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FUNDAMENTAL SOLUTIONS FOR DIRAC-TYPE OPERATORS

SWANHILD BERNSTEIN

Freiberg University of Mining and Technology
Faculty of Mathematics and Computer Science
Institute of Applied Mathematics I
D-09596 Freiberg, Germany

Abstract. We consider the Dirac-type operators $D + a$, a is a paravector in the Clifford algebra. For this operator we state a Cauchy-Green formula in the spaces $C^1(G)$ and $W_p^1(G)$. Further, we consider the Cauchy problem for this operator.

1. Preliminaries. Dirac and Dirac-type operators are considered in many papers. A significant selection of papers is contained in the bibliography. Most of them consider the operators in the quaternionic algebra. We want to consider the operator in the Clifford algebra. Thus our considerations differ in some sense from the considerations in the quaternionic algebra. We consider stationary problems and obtain as the main result the Cauchy-Green formula. In the case of nonstationary problems we consider the Cauchy problem.

2. Introduction. Let $(e_1, \ldots, e_m)$ be an orthonormal basis of $\mathbb{R}^m$, $m \in \mathbb{N}$, then by $\mathcal{C}$ we denote the 2^m-dimensional Clifford algebra obtained from the generating relations $e_j e_k + e_k e_j = 2\delta_{jk}$, $j, k = 1, \ldots, m$. Thus the quaternionic case is not contained. An element of $\mathcal{C}$ is of the form $a = \sum a_A e_A$, $a_A \in \mathbb{C}$, $e_A = e_{a_1 \ldots a_k} = e_{a_1} \cdot \ldots \cdot e_{a_k}$ for $A = \{a_1, \ldots, a_k\}$ with $a_1 < \ldots < a_k$, $e_0 = 1$ is the identity of $\mathcal{C}$. We identify a vector $x \in \mathbb{R}^m$ with the element $x = \sum_{j=1}^m x_j e_j$ of the Clifford algebra. Let G be a bounded domain of $\mathbb{R}^m$, with smooth boundary Γ. If F is a functionspace of complex-valued function, a function $u = \sum u_A e_A$ is an element of $F_{\mathcal{C}}$ iff u_A is a complex-valued function of the space F. We use the Sobolev-spaces $W_p^k(G)$, $1 < p < \infty$, $k \in \mathbb{N}$, the space of continuously differentiable functions $C^1(G)$ and the space $L_2[0, t; L_2(\mathbb{R}^m)]$ with the norm

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$(\int_0^T ||f||^2_{L_2(\mathbb{R}^m)} dt)^{1/2}$. We consider the Dirac operator

$$D_x + a = \sum_{j=1}^{m} e_j (\frac{\partial}{\partial x_j} + a_j) + e_0 a_0, \quad m \in \mathbb{N},$$

where $a = \sum_{j=1}^m e_j a_j = a_0 e_0 + \underline{a}$ and $D_x = \sum_{j=1}^m e_j \frac{\partial}{\partial x_j}$, then $D_x^2 = \nabla$, ∇ being the Laplacian. We want to illustrate the Dirac-operator in the Pauli-algebra. The basic vectors are e_1, e_2, e_3, $e_1^2 = e_2^2 = e_3^2 = 1$, the unit element of the algebra is e_0. We have the bivectors $e_{23} = e_2 e_3$, $e_{31} = e_3 e_1$ and $e_{12} = e_1 e_2$ and the pseudoscalar $e_{123} = e_1 e_2 e_3$. We consider a function

$$u = u_0 e_0 + u_1 e_1 + u_2 e_2 + u_3 e_3 + u_{23} e_{23} + u_{31} e_{31} + u_{12} e_{12} + u_{123} e_{123}.$$

