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A REMARK ON UNIQUENESS FOR QUASILINEAR ELLIPTIC EQUATIONS

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1. Introduction. Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 1$. Assume that $a(x,u)$ is a Carathéodory function satisfying

$$0 < \alpha \leq a(x,u) \leq \beta \quad \text{a.e. } x \in \Omega, \qquad \forall u \in \mathbf{R} \tag{1.1}$$

where α, β are two positive constants. For $f \in H^{-1}(\Omega)$, $g \in H^1(\Omega)$ we would like to consider here the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(a(x,u)\dfrac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega, \\ u - g \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

We use the summation convention and we refer to [GT] or [KS] for the definition of the Sobolev spaces used throughout the paper.

First, under the above assumptions, using a fixed point argument of Schauder type, it is very easy to show that (1.2) admits a solution (see for instance [CM]). We would like to investigate here the question of uniqueness. More precisely we would like to prove the following result:

THEOREM 1. *Assume that for some positive constant C one has*

$$|a(x,u) - a(y,u)| \leq C|x-y| \qquad \forall u \in \mathbf{R},\ \forall x, y \in \Omega \tag{1.3}$$

or

$$|a(x,u) - a(x,v)| \leq C|u-v| \qquad \forall u, v \in \mathbf{R},\ \text{a.e. } x \in \Omega \tag{1.4}$$

then the problem (1.2) *has a unique solution* ($|\ |$ *denotes the usual euclidean norm in* $\mathbf{R}^p$). *If* (1.3), (1.4) *fail then uniqueness can fail even if* $u \to a(x,u)$ *is Hölder continuous of any order.*

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Remark 1.1. Loosely speaking uniqueness holds if and only if either $|\nabla_x a(x,u)|$ or $\partial a(x,u)/\partial u$ are uniformly bounded. In fact, it has been pointed out to us by P. Bénilan that $\nabla_x a(x,u) \in L^2(\Omega)$ is enough to insure here uniqueness.

This kind of problems were considered before by several authors (see [CC], [CM], [M], [T]), however, even in this simple case the picture was not yet complete. In particular, no counterexample seems to be known.

2. The proof of uniqueness.

Let us first consider the case where (1.3) holds. Then set

$$A(x,s) = \int_0^s a(x,t)dt. \tag{2.1}$$

If $u \in H^1(\Omega)$ then it is clear that

$$A(x,u(x)) \in H^1(\Omega). \tag{2.2}$$

Moreover, in the distributional sense one has

$$\frac{\partial}{\partial x_i} A(x,u) = a(x,u)\frac{\partial u}{\partial x_i} + \int_0^{u(x)} \frac{\partial a(x,t)}{\partial x_i} dt. \tag{2.3}$$

Then let us prove:

Proposition 2.1. *Assume that* (1.3) *holds. Then* (1.2) *has a unique solution.*

Proof. Let us denote by u, v two solutions of (1.1). By subtraction we get

$$-\frac{\partial}{\partial x_i}\left(a(x,u)\frac{\partial u}{\partial x_i} - a(x,v)\frac{\partial v}{\partial x_i}\right) = 0 \quad \text{in } \Omega. \tag{2.4}$$

But thanks to (2.3) this reads also

$$-\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(A(x,u) - A(x,v)) - \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i} dt\right) = 0 \quad \text{in } \Omega. \tag{2.5}$$

If f is a function we denote by $[f > 0]$ the set defined by $[f > 0] = \{x \in \Omega \mid f(x) > 0\}$ and we use similar notation for $[0 < f \le \epsilon]$. Then we have

Lemma 2.1. *For any* $\xi \in C^1(\overline{\Omega})$

$$\int_{[u-v>0]} \left[\frac{\partial}{\partial x_i}(A(x,u) - A(x,v)) - \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i} dt\right] \frac{\partial \xi}{\partial x_i}\ dx = 0. \tag{2.6}$$

Let us postpone for the time being the proof of this lemma. Then, integrating by parts in (2.6) we obtain (recall that $A(x,u) - A(x,v) \in H_0^1(\Omega)$)

$$\int_{[u-v>0]} A(x,u) - A(x,v)\ \frac{\partial^2 \xi}{\partial x_i^2} + \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i} dt \frac{\partial \xi}{\partial x_i}\, dx$$

which reads also

$$\int_{[u-v>0]} \int_{v(x)}^{u(x)} \left[a(x,t)\frac{\partial^2 \xi}{\partial x_i^2} + \frac{\partial a(x,t)}{\partial x_i}\frac{\partial \xi}{\partial x_i}\right] dtdx = 0. \tag{2.7}$$

