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THE PUSH-FORWARD AND TODD CLASS OF FLAG BUNDLES

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Consider a connected reductive algebraic group G over an algebraically closed field k, and a principal G-bundle $\pi : X \to Y$, where X and Y are non-singular algebraic varieties over k. For any parabolic subgroup $P \subset G$, the map π factors through the flag bundle $h : X/P \to Y$. In this note, we describe the push-forward (or Gysin homomorphism) $h_* : A_*(X/P) \to A_*(Y)$ where A_* denotes the Chow group. Moreover, we compute the Todd class of the tangent bundle to h in $A_*(X/P)_{\mathbf{Q}}$. In the case when k is the field of complex numbers, our results hold when the Chow ring is replaced by the rational cohomology ring, and the proofs are the same.

The push-forward is described in [P] when G is the general linear group, and in [AC] for the canonical map $G/B \to G/P$ where G is arbitrary and B is a Borel subgroup of P. Note that this map is a flag bundle associated with the principal $P/R(P)$-bundle $G/R(P) \to G/P$, where $R(P)$ denotes the radical of P. Our formula for the Todd class seems to be new.

1. Complete flag bundles. Denote by G a connected reductive algebraic group, and by B a Borel subgroup. Choose a maximal torus $T \subset B$ with Weyl group W. Denote by $X^*(B)$ the character group of B, and by S the symmetric algebra of $X^*(B)$ over $\mathbf{Q}$. The root system of (G,T) is denoted by R; the set R^+ of positive roots consists in the opposites of roots of (B,T). Finally, denote by ρ the half-sum of positive roots, and by N their number.

Let $\pi : X \to Y$ be a principal G-bundle where X and Y are non-singular. Then π factors through the complete flag bundle $f : X/B \to Y$. The morphism f is smooth and proper of relative dimension N.

For any $\lambda \in X^*(B)$, we denote by $k\lambda$ the one-dimensional B-module with weight λ. Then $X \times^B k\lambda$ is the total space of a line bundle L_λ over X/B. We denote the first Chern class of L_λ by $c(\lambda) \in A^1(X/B)$. Since $L_{\lambda+\mu} \cong L_\lambda \otimes L_\mu$, we have $c(\lambda + \mu) = c(\lambda) + c(\mu)$.

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Therefore, c defines a ring homomorphism $c : S \to A^*(X/B)_{\mathbf{Q}}$ called the characteristic homomorphism; see [D1] and [D2].

PROPOSITION 1.1. *For any $u \in S$, we have*

$$f^* f_* c(u) = c\Big(\frac{\sum_{w\in W} \det(w) w(u)}{\prod\limits_{\alpha\in R^+} \alpha}\Big).$$

Proof. Choose a dominant weight λ. Then $f^* f_* L_\lambda$ is the vector bundle over X/B associated with the B-module $\Gamma(G/B, L_\lambda)$. Therefore, the Chern roots of $f^* f_* L_\lambda$ are the images by c of the weights of $\Gamma(G/B, L_\lambda)$. Now Weyl's character formula implies that

$$\mathrm{ch}(f^* f_* L_\lambda) = c\Big(\frac{\sum_{w\in W} \det(w) e^{w(\lambda+\rho)}}{\prod\limits_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2})}\Big).$$

Here, for $\mu \in X^*(B)$, we denote by e^μ the formal power series $\sum_{n=0}^{\infty} \mu^n/n!$. Observe that $c(e^\mu)$ makes sense in $A^*(X/B)$, because $c(\mu)$ is nilpotent.

On the other hand, we have by the Grothendieck-Riemann-Roch theorem:

$$\mathrm{ch}(f_* L_\lambda) = f_*\big(\mathrm{ch}(L_{-\lambda})\,\mathrm{td}(T_f)\big),$$

where $\mathrm{td}(T_f)$ is the Todd class of the relative tangent bundle. Observe that the Chern roots of T_f are $c(\alpha)$, $\alpha \in R^+$. It follows that

$$f^* f_* c\big(e^\lambda \prod_{\alpha\in R^+} \frac{\alpha}{1-e^{-\alpha}}\big) = c\Big(\frac{\sum_{w\in W} \det(w) e^{w(\lambda+\rho)}}{\prod\limits_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2})}\Big).$$