The scalar part is u_0, the pseudoscalar part u_{123}, the vector $\underline{u} = (u_1, u_2, u_3)$ and the bivector $\underline{v} = (u_{23}, u_{31}, u_{12})$. Then we obtain for the Dirac operator

$$\begin{gathered} Du = \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i} u = +(\frac{\partial}{\partial x_1} u_1 + \frac{\partial}{\partial x_2} u_2 + \frac{\partial}{\partial x_3} u_3) e_0 + \\ +(\frac{\partial}{\partial x_3} u_{31} - \frac{\partial}{\partial x_2} u_{12} + \frac{\partial}{\partial x_1} u_0) e_1 + (\frac{\partial}{\partial x_1} u_{12} - \frac{\partial}{\partial x_3} u_{23} + \frac{\partial}{\partial x_2} u_0) e_2 + \\ +(\frac{\partial}{\partial x_2} u_{31} - \frac{\partial}{\partial x_1} u_{31} + \frac{\partial}{\partial x_3} u_0) e_3 + \\ +(\frac{\partial}{\partial x_2} u_3 - \frac{\partial}{\partial x_3} u_2 + \frac{\partial}{\partial x_1} u_{123}) e_{23} + (\frac{\partial}{\partial x_3} u_1 - \frac{\partial}{\partial x_1} u_3 + \frac{\partial}{\partial x_2} u_{123}) e_{31} + \\ +(\frac{\partial}{\partial x_1} u_2 - \frac{\partial}{\partial x_2} u_1 + \frac{\partial}{\partial x_3} u_{123}) e_{12} + (\frac{\partial}{\partial x_3} u_{23} - \frac{\partial}{\partial x_2} u_{31} + \frac{\partial}{\partial x_3} u_{12}) e_{123}. \end{gathered}$$

This system is equivalent to another system

$$\begin{gathered} \mathrm{div}\underline{u} = - * \mathrm{curl} * \underline{v} \\ + \mathrm{curl} * \underline{u} - * \mathrm{grad} * u_{123} \\ - * \mathrm{div} * \underline{v}, \end{gathered}$$

where $*$ is the Hodge-operator (multiplication with the pseudo scalar e_{123}),

$$*e_0 = e_{123}, \quad *e_{123} = -e_0; \quad *e_2 = e_{31}, \quad *e_{31} = -e_2;$$

$$*e_1 = e_{23}, \quad *e_{23} = -e_1; \quad *e_3 = e_{12}, \quad *e_{12} = -e_3;$$

and the operator div is defined for vector $\underline{u}$

$$\mathrm{div}\underline{u} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i,$$

and the operator curl is also defined for vectors $\underline{u}$ in the following way

$$\mathrm{curl}\underline{u} = (\frac{\partial}{\partial x_2} u_3 - \frac{\partial}{\partial x_3} u_2) e_{23} + (\frac{\partial}{\partial x_3} u_1 - \frac{\partial}{\partial x_1} u_3) e_{31} + (\frac{\partial}{\partial x_1} u_2 - \frac{\partial}{\partial x_2} u_1) e_{12}$$

and the Hodge-operator transforms bivectors in vectors and vectors in bivectors. The operator grad is defined for scalars u_0

$$\mathrm{grad} u_0 = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_0 e_i$$

and the Hodge-operator transforms scalars in pseudoscalars and pseudoscalars in scalars.

3. The Dirac-type operator $D + a$. Let m be a natural number and

$$K_{a_0}(x) = K_{a_0}(|x|) = \frac{1}{\pi^{2/m} 2^{m/2}} (\frac{a_0}{|x|}^{m/2-1} K_{\frac{m}{2}-1}(a_0|x|))$$

where K denotes modified Bessel functions, the so-called MacDonald functions.

LEMMA. *The fundamental solution for* $\nabla - a_0^2$ *is* K_{a_0}.

The proof is contained in [0rt].

THEOREM. *The fundamental solution for* $D_x + a$ *is*

$$E_a(x) = \exp^{-<\underline{a},x>}\{(D_x - a_0)K_{a_0}(x)\} =$$
$$= \exp^{-<\underline{a},x>}\{\frac{1}{a_0(2\pi)^{m/2}} \cdot \sum_{j=1}^{m} \frac{x_j e_j}{|x|^m} (a_0|x|)^{m/2} K_{m/2}(a_0|x|) +$$
$$+ \frac{a_0}{(2\pi)^{m/2}} \cdot \frac{1}{|x|^{m-2}} (a_0|x|)^{m/2-1} K_{m/2-1}(a_0|x|)\}, \quad \textit{with } \langle \underline{a}, x \rangle = \textstyle\sum_{i=1}^{m} a_i x_i.$$

