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AN ALGEBRAIC METHOD FOR CALCULATING THE TOPOLOGICAL DEGREE

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1. Introduction. Effective methods for calculating the topological degree for a continuous mapping are very useful. In this paper we present an algebraic method which applies to polynomial mappings. We shall show that in this case the topological degree can be expressed in terms of signatures of some effectively defined bilinear forms (see Theorem 4.1).

The method may be derived from the theory of bilinear forms on finite intersection algebras given by Scheja & Storch [10], Eisenbud & Levine [5], Khimshiashvili [8], Kunz [7] and Cardinal [4]. All facts needed for the proof of Theorem 4.1 are presented in [2].

The complete proof requires some advanced facts concerning complete intersection algebras. In this paper we explain the method for polynomial mappings having only non-degenerate roots. This way we may avoid difficult details and make the main idea of the method to be more clear.

In the case of the local topological degree there is a similar formula (so called Eisenbud & Levine formula). One can find its proof in [1], [2], [5], [8]. In [9] one may find a description of an algorithm which has been used to create a computer program which can calculate the local topological degree.

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2. Preliminaries. In this section we shall collect some useful facts concerning bilinear forms and polynomial algebras.

Let R (resp. C) denote the field of real (resp. complex) numbers. Let V be a finite dimensional real vector space and let $\Phi : V \times V \to R$ be a bilinear symmetric form. Let V_+ (resp. V_-) denote a maximal subspace of V on which Φ is positive (resp. negative) definite, i.e. if $x \in V_+ - \{0\}$ (resp. $x \in V_- - \{0\}$) then $\Phi(x,x) > 0$ (resp. $\Phi(x,x) < 0$). We define

$$\operatorname{signature}\Phi = \dim V_+ - \dim V_- .$$

We shall say that Φ is *non-degenerate* if its matrix is non-singular.

LEMMA 2.1. *Let $\varphi : R \to R$ be an R-linear functional and let $\Phi : R \times R \to R$ be the bilinear form given by $\Phi(x,y) = \varphi(xy)$. Then* $\operatorname{signature}\Phi = \operatorname{sign}\varphi(1)$. *Moreover Φ is non-degenarate if and only if $\varphi(1) \neq 0$.*

Proof. Since φ is R-linear then for every $x \in R - \{0\}$ we have $\Phi(x,x) = \varphi(x^2) = \varphi(x^2 \cdot 1) = x^2\varphi(1)$. Because $x^2 > 0$ then $\operatorname{signature}\Phi = \operatorname{sign}\varphi(1)$.

LEMMA 2.2. *Let $\varphi : C \to R$ be an R-linear functional and let $\Phi : C \times C \to R$ be the bilinear form given by $\Phi(z,w) = \varphi(zw)$. Then* $\operatorname{signature}\Phi = 0$.

Proof. Let $V_+ \subset C$ denote a maximal R-subspace on which Φ is positive definite, i.e. $\Phi(z,z) = \varphi(z^2) > 0$ for every $z \in V_+ - \{0\}$. Then $\sqrt{-1}\, V_+$ is an R-subspace of C and if $w = \sqrt{-1}\, z \in \sqrt{-1}\, V_+ - \{0\}$ then $\Phi(w,w) = \varphi(w^2) = \varphi(-z^2) = -\varphi(z^2) < 0$. Hence $\dim V_- \geq \dim \sqrt{-1}\, V_+ = \dim V_+$.

