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# ON MARTINET'S SINGULAR SYMPLECTIC STRUCTURES

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**Introduction.** Let $V$ be a stratified subspace of $R^N$. We call it symplectic if there exists a differential 2-form $\omega$ on $R^N$ such that the restriction of $\omega$ to each stratum is a symplectic form. In the Marsden-Weinstein singular reduction theory these spaces were studied by several authors [5, 4, 9, 1]. In this paper we classify the symplectic spaces modelled on the so-called symplectic flag $S$. First we prove the corresponding Darboux theorem and then we show that the only reasonable symplectic structures on $S$ are those with underlying Martinet's singular symplectic structure of type $\Sigma_{2,0}$. Finally we find the normal form for this structure and show the similar result for an example of a stratified symplectic space with singular boundary of the maximal stratum.

**1. Singular symplectic spaces.** A stratified differential space with each stratum being a symplectic manifold is called a stratified symplectic space. This notion was introduced in [9] (see also [4]) in the context of standard symplectic reduction. For our purpose, in the first step we need embedded symplectic spaces.

DEFINTION 1.1. Let $S$ be a stratified subset of $R^N$ with each stratum $S_i$ (even dimensional) endowed with a symplectic structure $\omega_{S_i}$. We assume that there exists a closed two-form $\omega$ on $R^N$ such that $\omega|_{S_i} = \omega_{S_i}$. Then the pair $(S, \omega)$ is called a *singular symplectic* space.

A representative model of a singular symplectic space is a disjoint union of semialgebraic sets. We consider the following elementary symplectic flag:

$$S = S_{2n} \cup S_{2n-2} \subset R^{2n};$$
$$S_{2n} = \{(x, y) \in R^{2n} : x_1 > 0\}, \qquad S_{2n-2} = \{(x, y) \in R^{2n} : x_1 = 0, y_1 = 0\}$$

endowed with a symplectic structure $\omega$. By $\iota_k : S_i \to R^N$ we denote the canonical inclusions of $S_{2n-k}$. Here $S_{2n-1} = \{x \in R^{2n} : x_1 = 0\}$.

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EXAMPLE 1.1. Let $V \subset (M, \omega)$ be an algebraic hypersurface. Let $\mathcal{X}_V$ be its Whitney stratification. By $V^d$ we denote an element of $\mathcal{X}_V$, $V^d \in \mathcal{X}_V$, of dimension $d$. We say $V$ is a coisotropic hypersurface if and only if each stratum of $\mathcal{X}_V$ is a coisotropic or an isotropic submanifold of $(M, \omega)$. We easily see that a typical hypersurface $V$ defined by the polynomial equation $F(p) = 0$ is not coisotropic. As an example let us consider the cusp-edge surface $V$ in $R^{2n}$ endowed with a symplectic form $\omega$ in general position with respect to $V$. In this case $\omega|_{\mathrm{Sing}V}$ is a symplectic form. It is shown in [2] that $(V, \omega)$ is diffeomorphic to $(\{x_1^3 - y_1^2 = 0\}, \sum_{i=1}^n dx_i \wedge dy_i)$ and the reduced symplectic space of $V - \mathrm{Sing}\, V$ is isomorphic to the singular edge of $V$ (cf. [4]).

We conjecture that if $\mathrm{Sing}\, V$ is a coisotropic submanifold of $(R^{2n}, \omega)$, then $(V, \omega)$ is diffeomorphic to $(\{x_1^3 - x_2^2 = 0\}, \sum_{i=1}^n dx_i \wedge dy_i)$. Let $\Phi : R^{2n-1} \to R^{2n}$ be the parameterization of $\{x_1^3 - x_2^2 = 0\}$,

$$\Phi(s, y_1, y_2, x_3, y_3, \ldots, x_n, y_n) = (s^2, y_1, s^3, y_2, x_3, y_3, \ldots, x_n, y_n).$$

Then

$$\Phi^*\omega = ds \wedge d(3s^2 y_2 + 2sy_1) + \sum_{i=3}^n dx_i \wedge dy_i.$$

Let $\pi : R^{2n-1} \to R^{2n-2}$ be the mapping

$$\pi(s, y_1, y_2, x_3, y_3, \ldots, x_n, y_n) = (s, 3s^2 y_2 + 2sy_1, x_3, y_3, \ldots, x_n, y_n).$$

Let $S$ be the image of $\pi$. Then

$$S = \big\{(x, y) \in R^{2n-2} : x_1 \neq 0\big\} \cup \big\{(x, y) \in R^{2n-2} : x_1 = 0,\ x_2 = 0\big\}$$

and

$$\pi^*\Big(\sum_{i=1}^{n-1} dx_i \wedge dy_i\Big) = \Phi^*\omega.$$

The reduced space $S$ endowed with the Darboux form on $R^{2n-2}$ is a singular symplectic space.