Then, choosing $\xi = e^{\gamma x_1}$ in (2.7), we have for γ large enough and by (1.3)

$$a(x,t)\frac{\partial^2\xi}{\partial x_i^2} + \frac{\partial a(x,t)}{\partial x_i}\frac{\partial \xi}{\partial x_i} \ge e^{\gamma x_1}(\gamma^2\alpha - C\gamma) > 0 \tag{2.8}$$

and then in order for (2.7) to hold, $[u-v>0]$ must have measure 0. This leads to $u \le v$ and reversing the role of u and v to $u = v$.

Proof of the lemma. Let us denote by $(\ )^+$ the positive part of a function, by $\min[\ ,\ ]$ the minimum of two functions. Remark then that for $\xi \in C^1(\overline{\Omega})$, $\epsilon > 0$,

$$\min[(A(x,u) - A(x,v))^+/\epsilon, 1]\cdot \xi \in H_0^1(\Omega).$$

Thus, multiplying (2.5) by the above function and integrating over Ω we deduce (for simplicity we set below $A(x,u) = A(u)$),

$$\int_\Omega \left[\frac{\partial}{\partial x_i}(A(u)-A(v)) - \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i}dt\right] \min[(A(u)-A(v))^+/\epsilon]\frac{\partial \xi}{\partial x_i}\ dx \tag{2.9}$$
$$= -\int_{[0<A(u)-A(v)\le\epsilon]} \left[\frac{\partial}{\partial x_i}(A(u)-A(v)) - \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i}dt\right] \frac{\partial}{\partial x_i}\left(\frac{A(u)-A(v)}{\epsilon}\right)\xi\, dx.$$

Let us denote by χ_A the characteristic function of the set A. By (1.1) one has

$$[0 < u - v] = [0 < A(u) - A(v)].$$

So, when $\epsilon \to 0$

$$\min[(A(u)-A(v))^+/\epsilon, 1] \to \chi_{[0<u-v]} \quad \text{a.e. in } \Omega.$$

It follows, by the Lebesgue convergence theorem, that the first integral in (2.9) converges to

$$\int_{[0<u-v]} \left[\frac{\partial}{\partial x_i}(A(u)-A(v)) - \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i}dt\right] \frac{\partial \xi}{\partial x_i}\, dx.$$

Now, we claim that for $\xi \ge 0$

$$\lim_{\epsilon\to 0} \int_{[0<A(u)-A(v)\le\epsilon]} \left[\frac{\partial}{\partial x_i}(A(u)-A(v)) - \int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i}dt\right] \frac{\partial}{\partial x_i}\left(\frac{A(u)-A(v)}{\epsilon}\right)\xi\, dx \ge 0.$$

Indeed, this integral reads also

$$\frac{1}{\epsilon}\int_{[0<A(u)-A(v)\le\epsilon]} |\nabla(A(u)-A(v))|^2 \xi\ dx \tag{2.10}$$
$$- \int_{[0<A(u)-A(v)\le\epsilon]}\int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i}dt\frac{\partial}{\partial x_i}\left(\frac{A(u)-A(v)}{\epsilon}\right)\xi\ dx$$
$$\ge -\int_{[0<A(u)-A(v)\le\epsilon]}\int_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i}dt\frac{\partial}{\partial x_i}\left(\frac{A(u)-A(v)}{\epsilon}\right)\xi\ dx$$

$$\geq -\frac{1}{\epsilon} \int\limits_{[0<A(u)-A(v)\leq\epsilon]} \int\limits_{v(x)}^{u(x)} |\nabla a(x,t)| dt |\nabla(A(u)-A(v))| \xi \, dx.$$

Now, by (1.3) one has for some constant C

$$\int\limits_{v(x)}^{u(x)} |\nabla a(x,t)| dt \leq C(u(x)-v(x)).$$

Moreover, when $u \geq v$ one has

$$\alpha(u(x)-v(x)) \leq \int\limits_{v(x)}^{u(x)} a(x,t) dt = A(u) - A(v).$$

Hence on $[0 < A(u) - A(v) \leq \epsilon]$

$$\int\limits_{v(x)}^{u(x)} |\nabla a(x,t)| dt \leq \frac{C}{\alpha}(A(u)-A(v)) \leq \frac{C}{\alpha}\epsilon$$

and the last integral in (2.10) is bounded from below by

$$-\frac{C}{\alpha} \int\limits_{[0<A(u)-A(v)\leq\epsilon]} |\nabla(A(u)-A(v))| \; \xi \, dx$$

which goes to 0 with ϵ.