By similar arguments $\dim V_+ \geq \dim V_-$. Hence $\dim V_+ = \dim V_-$ and $\operatorname{signature}\Phi = 0$. ∎

Let

$$\mathcal{B} = R \oplus \cdots \oplus R \oplus C \oplus \cdots \oplus C = \overset{m}{\underset{1}{\oplus}}\, R \overset{r}{\underset{1}{\oplus}}\, C.$$

Then $\mathcal{B}$ is a finite dimensional R-algebra. Let $\varphi : \mathcal{B} \to R$ be an R-linear functional. Denote

$$\begin{aligned} s_1 &= \varphi(1 \oplus 0 \oplus \cdots \oplus 0), \\ &\ \ \vdots \\ s_m &= \varphi(0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0). \end{aligned}$$

From previous lemmas we get

PROPOSITION 2.3. *Let $\Phi : \mathcal{B} \times \mathcal{B} \to R$ be the bilinear form given by $\Phi(f,g) = \varphi(fg)$. Then*

$$\operatorname{signature}\Phi = \#\{1 \leq i \leq m : s_i > 0\} - \#\{1 \leq i \leq m : s_i < 0\}.$$

Moreover if Φ is non-degenerate then $s_1 \neq 0, \ldots, s_m \neq 0$.

Let $f_1, \ldots, f_n \in R[x_1, \ldots, x_n]$, let $F_R = (f_1, \ldots, f_n) : R^n \to R^n$ and let $F_C : C^n \to C^n$ be its complexification. Let

$$\mathcal{J} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$$

denote the determinant of the Jacobian matrix. Let $\mathcal{A} = R[x_1, \ldots, x_n] \,/\, \mathrm{I}$, where I is the ideal in $R[x_1, \ldots, x_n]$ generated by polynomials $f_1, \ldots, f_n$. Then $\mathcal{A}$ is an R-algebra.

From now on we shall assume that $d = \dim \mathcal{A} < \infty$ and that F_C has only non-degenerate complex roots, i.e. if $z \in F_C^{-1}(0)$ then $\mathcal{J}(z) \neq 0$.

The next two facts generalize the Fundamental Theorem of Algebra. They follow immediately from Corollary 1 in [6], p.57.

THEOREM 2.4. $\#\{z \in C^n : F_C(z) = 0\} = \dim \mathcal{A} = d.$

So there are d complex roots for F_C and we may assume that

$$F_C^{-1}(0) = \{p_1, \ldots, p_m, q_1, \bar{q}_1, \ldots, q_r, \bar{q}_r\},$$

where $p_1, \ldots, p_m \in R^n$, $q_1, \ldots, q_r \in C^n - R^n$ and $\bar{q}_i$ is the complex conjugate of q_i. Clearly $m + 2r = d$.

If $f \in \mathrm{I}$ then $f = 0$ on $F_C^{-1}(0)$. Then there is an R-homomorphism of algebras

$$\Psi : \mathcal{A} \to \mathcal{B} = \overset{m}{\underset{1}{\oplus}}\ R \overset{r}{\underset{1}{\oplus}}\ C$$

given by $\Psi(f) = f(p_1) \oplus \cdots \oplus f(p_m) \oplus f(q_1) \oplus \cdots \oplus f(q_r)$. It is easy to see that $\dim \mathcal{B} = m + 2r = d = \dim \mathcal{A}$.

THEOREM 2.5. *If $f = 0$ on $F_C^{-1}(0)$ then $f \in \mathrm{I}$. Hence $\Psi : \mathcal{A} \to \mathcal{B}$ is an isomorphism of R-algebras. Thus $g = h$ in $\mathcal{A}$ if and only if $g(p_i) = h(p_i)$ for $1 \le i \le m$ and $g(q_j) = h(q_j)$ for $1 \le j \le r$.*

3. The construction of bilinear forms. Denote $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Define $\mathcal{A}^2 = R[x, y] \,/\, \mathrm{I}_2$, where I_2 is the ideal in $R[x, y]$ generated by $f_1(x), \ldots, f_n(x)$, $f_1(y), \ldots, f_n(y)$. One may check that $\mathcal{A}^2$ is isomorphic to $\mathcal{A} \otimes \mathcal{A}$.

For $1 \le i, j \le n$ define

$$T_{ij}(x, y) \;=\; \frac{f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}$$

It is easy to see that each T_{ij} extends to a polynomial, thus we may assume that $T_{ij} \in R[x, y]$. Define

$$T(x, y) = \det\, [T_{ij}(x, y)].$$

It is easy to see that $\mathcal{J}(x) = T(x, x)$.