Proof. We prove that $E_a(x)$ is locally integrable and $(D+a)E_a = 0$ in $\mathbb{R}^m \setminus \{0\}$ and $(D + a)E_a = \delta$. On every compact subset of $\mathbb{R}^m$ the function $e^{-<\underline{a},x>}$ is bounded from above and positive,

$$K_{a_0} \sim \frac{1}{|x|^{m-2}}, \quad |x| \to 0, \qquad K_{a_0} \sim \frac{e^{-|\mathcal{R}a_0|\cdot|x|}}{|x|^{m/2-1/2}}, \quad |x| \to \infty,$$
$$\frac{\partial K_{a_0}(x)}{\partial x_j} \sim \frac{x_j}{|x|^m}, \quad |x| \to 0, \qquad \frac{\partial K_{a_0}(x)}{\partial x_j} \sim \frac{e^{-|\mathcal{R}a_0|\cdot|x|} x_j}{|x|^{m/2+1}}, \quad |x| \to \infty,$$

$\mathcal{R}a_0$ denotes the real part of the complex number a_0. Thus $E_a(x)$ is locally integrable in $\mathbb{R}^m$. Next,

$$(D_x + a)E_a(x) = (D_x + a)\{e^{-<\underline{a},x>}(D_a - a_0)K_{a_0}(x)\} =$$
$$= -\underline{a}e^{-<\underline{a},x>}(D_x - a_0)K_{a_0}(x) + e^{-<\underline{a},x>} D_x(D_x - a_0)K_{a_0}(x) +$$
$$+ae^{-<\underline{a},x>}(D_x - a_0)K_{a_0}(x) =$$
$$= e^{-<\underline{a},x>}(D_x + a_0)(D_x - a_0)K_{a_0}(x) = e^{-<\underline{a},x>}(\Delta - a_0^2)K_{a_0}(x) = 0$$

in $\mathbb{R}^m \setminus \{0\}$. On the other hand, let $\phi \in \mathcal{D}(\mathbb{R}^m)$:

$$((D + a)E_a(x), \phi(x)) = (e^{-<\underline{a},x>}(\Delta - a_0^2)K_{a_0}(x), \phi(x)) =$$
$$= ((\Delta - a_0^2)K_{a_0}(x), e^{-<\underline{a},x>}\phi(x)) = (\delta, e^{-<\underline{a},x>}\phi(x)) =$$
$$= \phi(0) = (\delta, \phi).$$

Remark. The case $a = a_0 \in \mathbb{C}$ is discussed in [Xu] if $e_i^2 = -1$ by using outer and inner monogenics.

We introduce the integral operators

$$T_a u := \int_G E_a(x-y)u(y)dy, \quad F_a u := -\int_\Gamma E_a(x-y)n(y)u(y)dy, \qquad x \notin \Gamma.$$

Here, $n(y)$ denotes the outward-normal at the point $y \in \Gamma$.

THEOREM. *The operator*

$$T_a : W^k_{p,c}(G) \to W^{k+1}_{p,C}(g), \qquad 1 < p < \infty,\ k = 0, 1, \ldots$$

is continuous.

Proof. T_a is a weakly singular integral operator with

$$|E_a(x-y)| \le \frac{C}{|x-y|^{m-1}}$$

and using [MP] we get thus the operator

$$T_a : L_{p,C}(G) \to L_{p,C}(g), \qquad 1 < p < \infty$$

is continuous. The rest is contained in the following lemmas.

LEMMA. *Let $u \in L_{p,C}(G)$, $1 < p < \infty$, $k = 0, 1, \ldots$. Then*

$$\frac{\partial}{\partial x_k} T_a u = \int_G \frac{\partial}{\partial x_k} E_a(x-y)u(y)dy + \frac{u(x)}{A_m} \int_{S_1} \frac{y-x}{|x-y|} \cdot \frac{y_k - x_k}{|x-y|} dS_1(y),$$

where the integral over S_1 is a constant only depending on the dimension m and k and $A_m = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}$ is the area of the unit sphere in $\mathbb{R}^m$.