Now we have a natural extension problem: let $\tilde{\omega}$ be a symplectic form on $S_{2n-2}$, we ask for the existence of the closed two-form on $R^N$ such that $\omega|_{S_{2n-2}} = \tilde{\omega}$ and $\omega|_{S_{2n}}$ is symplectic. The first step in approaching this problem is to classify singular symplectic spaces $(S, \omega)$, where $\omega$ provides a symplectic structure on $R^{2n}$.

By $G_S$ we denote the group of germs of diffeomorphisms $(R^{2n}, 0) \to (R^{2n}, 0)$ preserving $S$, i.e. if $\Phi \in G_S$ then $\Phi(S_{2n}) \subset S_{2n}$, and $\Phi(S_{2n-2}) \subset S_{2n-2}$.

Let $\Phi \in G_S$. Then using the standard setting of singularity theory (cf. [7]) we have

$$\Phi(x_1, y_1, \ldots, x_n, y_n) = (x_1\phi_1(x, y), x_1\phi_{12}(x, y) + y_1\phi_{22}(x, y), \phi_3(x, y), \ldots, \phi_{2n}(x, y)),$$

where $\phi_1, \phi_{12}, \phi_{22}, \phi_3, \ldots, \phi_{2n}$ are smooth germs of functions on $(R^{2n}, 0)$.

Let $\omega_1, \omega_2$ be two symplectic structures on $S$ (closed two-forms on $(R^{2n}, 0)$).

DEFINITION 1.2. We say that $\omega_1$ and $\omega_2$ are $S$-*equivalent* $(\omega_1 \sim_S \omega_2)$ if and only if there exists $\Phi \in G_S$ such that $\Phi^*\omega_1 = \omega_2$.

THEOREM 1.1 (Darboux form). *Let $\omega$ be a symplectic structure on $S$. Assume $\omega$ is a symplectic form on $R^{2n}$. Then $\omega$ is $S$-equivalent to the Darboux form*:

$$\omega \sim_S \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof. We take the homotopy (cf. [8]) $\omega_t = t\omega_1 + (1-t)\omega_0$, $t \in [0,1]$. One can check that $\omega_t$ is a nondegenerate form for every $t \in [0,1]$. We seek for a smooth family $t \to \Phi_t$ such that

$$\Phi_t^* \omega_t = \omega_0, \qquad \Phi_0 = id_{R^{2n}}. \tag{1}$$

Differentiating (1) we have

$$L_{V_t}\omega_t + \omega_1 - \omega_0 = 0,$$

where $L_{V_t}$ is the Lie derivative along the vector field $V_t$ generated by the flow $\Phi_t$. But

$$L_{V_t}\omega_t = d(V_t \rfloor \omega_t) + V_t \rfloor d\omega_t = d(V_t \rfloor \omega_t).$$

We have $d(\omega_0 - \omega_1) = 0$ and $\iota_{2n-1}^*(\omega_0 - \omega_1) = 0$. So by the relative Poincaré Lemma (see e.g. [11]) there exists a one-form $\alpha$ such that $d\alpha = \omega_0 - \omega_1$ and $\alpha$ vanishes on $S_{2n-1}$. Thus we have

$$V_t \rfloor \omega_t = \alpha \quad \text{and} \quad \alpha\,|_{(x,y)} = 0 \quad \text{for every } (x,y) \in S_{2n-1}. \tag{2}$$

Because $\omega_t$ is a nondegenerate form, (2) is always solvable with respect to $V_t$ and moreover $V_t(x,y) = 0$ for every $(x,y) \in S_{2n-1}$. We deduce $\Phi_t$ exists, $\Phi_t \in G_S$ and by compactness of the interval $[0,1]$ we have $\Phi^*\omega_1 = \omega_0$. ■

**2. Martinet's singular symplectic spaces.** Before we pass to the more detailed analysis of the degenerate case we recall the basic results on the standard classification of singularities of differential forms [6].