THEOREM 3.1. *For any polynomial $q(x)$ we have*

$$q(x)T(x, y) = q(y)T(x, y) \quad \text{in } \mathcal{A}^2.$$

Proof. Note B_j the j-th column of $[\, T_{ij}(x, y)]$. Then

$$(x_j - y_j)B_j = \begin{bmatrix} f_1(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_1(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n) \\ \vdots \\ f_n(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_n(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n) \end{bmatrix}$$

We do not change the determinant if we add to this column a linear combination of the form

$$\sum_{k \neq j} (x_k - y_k) B_k.$$

The j-th column then becomes

$$\sum_{k=1}^{n}(x_k - y_k)B_k = \begin{bmatrix} f_1(x_1,\ldots,x_n) - f_1(y_1,\ldots,y_n) \\ \vdots \\ f_n(x_1,\ldots,x_n) - f_n(y_1,\ldots,y_n) \end{bmatrix}$$

Developing this determinant relatively to the j-th column we get an element of the ideal I_2. Hence

$$(x_j - y_j)T(x,y) = 0 \qquad \text{in} \quad \mathcal{A}^2,$$

and then $x_jT(x,y) = y_jT(x,y)$ in $\mathcal{A}^2$. Hence

$$x_kx_jT(x,y) = x_ky_jT(x,y) = y_ky_jT(x,y) \qquad \text{in} \quad \mathcal{A}^2$$

and by induction

$$x_1^{a_1}\cdots x_n^{a_n}T(x,y) = y_1^{a_1}\cdots y_n^{a_n}T(x,y) \qquad \text{in} \quad \mathcal{A}^2.$$

So the theorem is true if $q(x)$ is a monomial. One gets the general case by linearity.

PROPOSITION 3.2. *Suppose that* $p,q \in F_C^{-1}(0)$. *If* $p = q$ *then* $T(p,q) = T(p,p) = \mathcal{J}(p)$, *if* $p \neq q$ *then* $T(p,q) = 0$.

Proof. We have already proved that $T(p,p) = \mathcal{J}(p)$. Suppose that $p \neq q$. There is a polynomial $Q(x) \in C[x]$ such that $Q(p) \neq 0$ and $Q(q) = 0$. Applying the same arguments as in the proof of the previous theorem one can see that there are $h_1,\ldots,h_n$, $g_1,\ldots,g_n \in C[x,y]$ such that

$$Q(x)T(x,y) = Q(y)T(x,y) + \sum_{i=1}^{n} h_i(x,y)f_i(x) + \sum_{j=1}^{n} g_j(x,y)f_j(y).$$

Since $f_1(p) = \ldots = f_n(p) = f_1(q) = \ldots = f_n(q) = 0$ then $Q(p)T(p,q) = Q(q)T(p,q) = 0$, and then $T(p,q) = 0$. ■

Suppose that $e_1(x),\ldots,e_d(x)$ form a basis in $\mathcal{A}$. Since $\mathcal{A}^2$ is isomorphic to $\mathcal{A}\otimes\mathcal{A}$ then $e_i(x)e_j(y)$ for $1 \le i,j \le d$ form a basis in $\mathcal{A}^2$. Hence there are $t_{ij} \in R$ such that

$$T(x,y) = \sum_{i,j=1}^{d} t_{ij}e_i(x)e_j(y) = \sum_{i=1}^{d} e_i(x)\hat{e}_i(y) \text{ in } \mathcal{A}^2,$$

where $\hat{e}_i = \sum_{j=1}^{d} t_{ij}e_j$.

THEOREM 3.3. $\hat{e}_1,\ldots,\hat{e}_d$ *form a basis in* $\mathcal{A}$.

