_2^1(\Omega))$, $i=1,\dots,m$, and (1.4) holds, the r.h.s. of (1.1) are in $L_2(\Omega^T)$. Hence, in view of [5] we have $u_i\in W_2^{2,1}(\Omega^T)$, $i=1,\dots,m$. Then by imbedding theorems $\nabla u_i\in L_{p_1}(\Omega^T)$ and $u_i\in L_{q_1}(\Omega^T)$, $i=1,\dots,m$, where $p_1\le\frac{2(n+2)}{n}$, $q_1\le\frac{2(n+2)}{n-2}$. Now the r.h.s. of (1.1) are in $L_{p_1}(\Omega^T)$, so in view of [5], $u_i\in W_{p_1}^{2,1}(\Omega^T)$, $i=1,\dots,m$. Then imbedding theorems imply that $\nabla u_i\in L_{p_2}(\Omega^T)$, $u_i\in L_{q_2}(\Omega^T)$, where $p_2\le\frac{p_1(n+2)}{n+2-p_1}$, $q_2\le\frac{p_1(n+2)}{n+2-2p_1}$. Continuing the considerations we get at the kth step $\nabla u_i\in L_{p_k}(\Omega^T)$, $u_i\in L_{q_k}(\Omega^T)$, $i=1,\dots,m$, and $p_k\le\frac{p_{k-1}(n+2)}{n+2-p_{k-1}}$, $q_k\le\frac{p_{k-1}(n+2)}{n+2-2p_{k-1}}$. By induction $p_s=\frac{2(n+2)}{n-2(s-1)}$ and $q_s=\frac{2(n+2)}{n-2s}$, $s=1,2,\dots$ Hence, at the sth step $u_i\in W_{p_s}^{2,1}(\Omega^T)$, $i=1,\dots,n$, so for sufficiently large s we conclude the proof.

In the case when a_{ij} are not continuous with respect to x and t the result of Solonnikov (see [5]) cannot be used. Then we obtain an L_∞-estimate by applying the method of Di Benedetto (see [3], Ch. 8, Sect. 2).

THEOREM 3.2. *Let S be Lipschitz continuous, let $a_{ij}=a\delta_{ij}$, $i,j=1,\dots,m$, $a=a(x,t,u,\nabla u)$ be measurable with respect to x, t and continuous with respect to $u,\nabla u$. Let the assumptions of Lemma 2.2 hold. Then the weak solution is bounded.*

Proof. We use the integral identity

$$(3.1)\qquad \sum_{i=1}^{m}\int_0^t\int_\Omega[\partial_t u_{ih}\phi_i+(a\cdot\nabla u_i)_h\cdot\nabla\phi_i]\,dx\,dy=\sum_{i=1}^{m}\int_0^t\int_\Omega f_{ih}\cdot\phi_i\,dx\,dt,$$

where $\phi_i=u_{ih}f(|u_h|)$, f is a nonnegative, nondecreasing function on $\mathbb{R}^+$ satisfying $\sup_{0\le s\le l}f'(s)<\infty$ for all $l>0$, and $f(\omega)=f_\varepsilon[(\omega-k)_+]$, where

$$f_\varepsilon(s)=\begin{cases}1 & \text{if } s\ge\varepsilon,\\ \varepsilon^{-1}s & \text{if } 0<s<\varepsilon,\\ 0 & \text{if } s\le 0,\end{cases}$$

and $k>k_0$, $k_0=\max\{|\omega|_{t=0}|_{L_\infty(\Omega)},|\omega|_S|_{L_\infty(S\times(0,T))}\}$.

Using the function ϕ_i in (3.1), integrating with respect to time and passing with h to zero we obtain

$$\frac{1}{2}\int_\Omega\int_0^\omega sf(s)\,ds+\frac{\alpha_1}{2}\int_{\Omega^t}|\nabla u|^2 f(\omega)\,dx\,dt+\alpha_1\int_{\Omega^t}|u_i\nabla u_i|^2\frac{f'(\omega)}{\omega}\,dx\,dt \le c\int_{\Omega^t}(1+\omega^2)f(\omega)\,dx\,dt+\frac{1}{2}\int_\Omega\int_0^\omega sf(s)\,ds|_{t=0}, \tag{3.2}$$

where $\omega=|u|$. Then passing with ε to zero we get

$$\int_\Omega(\omega-k)_+^2\,dx+\int_{\Omega^t}|\nabla(\omega-k)_+|^2\,dx\,dt\le A\int_{\Omega^t}\omega^2\chi\{(\omega-k)>0\}\,dx\,dt. \tag{3.3}$$

Now, using Lemma 2 of [6] we obtain $\sup_{\Omega^T}\omega\le 2k_0$. This concludes the proof.

4. Remarks

1. Using the technique of DiBenedetto we proved an L_∞-estimate for the system

$$\begin{aligned}&u_{it}-\operatorname{div}(a(x,t,u,\nabla u)\nabla u_i)=f_i(x,t,u,\nabla u),\quad i=1,\dots,m,\\ &u_i|_{t=0}=u_{i0},\quad u_i|_S=u_{bi},\quad i=1,\dots,m,\end{aligned} \tag{4.1}$$

where $|f_i|\le c_4|\nabla u|+c_5|u|+c_6$, $0<c_1\le a(x,t,u,\nabla u)\le c_2$, $|u_0|+|u_b|\le c_3$, c_1-c_6 are positive constants and $a(x,t,u,\nabla u)$ is measurable with respect to x, t and continuous with respect to u, ∇u. Continuity with respect to u and ∇u is necessary to prove existence of weak solutions.

2. Assuming continuity with respect to x and t in the principal part of the parabolic system we can prove regularity for weak solutions to the following system using the technique of Solonnikov:

$$\begin{aligned}&u_{it}-\sum_{j,l=1}^n\sum_{i,k=1}^m\partial_{x_j}(a_{ijkl}(x,t)u_{kx_l})=f_i(x,t,u,\nabla u),\quad i=1,\dots,m,\\ &u_i|_{t=0}=u_{i0},\quad u_i|_S=u_{bi},\quad i=1,\dots,m,\end{aligned} \tag{4.2}$$

where $a_{ijkl}=a_{ijkl}(x,t)$ are continuous with respect to x, t and satisfy the Legendre–Hadamard condition

$$a_{ijkl}\xi^{ij}\xi^{kl}\ge a_0|\xi|^2,\quad a_0>0,$$

where $|\ |$ is the euclidean norm in the linear space of matrices. The other assumptions are the same as in (4.1). Applying the technique of Solonnikov we can also show that $u_i\in L_\infty(\Omega^T)$ and $\nabla u_i\in L_\infty(\Omega^T)$, $i=1,\dots,m$. Moreover, Theorem 3.1 implies some Hölder continuity of ∇u also if data are sufficiently smooth.

In the above considerations the linear growth of f_i, $i=1,\dots,m$, with respect to ∇u plays the role of critical exponent.

In this case we can repeat the considerations of [7] implying an L_∞-estimate and we obtain the inequality $Y_{s+1}\le c\frac{2^{as}}{k^a}Y_s^{1+\alpha}$ (see (3.18) of [7]) but $\alpha>0$ holds for $n<2$ only.

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