Let $\omega$ be a germ of a closed two-form on $R^{2n}$ at zero. We denote

$$\Sigma_k(\omega) = \{x \in R^{2n} : rank\omega(x) = 2n - k\}, \qquad k \text{ is even.}$$

Let $\omega^n = f\Omega$, where $\Omega$ is the volume form on $R^{2n}$.

(i) If $f(0) \neq 0$ then $\omega$ is a symplectic form (according to the standard notation denoted by $\Sigma_0$) and by the Darboux theorem we obtain

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{3}$$

in local coordinates around zero.

(ii) Next we assume $f(0) = 0$ while $(df)(0) \neq 0$. We have $\Sigma_2(\omega) = \{f = 0\}$ and let $\iota : \Sigma_2(\omega) \to R^{2n}$ be the inclusion. If $\iota^*\omega^{n-1}(0) \neq 0$ then in local coordinates

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i \tag{4}$$

and this type of singular form $\omega$ is denoted by $\Sigma_{2,0}$ (and called Martinet's singular form).

Both types of forms $\Sigma_0$, $\Sigma_{2,0}$ are locally stable (see [6]) and this is why we use them in what follows.

PROPOSITION 2.1. *Let $\omega$ be a symplectic structure on $S$. Assume $f(0) = 0$ and $df_0 \neq 0$ (stability conditions), then $\omega$ is a singular form of type $\Sigma_{2,0}$ at zero, i.e. $\omega$ belongs to the standard orbit of* (ii) (4).

Remark 2.1. We see that the symplectic form $\omega$ on $S$ may be very singular in general. The singular set of $\omega$ is not visible from $S$ (see Fig. 1). The above proposition says that the typical symplectic forms on $S$ can only have $\Sigma_{2,0}$ or $\Sigma_0$ type singularities in the ambient space. Thus the two remaining stable cases $\Sigma_{2,2,0}$ are naturally excluded from our approach (cf. [3]).

Proof of Proposition 2.1. We see that $\omega$ is a symplectic form on $S_{2n-2}$. Let

$$\widetilde{S} = \Sigma_2(\omega) = \{f = 0\},$$

where $\omega^n = f\Omega$ and $\Omega$ is the standard volume form on $R^{2n}$. We have $T_0\widetilde{S} = T_0S_{2n-1}$, because $\omega$ is symplectic on $S_{2n}$. $S_{2n-2} \subset S_{2n-1}$ so $T_0S_{2n-2} \subset T_0S_{2n-1}$ and $T_0S_{2n-2} \subset T_0\widetilde{S}$. By assumption $\iota^*_{2n-2}\omega$ is symplectic. Thus $(\iota^*_{2n-2}\omega)^{n-1} \neq 0$ and this implies $(\iota^*\omega)^{n-1} \neq 0$, where $\iota : \widetilde{S} \to R^{2n}$ is the embedding of $\widetilde{S}$. ■

LEMMA 2.1. *By means of a diffeomorphism $\Phi \in G_S$ of the form*

$$\Phi(x, y) = (\phi(x, y), x_2, \ldots, x_n, y_1, \ldots, y_n)$$

*one can reduce $f$ to the following normal form:*

$$f(x_1, y_1, \ldots, x_n, y_n) = \pm(x_1 - \psi(y_1, x_2, y_2, \ldots, x_n, y_n)).$$

DEFINITION 2.1. We say that $\psi_1$, $\psi_2$ are *contact equivalent* if and only if there exists a diffeomorphism $\Phi : (R^{2n-1}, 0) \to (R^{2n-1}, 0)$ and a smooth function-germ $g : (R^{2n-1}, 0) \to R$, $g(0) \neq 0$, such that

$$\psi_1 = g \cdot (\psi_2 \circ \Phi).$$

Let $\omega_1, \omega_2$ be two symplectic forms on $S$. Let $f_1, f_2$ define their corresponding singular hypersurfaces, $\omega_1^n = f_1\Omega$ and $\omega_2^n = f_2\Omega$ and $\psi_1, \psi_2$ are as in Lemma 2.1. By straightforward check we obtain the following

PROPOSITION 2.2. *If $\omega_1$ and $\omega_2$ are $S$-equivalent then $\psi_1$ and $\psi_2$ are contact equivalent.*

Let $\omega$ be a symplectic form on $S$, $\omega^n = f\Omega$, $f(0) = 0$ and $df_0 \neq 0$. We see that $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial f}{\partial y_j}(0) = 0$ for $i = 2, \ldots, n, j = 1, \ldots, n$, so $\frac{\partial f}{\partial x_1}(0) \neq 0$. Thus

$$df \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n(0) \neq 0,$$

so $\{y_1, x_2, y_2, \ldots, x_n, y_n\}$ defines a coordinate system on

$$\widetilde{S} = \{f = 0\}.$$

Before we formulate the main theorem concerning the normal form of $\omega$ we need some necessary facts ([6]).