Now we set

$$u_0 := \prod_{\alpha\in R^+} \frac{\alpha}{e^{\alpha/2} - e^{-\alpha/2}}$$

(then u_0 is W-invariant) and $\mu := \lambda + \rho$ (then μ is dominant and regular). So we have

$$f^* f_* c(u_0 e^\mu) = c\Big(u_0 \frac{\sum_{w\in W} \det(w) e^{w(\mu)}}{\prod\limits_{\alpha\in R^+} \alpha}\Big).$$

By the lemma below, it follows that

$$f^* f_* c(u_0 u) = c\Big(\frac{\sum_{w\in W} \det(w) w(u_0 u)}{\prod\limits_{\alpha\in R^+} \alpha}\Big)$$

for any $u \in S$. Now observe that $u_0 - 1$ is a sum of classes of positive degree, to conclude the proof. ∎

LEMMA. *The $\mathbf{Q}$-vector space $c(S)$ is generated by $c(e^\mu)$, μ a dominant regular weight.*

Proof. First observe that the $\mathbf{Q}$-vector space S is generated by all non-negative powers of all dominant regular weights. Therefore, it suffices to show that $c(\mu)$ is a (finite) linear combination with rational coefficients of the $c(e^{n\mu})_{n\ge 1}$ for any regular dominant weight μ. There exists a sequence $(a_n)_{n\ge1}$ of rational numbers such that $\mu = \sum_{n\ge1} a_n (e^\mu - 1)^n$ as a formal power series. Furthermore, $c(e^\mu - 1)$ is nilpotent in $A^*(X/B)$ and this implies our statement. ∎

PROPOSITION 1.2. *For any* $u \in S^W$, *we have in* $A^*(X/B)_{\mathbf{Q}}$:

$$f^*f_*c\big(u\frac{\rho^N}{N!}\big) = c(u) = \frac{1}{|W|}f^*f_*c\big(u\prod_{\alpha\in R^+}\alpha\big).$$

P r o o f. By Proposition 1.1, we have

$$f^*f_*c\big(u\frac{\rho^N}{N!}\big) = c(u)c\Big(\frac{\sum_{w\in W}\det(w)w(\rho^N)}{N!\prod\limits_{\alpha\in R^+}\alpha}\Big).$$

On the other hand, the identity

$$\sum_{w\in W}\det(w)e^{w(\rho)} = \prod_{\alpha\in R^+}(e^{\alpha/2} - e^{-\alpha/2})$$

implies that

$$\sum_{w\in W}\det(w)w\big(\frac{\rho^N}{N!}\big) = \prod_{\alpha\in R^+}\alpha.$$

This proves the first equality. For the second one, we apply Proposition 1.1 to the anti-invariant element $u\prod_{\alpha\in R^+}\alpha$. ■

R e m a r k. Proposition 1.2 can be reformulated as follows: The restriction to invariants $c|_{S^W} : S^W \to A^*(X/B)_{\mathbf{Q}}$ is the composition of $c^W : S^W \to A^*(Y)_{\mathbf{Q}}$ with f^*, where

$$c^W(u) = f_*c\big(u\frac{\rho^N}{N!}\big) = \frac{1}{|W|}f_*c\big(u\prod_{\alpha\in R^+}\alpha\big).$$

Moreover, c^W is an algebra homomorphism, because f^* is injective.

PROPOSITION 1.3. *The Todd class of the relative tangent bundle of* $f : X/B \to Y$ *is given by*

$$\mathrm{td}(T_f) = e^{c(\rho)}f^*(f_*e^{c(\rho)})^{-1}.$$

Equivalently,

$$\mathrm{td}(T_f) = e^{c_1(T_f)/2}f^*(f_*e^{c_1(T_f)/2)})^{-1}.$$

P r o o f. With the notation of the proof of Proposition 1.1, we have

$$\mathrm{td}(T_f) = c\Big(\prod_{\alpha\in R^+}\frac{\alpha}{1-e^{-\alpha}}\Big) = e^{c(\rho)}c(u_0).$$

Furthermore, u_0 is invariant under W. Therefore, by Proposition 1.2, there exists $v \in A^*(Y)_{\mathbf{Q}}$ such that $c(u_0) = f^*v$. On the other hand, $f_*\,\mathrm{td}(T_f) = 1$ and hence $vf_*e^{c(\rho)} = 1$.