Proof. According to Theorem 2.5, $\mathcal{A}$ is isomorphic to the product $\mathcal{B} = \bigoplus_1^m R \bigoplus_1^r C$. Let $E_1,\ldots,E_d$ be the basis given by

$$\begin{gathered} E_1 = 1 \oplus 0 \oplus \cdots \oplus 0, \quad E_2 = 0 \oplus 1 \oplus \cdots \oplus 0, \ldots, \\ E_{m+1} = 0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0, \quad E_{m+2} = 0 \oplus \cdots \oplus \sqrt{-1} \oplus \cdots \oplus 0, \ldots, \\ E_{d-1} = 0 \oplus \cdots \oplus 0 \oplus 1, \quad E_d = 0 \oplus \cdots \oplus 0 \oplus \sqrt{-1}. \end{gathered}$$

Using Proposition 3.2 it is easy to see that elements $\hat{E}_1, \ldots, \hat{E}_d$ constructed as above form a basis. Moreover, since $e_1, \ldots, e_d$ are non-singular combinations of $E_1, \ldots, E_d$ then $\hat{e}_1, \ldots, \hat{e}_d$ are non-singular combinations of $\hat{E}_1, \ldots, \hat{E}_d$, and then they form a basis. ∎

Then there are $a_1, \ldots, a_d \in R$ such that $1 = a_1\hat{e}_1 + \cdots + a_d\hat{e}_d$ in $\mathcal{A}$. Hence if $p \in F_C^{-1}(0)$ then

$$a_1\hat{e}_1(p) + \cdots + a_d\hat{e}_d(p) = 1.$$

DEFINITION. Let $\varphi : \mathcal{A} \to R$ be the linear functional given by

$$\varphi(f) = a_1b_1 + \cdots + a_db_d,$$

for $f = b_1e_1 + \cdots + b_de_d \in \mathcal{A}$.

LEMMA 3.4. *If* $p_i \in F_R^{-1}(0)$ *for* $1 \le i \le m$ *and* $T_i(x) = T(x, p_i) \in \mathcal{A}$ *then* $\varphi(T_i) = 1$.

Proof. Since $T(x,y) = \sum_{j=1}^{d} e_j(x)\hat{e}_j(y)$ in $\mathcal{A}^2$ then there are $h_k, g_k \in R[x,y]$ such that

$$T(x,y) = \sum_{j=1}^{d} e_j(x)\hat{e}_j(y) + \sum_{k=1}^{n} (h_k(x,y)f_k(x) + g_k(x,y)f_k(y)).$$

Because $f_1(p_i) = \ldots = f_n(p_i) = 0$ then

$$T_i(x) = \sum_{j=1}^{d} e_j(x)\hat{e}_j(p_i) + \sum_{k=1}^{n} h_k(x,p_i)f_k(x),$$

and then $T_i = \hat{e}_1(p_i)e_1(x) + \cdots + \hat{e}_d(p_i)e_d(x)$ in $\mathcal{A}$. So $\varphi(T_i) = a_1\hat{e}_1(p_i) + \cdots + a_d\hat{e}_d(p_i) = 1$. ∎

Take $p_i \in F_R^{-1}(0)$. We have assumed that $\mathcal{J}(p) \neq 0$ for every $p \in F_C^{-1}(0)$, so $\mathcal{J}(p_i) \neq 0$. Let $t_i = T_i / \mathcal{J}(p_i) \in \mathcal{A}$. From Proposition 3.2, $t_i(p_i) = 1$ and $t_i(q) = 0$ for every $q \in F_C^{-1}(0)$, $q \neq p_i$. Let $\Psi : \mathcal{A} \to \mathcal{B}$ be the isomorphism of algebras defined before. Then $\Psi(t_i) = 0 \oplus \cdots \oplus 1 \oplus \cdots \oplus 0$, where 1 is in the i-th factor.

Let $\Phi : \mathcal{A} \times \mathcal{A} \to R$ be the bilinear form given by $\Phi(f,g) = \varphi(fg)$.

LEMMA 3.5. $\operatorname{signature}\Phi = \sum_{i=1}^{m} \operatorname{sign}\mathcal{J}(p_i)$.