Collecting the above results and letting $\epsilon \to 0$ in (2.9) we obtain for $\xi \in C^1(\overline{\Omega})$, $\xi \geq 0$

$$\int\limits_{[u-v>0]} \left[\frac{\partial}{\partial x_i}(A(x,u)-A(x,v)) - \int\limits_{v(x)}^{u(x)} \frac{\partial a(x,t)}{\partial x_i} dt \right] \frac{\partial \xi}{\partial x_i} \, dx \leq 0.$$

Changing ξ into $M - \xi$ for M large enough such that $M - \xi \geq 0$ leads to (2.6) for any $\xi \in C^1(\overline{\Omega})$.

Let us now turn to the case where (1.4) holds. In fact, we would like to use here a slightly more general assumption. Indeed let us set

$$\omega_a(t) = \sup_{x\in\Omega,\ |u-v|\leq t} |a(x,u)-a(x,v)| \tag{2.11}$$

and let us assume

$$\int\limits_{0^+} \frac{ds}{\omega_a(s)} = +\infty. \tag{2.12}$$

Clearly if (1.4) holds one has $\omega_a(t) \leq Ct$ and (2.12) holds.

Then we have:

PROPOSITION 2.2. *Assume that* (2.11), (2.12) *hold. Then* (1.2) *has a unique solution.*

P r o o f. Let us denote again by u, v two solutions of (1.1). Then for $\epsilon > 0$ let us set

$$F_\epsilon = \begin{cases} \int\limits_{\epsilon}^{x} ds/\omega(s)^2 & \text{when } x \geq \epsilon, \\ 0 & \text{when } x < \epsilon, \end{cases} \tag{2.13}$$

where for the sake of simplicity we have set $\omega_a = \omega$ (note that $\omega(t) > 0$ when $t > 0$ unless a is independent of u). From (2.4), multiplying by

$$F_\epsilon(u - v) \in H_0^1(\Omega) \tag{2.14}$$

and integrating by parts, we get

$$\int_\Omega \left(a(x,u)\frac{\partial u}{\partial x_i} - a(x,v)\frac{\partial v}{\partial x_i} \right) \cdot \frac{\partial}{\partial x_i} F_\epsilon(u - v)\; dx = 0. \tag{2.15}$$

This can be rewritten as

$$\begin{aligned} \int_\Omega a(x,u)\frac{\partial(u-v)}{\partial x_i} \cdot \frac{\partial}{\partial x_i} F_\epsilon(u - v)\, dx & \\ = - \int_\Omega (a(x,u) - a(x,v))\frac{\partial v}{\partial x_i} \cdot \frac{\partial}{\partial x_i} F_\epsilon(u - v)\; dx. \end{aligned} \tag{2.16}$$

From (2.13) we deduce

$$\begin{aligned} \int_{[u-v>\epsilon]} a(x,u)\frac{|\nabla(u-v)|^2}{\omega^2(u-v)}\; dx & \\ &= - \int_{[u-v>\epsilon]} \frac{(a(x,u) - a(x,v))}{\omega^2(u-v)}\frac{\partial v}{\partial x_i} \cdot \frac{\partial}{\partial x_i}(u - v)\; dx \\ &\leq \int_{[u-v>\epsilon]} \frac{\omega(u-v)}{\omega^2(u-v)}|\nabla v||\nabla(u-v)|\; dx \\ &= \int_{[u-v>\epsilon]} |\nabla v|\frac{|\nabla(u-v)|}{\omega(u-v)}\; dx. \end{aligned} \tag{2.17}$$

Hence by (1.1) and the Cauchy–Schwarz inequality we obtain

$$\alpha \int_{[u-v>\epsilon]} \frac{|\nabla(u-v)|^2}{\omega^2(u-v)}\, dx \leq \left[\int_{[u-v>\epsilon]} |\nabla v|^2\; dx \right]^{\frac{1}{2}} \left[\int_{[u-v>\epsilon]} \frac{|\nabla(u-v)|^2}{\omega^2(u-v)}\, dx \right]^{\frac{1}{2}}$$

from which it follows

$$\int_{[u-v>\epsilon]} \frac{|\nabla(u-v)|^2}{\omega^2(u-v)}\, dx \leq \frac{1}{\alpha^2} \int_\Omega |\nabla v|^2\; dx.$$