Proof. We have $E_a(x-y) = e^{-<\underline{a}, x-y>}(D_x - a_0)K_{a_0}(x-y)$,

$$a_0 K_{a_0}(x-y) \sim \frac{1}{|x-y|^{m-2}}, \quad x \to y,$$

$$D_x K_{a_0}(x-y) = \frac{1}{A_m} e^{-<\underline{a}, x-y>} \frac{x-y}{|x-y|}, \quad x \to y.$$

Now, we get

$$\begin{aligned}
&\frac{\partial}{\partial x_k} \int_{G_\epsilon} e^{-<\underline{a}, x-y>}(D_x - a_0)K_{a_0}(x-y)u(y)dy = \\
&\int_{G_\epsilon} \frac{\partial}{\partial x_k} \{e^{-<\underline{a}, x-y>}(D_x - a_0)K_{a_0}(x-y)\}u(y)dy + \\
&- \int_{r=\epsilon=|x-y|} e^{-<\underline{a}, x-y>}(D_x - a_0)K_{a_0}(x-y)u(y)\cos(r, x_k)dS_\epsilon \\
&= \int_{G_\epsilon} \frac{\partial}{\partial x_k} \{e^{-<\underline{a}, x-y>}(D_x - a_0)K_{a_0}(x-y)\}u(y)dy + \\
&+ \int_{\epsilon=|x-y|} e^{-<\underline{a}, x-y>} a_0 K_{a_0}(x-y)u(y)\cos(r, x_k)dS_\epsilon + \\
&- \int_{\epsilon=|x-y|} e^{-<\underline{a}, x-y>} D_x K_{a_0}(x-y)u(y)\cos(r, x_k)dS_\epsilon = \\
&= \int_{G_\epsilon} \frac{\partial}{\partial x_k} \{e^{-<\underline{a}, x-y>}(D_x - a_0)K_{a_0}(x-y)\}u(y)dy +
\end{aligned}$$

$$-\int_{|x-y|=1} \epsilon^{m-1} e^{-<\underline{a},x-(x+\epsilon\theta)>} a_0 K_{a_0}(x-y)u(x+\epsilon\theta)\cos(r,x_k)dS_1 +$$
$$-\int_{|x-y|=1} e^{-<\underline{a},x-(x+\epsilon\theta)>} D_x K_{a_0}(x-y)u(x+\epsilon\theta)\epsilon^{m-1}\cos(r,x_k)dS_1.$$

We take $\epsilon \to 0$:

$$= \int_G \frac{\partial}{\partial x_k} E_a(x-y)u(y)dy + \frac{u(x)}{A_m}\int_{S_1} \frac{y-x}{|x-y|}\cdot\frac{y_k-x_k}{|x-y|}dS_1(x),$$
$$e^{-<\underline{a},x-(x+\epsilon\theta)>} = e^{\epsilon<\underline{a},\theta>} \to 1).$$

LEMMA. *The operator* $\frac{\partial}{\partial x_k}T_a : L_{p,C}(g) \to L_{p,C}$, $1 < p < \infty$, *is continuous.*

Proof. From the Lemma above we get

$$\frac{\partial}{\partial x_k}T_a = \int_G \frac{\partial}{\partial x_k}E_a(x-y)u(y)dy + C(k)u(x),$$

where $C(k)$ is a constant only depending on k. Thus $C(k)u \in L_{p,C}$ if $u \in L_{p,C}$. We consider the first term.

$$\frac{\partial}{\partial x_k}E_a(x-y) =$$
$$-\underline{a}E_a(x-y) - e^{-<\underline{a},x-y>}\frac{a_0}{(2\pi)^{m/2}}\cdot\frac{x_k-y_k}{|x-y|^m}(a_0|x-y|)^{m/2}K_{m/2}(a_0|x-y|)+$$
$$e^{-<\underline{a},x-y>}\frac{a_0}{(2\pi)^{m/2}}\cdot\sum_{j=1}^{m}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^m}\cdot(a_0|x-y|)^{m/2-1}K_{m/2-1}(a_0|x-y|)+$$
$$e^{-<\underline{a},x-y>}\frac{1}{a_0(2\pi)^{m/2}}K_{m/2-}(a_0|x-y|)\cdot\sum_{j=1}^{m}\frac{\partial}{\partial x_k}(\frac{(x_j-y_j)e_j}{|x-y|}).$$

We only have to consider the last part, because the other parts lead to weakly singular kernels. We prove that the last part creates a singular kernel of a Calderon-Zygmund operator. We choose $k=1$ and use sphericals cordinates in the following way:

$$y_1 = x_1 + r\cos\theta_1,$$
$$y_2 = x_2 + r\sin\theta_1\cos\theta_1,$$
$$\ldots\ldots,$$
$$y_{m-1} = x_{m-1} + r\sin\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\cos\theta_{m-1},$$
$$y_m = x_m + r\sin\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\sin\theta_{m-1},$$