Proof. From Lemma 3.4, $\varphi(t_i) = \varphi(T_i / \mathcal{J}(p_i)) = \mathcal{J}(p_i)^{-1}\varphi(T_i) = \mathcal{J}(p_i)^{-1}$ for $1 \le i \le m$. Then $\operatorname{sign}\varphi(t_i) = \operatorname{sign}\mathcal{J}(p_i)$. Now it is enough to apply Proposition 2.3. ∎

Let $M : R^n \to R$ be a polynomial, let $\varphi_M : \mathcal{A} \to R$ be the linear functional given by $\varphi_M(f) = \varphi(Mf)$, let $\Phi_M : \mathcal{A} \times \mathcal{A} \to R$ be the bilinear form given by $\Phi_M(f,g) = \varphi_M(fg) = \varphi(Mfg)$.

LEMMA 3.6. $\operatorname{signature}\Phi_M = \sum_{i=1}^{m} \operatorname{sign} M(p_i)\mathcal{J}(p_i)$. *If* Φ_M *is non-degenerate then* $M(p_i) \neq 0$ *for every* $1 \le i \le m$.

Proof. Using the same arguments as in the proof of the previous lemma one can show that $\varphi_M(t_i) = M(p_i) \,/\, \mathcal{J}(p_i)$. From Proposition 2.3,

$$\operatorname{signature} \Phi_M = \sum_{i=1}^{m} \operatorname{sign} M(p_i)\mathcal{J}(p_i).$$

Moreover, if Φ_M is non-degenerate then $0 \neq \varphi_M(t_i) = M(p_i) \,/\, \mathcal{J}(p_i)$.

4. A formula for the topological degree. Let $F_R = (f_1, \ldots, f_n) : R^n \to R^n$ be a polynomial mapping, let $M : R^n \to R$ be a polynomial and let $B = \{ x \in R^n : M(x) > 0 \}$. If B is bounded and $\partial B \cap F_R^{-1}(0) = \emptyset$ then $\deg(F_R, B, 0)$ will denote the topological degree of F_R with respect to B and $0 \in R^n$.

Let $\mathcal{A} = R[x_1, \ldots, x_n] \,/\, \mathrm{I}$, where I is the ideal in $R[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_n$. If $\dim \mathcal{A} < \infty$ then one may define bilinear forms Φ and $\Phi_M : \mathcal{A} \times \mathcal{A} \to R$ the same way as in Section 3.

Theorem 4.1 (A formula for the topological degree). *If Φ_M is non-degenerate then $\partial B \cap F_R^{-1}(0) = \emptyset$. So if B is bounded then* $\deg(F_R, B, 0)$ *is defined and*

$$\deg(F_R, B, 0) = \frac{1}{2}(\operatorname{signature} \Phi + \operatorname{signature} \Phi_M).$$

In this paper we shall give the proof under the additional assumption that all complex roots are non-degenerate, i.e. if $p \in F_C^{-1}(0)$ then $\mathcal{J}(p) \neq 0$. We want to point out that this assumption is not necessary.

Proof. From Lemma 3.6, $M^{-1}(0) \cap F_R^{-1}(0) = \emptyset$. Since $\partial B \subset M^{-1}(0)$ then $\partial B \cap F_R^{-1}(0) = \emptyset$. According to Theorem 2.4, $F_R^{-1}(0)$ is finite. In that case

$$\deg(F_R, B, 0) = \sum_{i \in P} \operatorname{sign} \mathcal{J}(p_i),$$

where $P = \{1 \leq i \leq m : M(p_i) > 0 \}$. From Lemmas 3.5 and 3.6 it is easy to deduce that

$$\deg(F_R, B, 0) = \frac{1}{2}(\operatorname{signature} \Phi + \operatorname{signature} \Phi_M). \ \blacksquare$$

Using the same arguments one can prove

Theorem 4.2. *Let $D \subset R^n$ be an open bounded set containing all $F_R^{-1}(0)$. Then*

$$\deg(F_R, D, 0) = \operatorname{signature} \Phi.$$

5. The algorithm and computations. In this section we will present a method of calculating the matrix of the bilinear form presented in previous sections, and we will illustrate the method on one simple example.