LEMMA 2.2. *Let $\tau$ be a $k$-form on $R^n$ satisfying*

$$\frac{\partial}{\partial x_1} \rfloor \tau = 0, \qquad \frac{\partial}{\partial x_1} \rfloor d\tau = 0. \tag{5}$$

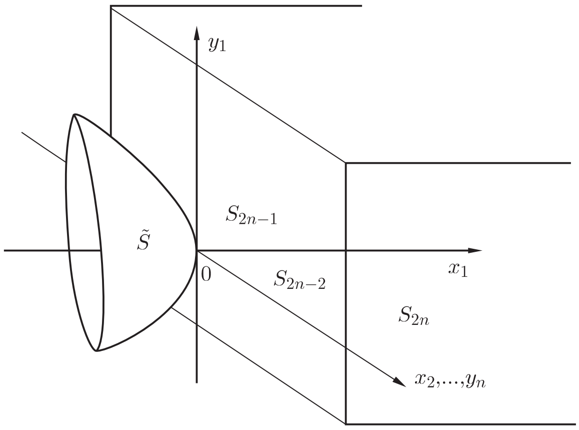


Fig. 1

*Then* $\tau = \pi^*\iota^*\tau$, *where*

$$\pi : R^n \to \{x_1 = 0\}, \qquad \pi(x_1, x_2, \ldots, x_n) = (0, x_2, \ldots, x_n),$$
$$\iota : \{x_1 = 0\} \to R^n, \qquad \iota(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n).$$

LEMMA 2.3. *Let* $\tau$ *be a* $k$*-form on* $R^n$ *satisfying*

$$\frac{\partial}{\partial x_1} \rfloor \tau = 0, \qquad \frac{\partial}{\partial x_1} \rfloor d\tau = \varphi\tau, \tag{6}$$

*where* $\varphi$ *is a smooth function on* $R^n$. *Then*

$$\tau = \zeta\pi^*\iota^*\tau,$$

*where* $\zeta$ *is a smooth function on* $R^n$, *and* $\zeta|_{\{x_1=0\}} = 1$.

It is easy to prove the following lemmas.

LEMMA 2.4. *Let* $\alpha$ *be a germ of a closed* $(n-1)$*-form on* $R^n$ *at* $0$ *satisfying the following conditions*:

1. $\alpha_0 \neq 0$,
2. *a germ of a vector field* $X$ *at* $0$ *such that* $X\rfloor\alpha = 0$ *and* $X(0) \neq 0$ *meets* $\{x_1 = 0\}$ *transversally at* $0$.

*Then there exists a germ of diffeomorphism* $\Phi : (R^n, 0) \to (R^n, 0)$, *which preserves* $\{x_1 = 0\}$ *and*

$$\Phi^*\alpha = dx_2 \wedge \ldots \wedge dx_n,$$

*where* $(x_1, \ldots, x_n)$ *is a coordinate system on* $R^n$.

LEMMA 2.5. *Let* $\alpha$ *be a germ of a* 1*-form on* $R^{2k+1}$ *at* $0$ *satisfying the following conditions*:

1. $\alpha \wedge (d\alpha)^k_0 \neq 0$,
2. *a germ of a vector field* $X$ *at* $0$ *such that*

$$X\rfloor\alpha \wedge (d\alpha)^k = (d\alpha)^k$$

*meets* $\{z=0\}$ *transversally at* 0,

3. $\iota^*\alpha_0 \neq 0$, *where* $\iota : \{z=0\} \hookrightarrow R^{2k+1}$ *is the canonical inclusion.*

*Then there exists a germ of diffeomorphism* $\Phi : (R^{2k+1}, 0) \to (R^{2k+1}, 0)$, *which preserves* $\{z=0\}$ *and*

$$\Phi^*\alpha = dz + dy_1 + \sum_{i=1}^{k} x_i dy_i,$$

*where* $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$ *is a coordinate system on* $R^n$.