Set

$$G_\epsilon = \begin{cases} \displaystyle\int_\epsilon^x \frac{ds}{\omega(s)} & \text{when } x \geq \epsilon, \\ 0 & \text{when } x < \epsilon. \end{cases}$$

Then the above inequality reads

$$\int_\Omega |\nabla G_\epsilon(u - v)|^2\, dx \leq \frac{1}{\alpha} \int_\Omega |\nabla v|^2\, dx.$$

Hence, by the Poincaré Inequality (see [BKS] for a similar argument), for some positive constant C

$$\int_{\Omega} |G_\epsilon(u-v)|^2 \, dx \le C \int_{\Omega} |\nabla v|^2 \, dx.$$

Letting ϵ go to 0 we deduce by (2.12) that $u \le v$ and the result follows by exchanging the role of u and v.

So, we have established the part of Theorem 1 regarding uniqueness. Let us turn now to the second part of this theorem.

3. A class of counterexamples. We are going to construct one dimensional counterexamples. So, for $\Omega = (a_1, a_2)$ we will consider the problem

$$\begin{cases} -(a(x,u)u')' = f & \text{in } \Omega, \\ u(a_1) = A_1, \quad u(a_2) = A_2, \end{cases} \tag{3.1}$$

where a_1, a_2, A_1, A_2 are constants. Let us prove:

PROPOSITION 3.1. *Assume that* (1.3), *or* (2.11), (2.12) *fail*, *then* (1.2) *or* (3.1) *may have several solutions even if* $u \to a(x,u)$ *is Hölder continuous of any order* γ, $\gamma \in (0,1)$ *i.e. even if*

$$|a(x,u) - a(x,v)| \le C|u-v|^\gamma \qquad \forall u, v \in \mathbf{R}, \ \ a.e. \ x \in \Omega. \tag{3.2}$$

Proof. Let ω be a nondecreasing, continuous function such that

$$\omega(0) = 0, \quad \omega(t) > 0 \quad \forall t > 0, \qquad \int_{0^+} \frac{ds}{\omega(s)} < +\infty, \tag{3.3}$$

$$\omega(t)/t \text{ is nonincreasing.} \tag{3.4}$$

We are going to construct a counterexample to uniqueness of the type of (3.1) with an a having a modulus of continuity ω_a equivalent to ω. Set

$$\theta(s) = \int_0^s \frac{ds}{\omega(s)}. \tag{3.5}$$

Then, θ is a one-to-one mapping from $[0,T]$ into $[0,\theta(T)]$ for any $T > 0$. Let us denote by θ^{-1} its inverse. One has clearly

$$\frac{d\theta^{-1}}{dy}(y) = \omega(\theta^{-1}(y)) \qquad \forall y > 0. \tag{3.6}$$

Let u be a smooth increasing function defined on (a_1, a_2), and such that $u(a_1) = A_1 < A_2 = u(a_2)$. Then, let us define v by

$$v = \begin{cases} u + \theta^{-1}(x - a_1) & \text{in a neighbourhood of } a_1, \\ u + \theta^{-1}(a_2 - x) & \text{in a neighbourhood of } a_2, \end{cases} \tag{3.7}$$

v being smooth and such that

$$v > u, \quad v' > 0 \quad \text{on } (a_1, a_2). \tag{3.8}$$

It is clear that such a definition is always possible. Now, let us define $a(x,u)$ by setting

$$a(x,u) = \begin{cases} 1 & \text{if } x \notin (a_1,a_2), \\ 1 & \text{if } x \in (a_1,a_2) \text{ and if } u \le u(x), \\ \dfrac{u'(x)}{v'(x)} & \text{if } x \in (a_1,a_2) \text{ and if } u \ge v(x), \\ \delta + (1-\delta)\dfrac{u'(x)}{v'(x)} & \text{if } x \in (a_1,a_2) \\ & \text{and if } u = \delta u(x) + (1-\delta)v(x). \end{cases} \tag{3.9}$$

It is clear that a defined that way is continuous in both variables x, u (note that $u'(a_1) = v'(a_2)$, $u'(a_2) = v'(a_2)$). Moreover, (1.1) holds.