First of all, we will briefly describe a notion of a Gröbner basis. In this article we only present some of the aspects of a Gröbner basis, the reader can find more details in [3]. A Gröbner basis of an ideal I is a set of its special generators which is useful to express the residue class of a polynomial in $R[x]/I$. Gröbner bases also enable to find the dimension of $R[x]/I$ (as a vector space) and its basis. We also describe the Buchberger algorithm for calculating Gröbner bases. In many computer algebra systems there exist implementations of that algorithm, for example in Axiom, Macsyma, MAS, Maple, Mathematica, Reduce.

Let us denote $x = (x_1, x_2, \dots, x_n)$ and $N = \{0, 1, 2, \dots\}$. For $\alpha, \beta \in R^n$ let $\alpha\beta$ denote the standard scalar product. Let $\leq$ be a semigroup ordering in the set of monomials $T = \{x^\alpha : \alpha \in N^n\}$. That means $\leq$ is a linear ordering and

$$x^\alpha \leq x^\beta \ \text{ implies } \ x^\alpha x^\gamma \leq x^\beta x^\gamma \ \text{ for any } \ \gamma \in N^n.$$

Any such ordering can be obtained by a matrix $A \in GL(n, R)$ in the following way: if $a_1, \dots, a_n$ are the rows of the matrix A, then $x^\alpha \leq x^\beta$ iff $\alpha = \beta$ or there exists i such that $a_i\alpha < a_i\beta$ and $a_j\alpha = a_j\beta$ for all $j < i$. That means that $A\alpha$ is lexicographically smaller than $A\beta$ as a column vector in R^n. Two of the most important and commonly used orderings are:

(a) the lexicographical ordering obtained by the identity matrix.

(b) the total degree ordering obtained by the matrix $\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix}$.

We also assume that the ordering $\leq$ satisfies

$$1 \leq x^\alpha \ \text{ for any } \ \alpha \in N^n.$$

That means that in the corresponding matrix in each column the first nonzero element is positive. This condition also implies that if $x^\alpha \mid x^\beta$, then $x^\alpha \leq x^\beta$. Thus $\leq$ is an extension of the partial ordering $\mid$ to a linear ordering. Because for the ordering $\mid$ we have:

LEMMA 5.1. (Dickson's lemma) *For every set $A \subset T$ there exists a finite subset $B \subset A$ such that for every $x^\alpha \in A$ there is $x^\beta \in B$ with $x^\beta \mid x^\alpha$.*

Therefore $\leq$ is a wellordering.

For the fixed ordering $\leq$ and for any polynomial $f = \sum a_\alpha x^\alpha$ we define: the set of terms $T(f)$, the head term $HT(f)$ and the head monomial $HM(f)$

$$\begin{aligned} T(f) &= \{x^\alpha : a_\alpha \neq 0\} \\ HT(f) &= \max T(f) \\ HM(f) &= a_\alpha x^\alpha \quad \text{where} \quad x^\alpha = HT(f). \end{aligned}$$

For a set $P \subset R[x]$ we define the set of head terms

$$HT(P) = \{HT(f) : f \in P\}.$$

For $S \subset T$ we define

$$mult(S) = \{t \in T : s \mid t \ \text{ for some } \ s \in S\}.$$

For $f, g \in R[x]$ we define the s-polynomial

$$spol(f, g) = \frac{HM(g)f - HM(f)g}{lcm(HT(f), HT(g))}$$

where $lcm(x^\alpha, x^\beta) = x^\gamma$ and $\gamma = (\min(\alpha_1, \beta_1), \dots, \min(\alpha_n, \beta_n))$.

The set of head terms of an ideal I is also a kind of ideal. That means that if $s \in HT(I)$, then $st \in HT(I)$ for any $t \in T$. A Gröbner basis is a set of generators of $HT(I)$.