Now we prove the main theorem obtaining the normal form (with moduli) of the symplectic structure on $S$. The geometrical contents of this theorem is illustrated in Fig. 1.

THEOREM 2.1. *Let* $\omega$ *be a symplectic structure on* $S$. *Assume* $f(0) = 0$ *and* $df_0 \neq 0$. *Then* $\omega$ *is* $S$*-equivalent to the form*

$$(x_1 - \psi(x_2, \ldots, x_n, y_1, \ldots, y_n))d(x_1 - \psi(x_2, \ldots, x_n, y_1, \ldots, y_n)) \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i, \tag{7}$$

*where* $\psi$ *is a germ at* 0 *of a smooth function,* $\psi(0) = 0$, $\frac{\partial \psi}{\partial x_i}(0) = 0$, $i = 2, \ldots, n$, $\frac{\partial \psi}{\partial y_i}(0) = 0$, $i = 1, \ldots, n$.

Proof. By Lemma 2.1 we have $f = \pm(x_1 - q)$, where $q$ does not depend on $x_1$. We are searching for a 1-form $\alpha$ satisfying the following conditions:

1. $d\alpha = \omega$,
2. $\iota^*\alpha \wedge (d\iota^*\alpha)_0^{n-1} \neq 0$, where $\iota : \tilde{S} \hookrightarrow R^{2n}$ is the canonical inclusion,
3. $\tilde{\iota}^*\alpha_0 \neq 0$, where $\tilde{\iota} : \tilde{S} \cap \{y_1 = 0\} \hookrightarrow R^{2n}$ is the canonical inclusion.

$\omega$ is closed, then there exists a 1-form $\alpha$ such that $d\alpha = \omega$. If $\alpha$ fails to satisfy condition 3 then we replace it by the 1-form $\alpha + dy_2$, which satisfies conditions 1 and 3.

Since $S_{2n-2}$ is symplectic and $T_0 S_{2n-2} = T_0(\tilde{S} \cap \{y_1 = 0\})$, we have $(\tilde{\iota}^* d\alpha)_0^{n-1} = (\tilde{\iota}^*\omega)_0^{n-1} \neq 0$. Hence by Lemma 2.4, we obtain

$$\delta^*\iota^*(d\alpha)^{n-1} = dx_2 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n,$$

where $\delta : (\tilde{S}, 0) \to (\tilde{S}, 0)$ is a diffeomorphism which preserves $\tilde{S} \cap \{y_1 = 0\}$. Therefore

$$\iota^* d(\Delta^*\alpha)^{n-1} = dx_2 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n,$$

where $\Delta \in G_S$ and

$$\Delta(x, y) = (x_1, \delta(x_2, \ldots, x_n, y_1, \ldots, y_n)).$$

If $\Delta^*\alpha$ fails to satisfy condition 2, then we replace it by the 1-form $\Delta^*\alpha + dy_1$, which satifies all the conditions.

From condition 2 it follows that a vector field $X$ which satisfies the conditions

$$X \rfloor \alpha \wedge (d\alpha)^{n-1} = 0, \qquad X(0) \neq 0,$$

meets $\tilde{S}$ transversally at 0. Hence $X$ also meets $S_{2n-1}$ transversally at 0. Therefore by means of elements from $G_S$ one can reduce $X$ to the form $\pm\frac{\partial}{\partial x_1}$. Thus $\tilde{S}$ is locally a graph of a smooth function $\theta : (S_{2n-1}, 0) \to (R, 0)$. Hence $(x_2, \ldots, x_n, \ldots, y_1, \ldots, y_n)$ define a coordinate system on $\tilde{S}$. From 2 and 3 it follows that $\iota^*\alpha$ satisfies the assumptions of Lemma 2.5. Therefore we have

$$\phi^*\iota^*\alpha = dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i,$$

where $\phi : (\tilde{S}, 0) \to (\tilde{S}, 0)$ is a diffeomorphism which preserves $\tilde{S} \cap \{y_1 = 0\}$. Let $\Phi \in G_S$ be such that

$$\Phi(x, y) = (x_1, \phi(x_2, \ldots, x_n, y_1, \ldots, y_n)).$$

Hence we obtain

$$\iota^*\Phi^*\alpha = dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i.$$