From this choice of a one has obviously

$$a(x,v)v' = a(x,u)u' = u' \qquad \text{on } (a_1,a_2) \tag{3.10}$$

so that u and v are both solution to (3.1) with $f = -u''$.

Now, for t small enough, there exists x close to a_1 or a_2 such that $|u(x) - v(x)| = t$, then by (3.9),

$$a(x,u(x)) - a(x,v(x)) = 1 - \frac{u'(x)}{v'(x)} = \frac{v'(x) - u'(x)}{v'(x)}.$$

But, in the neighbourhood of a_1 or a_2 one has (for instance for a_1, and by (3.6))

$$(v-u)' = \omega(\theta^{-1}(x-a_1)) = \omega(v-u) \tag{3.11}$$

and thus,

$$a(x,u(x)) - a(x,v(x)) = \frac{1}{v'(x)}\, \omega(v(x) - u(x)) = \frac{1}{v'(x)}\, \omega(t).$$

So, for t small $\omega_a(t) \ge C\omega(t)$ for some constant C, hence

$$\int_{0^+} \frac{ds}{\omega_a(s)} \le \frac{1}{C} \int_{0^+} \frac{ds}{\omega(s)}$$

and (2.12) fails. Of course, one can show that (1.3) fails as well.

In the case where (3.4) holds, let us now prove that, for some constant C, one has also

$$\omega_a(t) \le C\omega(t). \tag{3.12}$$

For that, remark that if P denotes the projection of $\mathbf{R}$ onto $[u(x), v(x)]$, i.e. if

$$P(y) = \begin{cases} u(x) & \text{if } y \le u(x), \\ y & \text{if } y \in [u(x), v(x)], \\ v(x) & \text{if } y \ge v(x), \end{cases} \tag{3.13}$$

then, by definition of a one has

$$a(x,y) = a(x,P(y)). \tag{3.14}$$

So, if we prove that

$$|a(x,z) - a(x,z')| \le C\omega(|z-z'|) \qquad \forall z,z' \in [u(x),v(x)], \text{ a.e. } x \in \Omega \tag{3.15}$$

we will have (3.12) since from (3.14) it will follow

$$|a(x,u)-a(x,v)| = |a(x,P(u))-a(x,P(v))| \le C\omega(|P(u)-P(v)|) \le C\omega(|u-v|) \quad \forall u,v \in \mathbf{R}, \text{ a.e. } x \in \Omega. \tag{3.16}$$

To prove (3.15) consider for $\delta, \delta' \in [0,1]$

$$z = \delta u(x) + (1-\delta)v(x), \qquad z' = \delta' u(x) + (1-\delta')v(x). \tag{3.17}$$

From (3.9) one has

$$a(x,z) - a(x,z') = (\delta - \delta')[1 - u'(x)/v'(x)] \tag{3.18}$$

and

$$z - z' = (\delta - \delta')(u(x) - v(x)). \tag{3.19}$$

So, for x outside of neighbourhoods of a_1 and a_2 one has

$$|a(x,z) - a(x,z')| = |z-z'||1 - u'(x)/v'(x)|/|u-v| \le C|z-z'|. \tag{3.20}$$

Let us fix some $t_0 > 0$. Since by (3.4) the function $\omega(t)/t$ is nonincreasing, one has

$$\omega(t)/t \ge \omega(t_0)/t_0 \qquad \forall t \le t_0 \tag{3.21}$$

hence for some constant C

$$\omega(t) \ge Ct \qquad \forall t \le t_0. \tag{3.22}$$

It then follows from (3.20) that

$$|a(x,z) - a(x,z')| \le C'\omega(|z-z'|). \tag{3.23}$$

C' depends of course on the neigbourhoods of a_1, a_2 considered. Note also that a being bounded we need only to establish (3.23) for small values of $|z-z'|$. Now, in the neighbourhood of a_1 or a_2, by (3.11), (3.18) one has

$$\begin{aligned} |a(x,z) - a(x,z')| &= |\delta - \delta'|\left|\frac{v'-u'}{v'}\right| = |\delta-\delta'|\frac{\omega(v-u)}{v'} \\ &= \frac{1}{v'}\frac{|z-z'|}{v-u}\omega(v-u) \\ &= \frac{1}{v'}\omega(|z-z'|)\frac{|z-z'|}{\omega(|z-z'|)}\frac{\omega(v-u)}{v-u}. \end{aligned} \tag{3.24}$$

Using (3.4) and the fact that $|z-z'| \le v-u$ we derive

$$|a(x,z) - a(x,z')| \le \frac{1}{v'}\omega(|z-z'|).$$

So, combining this with (3.23), we get for some constant C

$$|a(x,z) - a(x,z')| \le C\omega(|z-z'|) \tag{3.25}$$

and (3.12) follows.