DEFINITION 5.2. *A Gröbner basis* of an ideal I is a finite set $G \subset I$ such that $HT(I) = mult(HT(G))$.

By Dickson's lemma, Gröbner bases of an ideal I exist. It is proper to add that the set of head terms depends on the choice of ordering, so it may happen that a set of polynomials being a Gröbner basis with respect to one ordering cannot be a Gröbner basis with respect to another ordering.

Consider the following algorithm:

```
h:=NFBuchberger(f,G);
BEGIN
   h:=f;
   WHILE HT(h) ∈ mult(HT(G)) DO
      choose any g ∈ G such that HT(g) | HT(h)
      h := h - HM(h)/HM(g) g;
   END
END
```

The while-loop in this algorithm terminates because the head term of the polynomial h becomes smaller and smaller and $\leq$ is a wellordering. If G is a Gröbner basis of an ideal I, then the algorithm finds a polynomial of the smallest head term in the residue class of f. In particular, if $f \in I$, then the result is 0. This shows that if G is a Gröbner basis of an ideal I, then the ideal generated by G equals I and the monomials $T - HT(I)$ form a basis of $R[x]/I$. The following algorithm finds the presentation of a polynomial f in the basis of monomials:

```
h := Presentation(f, G);
BEGIN
   h:=0; f:=NFBuchberger(f,G);
   WHILE f ≠ 0 DO
      h := h + HM(f);
      f := f - HM(f);
      f := NFBuchberger(f,G);
   END
END.
```

The next proposition shows a method of calculating a Gröbner basis:

PROPOSITION 5.3. *A finite set $G \subset R[x]$ is a Gröbner basis of an ideal I iff $G \subset I$ and $NFBuchberger(spol(g_1, g_2), G) = 0$ for every $g_1, g_2 \in G$.*

And here is the Buchberger algorithm for finding a Gröbner basis of the ideal generated by S:

```
G := Gröbner (S);
BEGIN
   G:=S;
   P:= { (u,v) :  u, v ∈ S };
   WHILE P ≠ ∅ DO
      choose any (u,v) ∈ P
      P := P - {(u,v)};
      h := NFBuchberger( spol(u,v) , G);
      IF h ≠ 0 THEN
         P := P ∪{(h,g) :  g ∈ G };
         G := G ∪ {h}
      END
   END
END.
```

Note that some of the polynomials which are outputs of that algorithm are redundant. Their head terms are divided by other head terms from the Gröbner basis G, so they can be deleted. It shows that a Gröbner basis of an ideal I is not determined. But even if we remove all redundant polynomials, then two Gröbner bases can be different. For example, for a fixed ordering $\leq$ in the presented algorithm we can get various Gröbner bases, if we change the order of calculating s-polynomials of the pairs from the set P (called critical pairs). The choices of polynomials from G in NFBuchberger also influence outputs of the algorithm. The choices we make during computations have also effect on the time of calculations. There exist selecting strategies to make computations faster.

Most of existing algorithms for finding Gröbner bases use two criteria of deleting some critical pairs. The reader can find this powerful method of reducing the number of calculations together with the algorithm in [3].

It is also proper to add that the choice of term order influences the time of calculations of a Gröbner basis. From the two described orderings, i.e. the lexicographical and the total degree, the first one is slower in most of examples.

The next example shows a method of calculating a matrix of the bilinear form using Gröbner bases. We used Maple to calculate it but it can also be done by hand. Let $F : R^2 \to R^2$ be a map given by the formula

$$F(x_1, x_2) = (f_1, f_2) = (x_1 x_2^2 - x_1, x_2^3 - x_1 x_2 + 1).$$

Polynomials

$$x_1 - x_2^4 - x_2 \ , \ x_2^5 - x_2^3 + x_2^2 - 1$$

are a Gröbner basis of $I = (f_1, f_2)$ with respect to the lexicographical ordering. Thus $T - HT(I) = \{1, x_2, x_2^2, x_2^3, x_2^4\}$ and $e_1 = 1$, $e_2 = x_2$, $e_3 = x_2^2$, $e_4 = x_2^3$, $e_5 = x_2^4$ are a basis of $R[x]/(f_1, f_2)$. We have