It is easy to check that the vector field $X$ satisfies the following conditions:

$$X \rfloor \alpha = 0 \quad \text{and} \quad X \rfloor d\alpha = \varphi\alpha, \tag{8}$$

where $\varphi : R^{2n} \to R$ is a smooth function. Thus by Lemma 2.3, we obtain

$$\alpha = h\Big(dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i\Big),$$

where $h : R^n \to R$ is a smooth function such that $h|_{\tilde{S}} = 1$. We have

$$(d\alpha)^n = n! h^{n-1} \frac{\partial h}{\partial x_1} \Omega.$$

On the other hand, by Lemma 2.1, $\omega^n = \pm(x_1 - g)\Omega$. Hence $n! h^{n-1} \frac{\partial h}{\partial x_1} = \pm(x_1 - g)$, and

$$\frac{\partial h^n}{\partial x_1} = \pm \frac{1}{(n-1)!}(x_1 - g)$$

with an extra condition $h|_{\{x_1 = g\}} = 1$. Solving this equation we get

$$h = \sqrt[n]{\frac{\pm 1}{2(n-1)!}(x_1 - g)^2 + 1}.$$

By the diffeomorphism $\Lambda^{-1} \in G_\Sigma$, where

$$\Lambda(x, y) = (x_1, h(x, y)x_2, \ldots, h(x, y)x_n, y_1, \ldots, y_n),$$

we reduce $\alpha$ to

$$\alpha = h(dy_1 + dy_2) + \sum_{i=2}^{n} x_i dy_i,$$

The diffeomorphism

$$\Upsilon(x, y) = \left( (x_1 - \zeta)\sqrt{(n-1)! \sum_{i=0}^{n-1} \binom{n}{i+1} \Big(\frac{\pm(x_1 - \zeta)^2}{2}\Big)^i} - g, y_1, \ldots, x_n, y_n \right),$$

where $\zeta$ is a function which does not depend on $x_1$ and satisfies

$$\sqrt[n]{\frac{\pm 1}{2(n-1)!} g^2 + 1} = \pm \frac{\zeta^2}{2} + 1,$$

preserves the sets $S_{2n-1}$, $S_{2n-2}$ and

$$\Upsilon^*\alpha = \left(\pm\frac{(x_1-\zeta)^2}{2}+1\right)(dy_1+dy_2)+\sum_{i=2}^{n} x_i dy_i.$$

If $\Upsilon$ does not belong to $G_\Sigma$ then we replace it by $\Theta\circ\Upsilon$, where

$$\Theta(x,y) = (-x_1, x_2, \ldots, x_n, y_1, \ldots, y_n).$$

Hence we obtain

$$\alpha = \left(1\pm\frac{1}{2}(x_1-\psi)^2\right)(dy_1+dy_2)+\sum_{i=2}^{n} x_i dy_i.$$

Therefore

$$\omega = d\alpha = \pm(x_1-\psi)d(x_1-\psi)\wedge dy_1 + d\left(x_2\pm\frac{1}{2}(x_1-\psi)^2\right)\wedge dy_2+\sum_{i=3}^{n} dx_i\wedge dy_i.$$

Finally, by means of $\Xi\in G_\Sigma$, where

$$\Xi(x,y) = \left(x_1, x_2\pm\frac{1}{2}(x_1-\psi)^2, x_3,\ldots,x_n,\pm y_1, y_2, y_3,\ldots,y_n\right),$$

we reduce $\omega$ to the form 7. ∎

Now we pass to the investigation of stability properties of symplectic structures on $S$.

DEFINITION 2.2. Let $\omega$ be a symplectic form on $S$. Then $\omega$ is *stable* at $p\in S_{2n-2}$ if for any neighbourhood $U$ of $p$ in $S_{2n-2}$ there is a neighbourhood $V$ of $\omega$ (in the $C^\infty$ topology on closed 2-forms) such that if $\beta$ is in $V$, then there is a point $q\in U$ and a germ of a diffeomorphism $\Phi:(R^{2n},q)\to(R^{2n},p)$ which preserves $S$ and $\Phi^*\beta=\omega$.

It is easy to see that the Darboux form on $S$ is stable.