R e m a r k. The class $f^*f_*e^{c(\rho)} \in A^*(X/B)_{\mathbf{Q}}$ is even, and its part of degree at most two is $1 + \frac{1}{24}c(\sum_{\alpha\in R^+}\alpha^2)$. Indeed, we have by Proposition 1.1:

$$f^*f_*e^{c(\rho)} = c\Big(\frac{\sum_{w\in W}\det(w)e^{w(\rho)}}{\prod\limits_{\alpha\in R^+}\alpha}\Big) = c\big(\prod_{\alpha\in R^+}\frac{e^{\alpha/2}-e^{-\alpha/2}}{\alpha}\big).$$

Moreover, the formal power series

$$\frac{e^{x/2}-e^{-x/2}}{x} = 1 + \frac{x^2}{24} + \cdots$$

is even.

2. General flag bundles. Let $P \supset B$ be a parabolic subgroup of G. Denote by L the Levi subgroup of P which contains T, with root system R_L and Weyl group W_L. The morphism $f : X/B \to Y$ is the composition of $g : X/B \to X/P$ with $h : X/P \to Y$. Observe that g is the complete flag bundle associated with the principal L-bundle $X/R_u(P) \to X/P$. Therefore, we have a homomorphism $c^{W_L} : S^{W_L} \to A^*(X/P)$. We will describe h_* and the Todd class of the relative tangent bundle to h as well.

Proposition 2.1. *For any $u \in S^{W_L}$, we have*

$$h^* h_* c^{W_L}(u) = c^{W_L}\Big(\sum_{w \in W/W_L} w(u/\prod_{\alpha \in R^+ \setminus R_L} \alpha)\Big).$$

The right-hand side makes sense, because both u and $\prod_{\alpha \in R^+ \setminus R_L} \alpha$ are invariant under W_L.

Proof. By the remark after Proposition 1.2, we have

$$c^{W_L}(u) = \frac{1}{|W_L|} g_* c\big(u \prod_{\alpha \in R_L^+} \alpha\big).$$

It follows that

$$\begin{aligned} g^* h^* h_* c^{W_L}(u) &= \frac{1}{|W_L|} f^* f_* c(u \prod_{\alpha \in R_L^+} \alpha) \\ &= \frac{1}{|W_L|} c\big(\sum_{w \in W} \det(w) w(u \prod_{\alpha \in R_L^+} \alpha)/\prod_{\alpha \in R^+} \alpha\big) \\ &= c\big(\sum_{w \in W/W_L} w(u/\prod_{\alpha \in R^+ \setminus R_L} \alpha)\big). \end{aligned}$$

Proposition 2.2. *The Todd class of the relative tangent bundle of $h : X/P \to Y$ is given by*

$$\mathrm{td}(T_h) = c^{W_L}(u) h^*\big(h_* c^{W_L}(u)\big)^{-1}$$

where u stands for

$$e^{\rho - \rho_L} \sum_{w \in W_L} \det(w) e^{w(\rho_L)} / \prod_{\alpha \in R_L^+} \alpha.$$

Proof. Observe that $\mathrm{td}(T_f) = \mathrm{td}(T_g) g_* \mathrm{td}(T_h)$ and that $g_* \mathrm{td}(T_g) = 1$, whence $\mathrm{td}(T_h) = g_* \mathrm{td}(T_f)$. Furthermore, by Proposition 1.3, we have

$$\mathrm{td}(T_f) f^*(f_* e^{c_1(T_f)/2}) = e^{c_1(T_f)/2}.$$

It follows that

$$\mathrm{td}(T_h) h^*(h_* g_* e^{c_1(T_f)/2}) = g_* e^{c_1(T_f)/2}.$$

Now $c_1(T_f) = c_1(T_g) + g^* c_1(T_h)$. Therefore, we have $\mathrm{td}(T_h) h^*(h_* v) = v$ where $v := e^{c_1(T_h)/2} g_* e^{c_1(T_g)/2}$. But $c_1(T_h) = 2c(\rho - \rho_L)$ and moreover

$$g_* e^{c_1(T_g)/2} = c^{W_L}\big(\sum_{w \in W_L} \det(w) e^{w(\rho_L)} / \prod_{\alpha \in R_L^+} \alpha\big)$$

by Proposition 1.1 applied to the complete flag bundle g. ∎

3. The case of classical groups. For any root system R, we set

$$u(R) := \frac{\sum_{w \in W} \det(w) e^{w(\rho)}}{\prod_{\alpha \in R^+} \alpha} = \prod_{\alpha \in R^+} \frac{e^{\alpha/2} - e^{-\alpha/2}}{\alpha}$$

where W is the Weyl group, R^+ is a set of positive roots, and ρ is the half-sum of positive roots. This defines $u(R)$ as a formal sum of Weyl group invariants, independently of the choice of R^+. To finish the computation of the Todd class of flag bundles, we need formulas for $u(R)$: for example, it follows from Proposition 2.2 that

$$\operatorname{td}(T_{G/P}) = c^{W_L}\big(e^{\rho - \rho_L} u(R_L)\big).$$

Observe that $u(R)$ is the product of the $u(R_i)$ over all irreducible components R_i of R. For R an irreducible root system of type A, B, C or D, we will obtain a determinantal formula and an expansion of $u(R)$ into S-functions (for these, see [M] 1.3).