In particular, when $\omega(t) = t^\gamma$, $\gamma \in (0,1)$ (note that for such an ω (3.3), (3.4) hold) (3.16) reads

$$|a(x,u) - a(x,v)| \le C|u-v|^\gamma \qquad \forall u,v \in \mathbf{R}, \text{ a.e. } x \in \Omega$$

which is (3.2). This completes the proof of proposition 3.1.

Remark 3.1. One can produce examples with more than two solutions by piling up different functions v.

Remark 3.2. When in (1.2) $g = 0$, it is still possible to produce examples of nonuniqueness. For instance consider the construction we just made on $(a_1, a_2) = (-1, 0)$ and with $0 = A_1 < A_2$. Then, symmetrise u and v on $(0, 1)$. It is clear that we are producing that way a counterexample to uniqueness on $(-1, 1)$ with homogeneous boundary data i.e. with $g = 0$.

4. Concluding remarks. In fact Theorem 1 can also be rephrased into a comparison principle. More precisely we have:

THEOREM 1. *Assume that for some positive constant C one has*

$$|a(x,u) - a(y,u)| \le C|x-y| \qquad \forall u \in \mathbf{R},\ \forall x, y \in \Omega \tag{1.3}$$

or

$$|a(x,u) - a(x,v)| \le C|u-v| \qquad \forall u, v \in \mathbf{R},\ a.e.\ x \in \Omega. \tag{1.4}$$

Let us denote by u_1, u_2 the solution to (1.2) *corresponding respectively to the data (f_1, g_1), (f_2, g_2). Then if $f_1 \le f_2$ and $g_1 \le g_2$, one has $u_1 \le u_2$. ($f_1 \le f_2$ is for instance taken in the H^{-1} or in the measures sense).*

Proof. In the case when (1.3) holds, (2.5) becomes

$$-\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(A(x,u_1) - A(x,u_2)) - \int_{u_2(x)}^{u_1(x)} \frac{\partial a(x,t)}{\partial x_i} dt\right) \le 0 \quad \text{ in } \Omega.$$

So, one can establish (2.6) (with u, v replaced by u_1, u_2) as in section 2 and conclude the same way that $u_1 \le u_2$.

In the case where (1.4) holds, since $F_\epsilon(u_1 - u_2) \in H_0^1(\Omega)$ and $F_\epsilon(u_1 - u_2) \ge 0$, (2.16) becomes

$$\int_\Omega a(x,u_1)\frac{\partial(u_1 - u_2)}{\partial x_i} \cdot \frac{\partial}{\partial x_i} F_\epsilon(u_1 - u_2)\, dx \le -\int_\Omega (a(x,u_1) - a(x,u_2))\frac{\partial u_2}{\partial x_i} \cdot \frac{\partial}{\partial x_i} F_\epsilon(u_1 - u_2)\, dx$$

and the proof is the same.

The situation is quite different for the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\left(a(x,u)\dfrac{\partial u}{\partial x_i}\right) + \lambda(x)u = f & \text{in } \Omega, \\ u - g \in H_0^1(\Omega). \end{cases}$$

where $\lambda(x)$ is a positive function. Here uniqueness can hold even when both (1.3), (1.4) fail. We refer the reader to [A], [AC].

With a similar technique uniqueness and nonuniqueness results are also available for more general nonlinear problems as for instance variational inequalities associated to nonlinear operators of the type considered here (see [CM], [M]), or systems (see [A], [CFM]).

It is also possible to consider the parabolic analogue of our problem i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x_i}\left(a(x,u)\dfrac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega \times (0,T), \\ u - g \in H_0^1(\Omega) \quad t \in (0,T), \qquad u(.,0) = u_0. \end{cases}$$

In this situation uniqueness may also hold even if (1.3), (1.4) fail (see [Ar], [CR], [R]).

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