PROPOSITION 2.3. *Let $\omega$ be a symplectic structure on $S$. Assume $f(0)=0$ and $df_0\neq 0$. Then $\omega$ is not stable at* 0.

Proof. From Theorem 2.1 it follows that $\omega$ can be reduced to the form

$$(x_1-\psi)d(x_1-\psi)\wedge dy_1+\sum_{i=2}^{n} dx_i\wedge dy_i.$$

Suppose the proposition is false. Let $U$ be a neigbourhood of $0\in R^{2n}$. $\psi(0)=0\in R$ is a critical value of $\psi|_U$. From the Sard theorem we see that there is $\epsilon\in R$ which is not a critical value of $\psi|_U$, in any neighbourhood of $0\in R$. Let $\beta=\alpha+\epsilon d(x_1-\psi)\wedge dy_1$. Then we can find a diffeomorphism $\Phi$ which preserves $S$ and $\Phi^*\beta=\omega$. Hence

$$\Phi^*\beta^n = \Phi^*((x_1-\psi+\epsilon)\Omega) = \omega^n = (x_1-\psi)\Omega.$$

Since $\Sigma_2(\omega)$ is tangent to $S_{2n-1}$ at 0, $\Sigma_2(\beta)$ is tangent to $S_{2n-1}$ at $q=\Phi(0)\in S_{2n-2}$. Therefore, we obtain

$$\psi(q)=\epsilon,\qquad d\psi_q = 0,$$

which contradicts the fact that $\epsilon$ is not a critical value of $\psi|_U$. ∎

**2.1.** Remark. Let us consider the following semialgebraic set:

$$S = S_{2n} \cup S_{2n-2} \subset R^{2n};$$

$$S_{2n} = \{(x, y) \in R^{2n} : x_1^3 > y_1^2\}, \qquad S_{2n-2} = \{(x, y) \in R^{2n} : x_1 = 0, y_1 = 0\}.$$

We notice the difference with the previous space: $\partial S_{2n}$ is a singular set.

We endow $S$ with a symplectic structure $\omega$. As before $G_S$ denotes the group of diffeomorphisms $(R^{2n}, 0) \to (R^{2n}, 0)$ preserving $S$. Let $\omega_1$, $\omega_2$ be two symplectic structures on $S$. We say that $\omega_1$ and $\omega_2$ are $S$-equivalent if and only if $\Phi^*\omega_1 = \omega_2$ for some $\Phi \in G_S$. Now we can show the following

PROPOSITION 2.4. *Let $\omega$ be a symplectic structure on $S$. Assume $f(0) = 0$ and $df_0 \neq 0$. Then $\omega$ is a singular form of type $\Sigma_{2,0}$ at zero.*

Proof. By straightforward use of the proof of Proposition 2.1.

An analogous Darboux theorem for the space $S$ is proved by Arnold ([2]): Let $\omega$ be a symplectic structure on $R^{2n}$. Then $\omega$ is $S$-equivalent with respect to formal equivalence to the Darboux form:

$$\omega \sim \sum_{i=1}^{n} dx_i \wedge dy_i.$$

**References**


[1] R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd ed., Benjamin/Cummings, Reading, 1978.

[2] V. I. Arnold, *Lagrangian manifolds with singularities, asymptotic rays and the open swallow tail*, Functional Anal. Appl. 15 (1981), 235–246.

[3] M. Golubitsky and D. Tischler, *An example of moduli for singular symplectic forms*, Invent. Math. 38 (1977), 219–225.

[4] S. Janeczko, *Coisotropic varieties and their generating families*, Ann. Inst. H. Poincaré 56 (1992), 429–441.

[5] E. Lerman, R. Montgomery and R. Sjamaar, *Examples of singular reduction*, preprint, 1991.

[6] J. Martinet, *Sur les singularités des formes différentielles*, Ann. Inst. Fourier (Grenoble) 20 (1) (1970), 95–178.

[7] —, *Singularities of Smooth Functions and Maps*, Cambridge Univ. Press, Cambridge, 1982.

[8] J. Moser, *On volume elements on manifolds*, Trans. Amer. Math. Soc. 120 (1965), 280–296.

[9] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. 134 (1991), 375–442.

[10] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge Univ. Press, Cambridge, 1984.

[11] A. Weinstein, *Lectures on Symplectic Manifolds*, CBMS Regional Conf. Ser. in Math. 29, Amer. Math. Soc., Providence, R.I., 1977.