Type A_n: The positive roots are the $x_i - x_j$ $(1 \le i < j \le n+1)$. We claim that

$$\begin{aligned} u(A_n) = {} & \det\big(e^{(n-2i+2)x_j/2}\big)_{1 \le i,j \le n+1} \prod_{1 \le i < j \le n+1} (x_i - x_j)^{-1} \\ = {} & \sum_{\lambda_1 \ge \dots \lambda_{n+1} \ge 0} \frac{n!(n-1)! \cdots 1!}{2^{\lambda_1 + \dots + \lambda_{n+1}} (\lambda_1 + n)!(\lambda_2 + n - 1)! \cdots \lambda_{n+1}!} \\ & \times s_\lambda(n, n-2, \dots, -n) s_\lambda(x_1, \dots, x_{n+1}). \end{aligned}$$

Indeed, $u(A_n)$ can be written as

$$\prod_{1 \le i < j \le n+1} \big(e^{(x_i - x_j)/2} - e^{-(x_i - x_j)/2}\big) \prod_{1 \le i < j \le n+1} (x_i - x_j)^{-1}$$

and the first formula follows by the classical expression of the Vandermonde determinant. To obtain the second formula, we simply expand each exponential in the determinant into its power series.

Type B_n: The positive roots are the $x_i + x_j$, $x_i - x_j$ $(1 \le i < j \le n)$ and $x_1, \dots, x_n$. We obtain similarly

$$\begin{aligned} u(B_n) = {} & 2^n \det\big(\operatorname{sh}((n - i + 1/2)x_j)/x_j\big)_{1 \le i,j \le n} \prod_{1 \le i < j \le n} (x_i^2 - x_j^2)^{-1} \\ = {} & \sum_{\lambda_1 \ge \dots \ge \lambda_n \ge 0} \frac{(2n-1)!(2n-3)! \cdots 1!}{(2n-1+2\lambda_1)!(2n-3+2\lambda_2)! \cdots (1+2\lambda_n)!} \\ & \times s_\lambda\Big(\big(n - \frac{1}{2}\big)^2, \big(n - \frac{3}{2}\big)^2, \dots, \big(\frac{1}{2}\big)^2\Big) s_\lambda\big(x_1^2, \dots, x_n^2\big). \end{aligned}$$

Type C_n: The positive roots are the x_i+x_j, x_i-x_j $(1\le i<j\le n)$ and $2x_1,\dots,2x_n$. We have

$$\begin{aligned}
u(C_n) =&\ \det\Big(\frac{\mathrm{sh}((n-i+1)x_j)}{x_j}\Big)_{1\le i,j\le n}\prod_{1\le i<j\le n}(x_i^2-x_j^2)^{-1}\\
=&\sum_{\lambda_1\ge\cdots\lambda_n\ge 0}\frac{(2n-1)!(2n-3)!\cdots 1!}{(2n-1+2\lambda_1)!(2n-3+2\lambda_2)!\cdots(1+2\lambda_n)!}\\
&\times s_\lambda(n^2,(n-1)^2,\dots,1^2)s_\lambda(x_1^2,\dots,x_n^2).
\end{aligned}$$

Type D_n: The positive roots are the x_i+x_j, x_i-x_j $(1\le i<j\le n)$. We have

$$\begin{aligned}
u(D_n) =&\ 2^{n-1}\det\big(\mathrm{ch}((n-i)x_j)\big)_{1\le i,j\le n}\prod_{1\le i<j\le n}(x_i^2-x_j^2)^{-1}\\
=&\sum_{\lambda_1\ge\cdots\ge\lambda_{n-1}\ge 0}\frac{(2n-2)!(2n-4)!\cdots 2!}{(2n-2+2\lambda_1)!(2n-4+2\lambda_2)!\cdots(2+2\lambda_{n-1})!}\\
&\times s_\lambda((n-1)^2,(n-2)^2,\dots,1^2)s_\lambda(x_1^2,\dots,x_n^2).
\end{aligned}$$

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