$$T(x,y) = \begin{bmatrix} x_2^2 - 1 & -x_2 \\ x_2 y_1 + y_1 y_2 & x_2^2 + x_2 y_2 + y_2^2 - y_1 \end{bmatrix}$$

$$= x_2^4 + x_2^3 y_2 + x_2^2 y_2^2 - x_2^2 - x_2 y_2 - y_2^2 + y_1 + x_2 y_1 y_2$$

$$\equiv (y_2 - y_2^2 + y_2^4) + x_2(1 - y_2 + y_2^3) + x_2^2(-1 + y_2^2) + x_2^3 y_2 + x_2^4.$$

Thus the dual basis is equal to $\hat{e}_1 = x_2 - x_2^2 + x_2^4$, $\hat{e}_2 = 1 - x_2 + x_2^3$, $\hat{e}_3 = -1 + x_2^2$, $\hat{e}_4 = x_2$, $\hat{e}_5 = 1$. Since $1 = \hat{e}_5$, we have $\varphi(x_2^i) = 0$ for $i = 0, 1, 2, 3$ and $\varphi(x_2^4) = 1$. It is easy to verify the following congruences in I

$$x_2^5 \equiv x_2^3 - x_2^2 + 1$$

$$x_2^6 \equiv x_2^4 - x_2^3 + x_2$$

$$x_2^7 \equiv -x_2^4 + x_2^3 + 1$$

$$x_2^8 \equiv x_2^4 - x_2^3 + x_2^2 + x_2 - 1.$$

Thus $\varphi(x_2^5) = 0$, $\varphi(x_2^6) = \varphi(x_2^8) = 1$, $\varphi(x_2^7) = -1$, and the matrix of the bilinear form $\Phi(g, h) = \varphi(gh)$ in the basis e_i equals

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

Its signature is 1 and by Proposition 4.2 the degree of the map F on a bounded set containing all zeros equals 1.

Let $M = 4 - x_1^2 - x_2^2$. Because

$$M = 4 - x_1^2 - x_2^2 \equiv 4 - (x_2^4 + x_2)^2 - x_2^2 \equiv -x_2^4 - x_2^3 - x_2^2 - x_2 + 3,$$

then

$$\varphi_M(1) = \varphi(M) = -1.$$

We also have

$$Mx_2 \equiv x_2(-x_2^4 - x_2^3 - x_2^2 - x_2 + 3) \equiv -x_2^4 - 2x_2^3 - x_2 - 2$$

Thus

$$\varphi_M(x_2) = \varphi(Mx_2) = -1.$$

In the same way we can calculate

$$\varphi_M(x_2^2) = \varphi(-2x_2^4 - x_2^3 + 4x_2^2 - x_2 - 1) = -2.$$

$$\varphi_M(x_2^3) = \varphi(-x_2^4 - 2x_2^3 + x_2^2 - x_2 - 2) = -1.$$

$$\varphi_M(x_2^4) = \varphi(2x_2^4 - x_2 - 1) = 2.$$

Thus the matrix of Φ_M is equal to

$$\begin{bmatrix} -1 & -1 & -2 & -1 & 2 \\ -1 & -2 & -1 & 2 & 0 \\ -2 & -1 & 2 & 0 & 2 \\ -1 & 2 & 0 & 2 & -4 \\ 2 & 0 & 2 & -4 & 1 \end{bmatrix}.$$

Its signature is -1 and by Proposition 4.1 the degree of the map F on the ball of radius 2 centered at the orgin equals $\frac{1}{2}(\text{signature}\,\Phi + \text{signature}\,\Phi_M) = 0$.

The reader can also verify that F has two real zeros $(2, 1)$, $(0, -1)$. Their multiplicities are 1, 2 and the local topological degrees at these points are 1, 0.

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