$r = |x-y|$, $\theta_1 \in [0,\pi]$, $\theta_i \in [-\pi,\pi]$, $i = 2,3,\ldots,m-1$. We have

$$\frac{\partial r}{\partial x_1} = \frac{x_1-y_1}{r} = -\cos\theta_1.$$

Thus

$$-\frac{\partial}{\partial x_1}\cos\theta_1 = \frac{1}{r} - \frac{(x_1-y_1)^2}{r^3}\frac{1}{r} - \frac{\cos^2\theta_1}{r} = \frac{\sin^2\theta_1}{r} = \sin\theta_1\frac{\partial\theta_1}{\partial x_1}$$

and we obtain

$$\frac{\partial\theta_1}{\partial x_1} = \frac{\sin\theta_1}{r}$$

because of

$$\theta_1 \in [0,\pi].$$

Further

$$\frac{\partial}{\partial x_1}(\frac{y_2-x_2}{r}) = \frac{\partial}{\partial x_1}(\sin\theta_1\cos\theta_2)$$

$$= \frac{\partial}{\partial\theta_1}(\sin\theta_1\cos\theta_2)\frac{\partial\theta_2}{\partial x_1} = \frac{\sin\theta_1}{r}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\frac{\partial\theta_2}{\partial x_1}$$

$$\cos\theta_1\cos\theta_2\frac{\partial\theta_1}{\partial x_1} - \sin\theta_1\sin\theta_2\frac{\partial\theta_2}{\partial x_1} = \frac{\sin\theta_1}{r}\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\frac{\partial\theta_2}{\partial x_1}.$$

On the other hand

$$\frac{\partial}{\partial x_1}(\frac{y_2-x_2}{r}) = \frac{(y_2-x_2)}{r}\cdot(\frac{(y_1-x_1)}{r})\cdot\frac{1}{r} = \frac{\sin\theta_1}{r}\cos\theta_1\cos\theta_2$$

and we obtain

$$-\sin\theta_1\sin\theta_2\frac{\partial\theta_2}{\partial x_1} = 0$$

and thus $\frac{\partial\theta_2}{\partial x_1} = 0$. Now, let $\frac{\partial\theta_j}{\partial x_1} = 0$, $j = 2, \cdot, l-1$, $l \le m-1$, then

$$\frac{\partial}{\partial x_1}(\frac{y_2-x_2}{r}) = \frac{\partial}{\partial x_1}(\sin\theta_1\cdot\ldots\cdot\sin\theta_{l-1}\cos\theta_l)$$

$$= \frac{\partial}{\partial\theta_1}(\sin\theta_1\cdot\ldots\cdot\sin\theta_{l-1}\cos\theta_l)\frac{\partial\theta_1}{\partial x_1} + \frac{\partial}{\partial\theta_1}(\sin\theta_1\cdot\ldots\cdot\sin\theta_{l-1}\cos\theta_l)\frac{\partial\theta_l}{\partial x_1} =$$

$$\frac{\sin\theta_1}{r}\cos\theta_1\sin\theta_2\ldots\sin\theta_{l-1}cos\theta_l - \sin\theta_1\ldots\sin\theta_{l-1}\sin\theta_1\frac{\partial\theta_1}{\partial x_1}.$$

On the other hand

$$\frac{\partial}{\partial x_1}(\frac{y_1-x_1}{r}) = \frac{(y_1-x_1)}{r}\cdot(\frac{(y_1-x_1)}{r})\cdot\frac{1}{r} = \frac{\sin\theta_1}{r}\cos\theta_1\sin\theta_2\ldots\sin\theta_{l-1}\cos\theta_l$$

and we get $\frac{\partial\theta_l}{\partial x_1} = 0$, $l = 2,\ldots,m-1$. Though, we obtain

$$\frac{\partial}{\partial x_1}(\frac{x}{|x|^m}) = \frac{\partial}{\partial x_1}(\sum_{j=1}^m\frac{(x_j-y_j)}{|x-y|^m}e_j) = -\frac{\partial}{\partial x_1}(\sum_{j=1}^m\frac{(x_j-y_j)}{r}\cdot\frac{1}{r^m}e_j) =$$

$$= \frac{(m-1)}{r^m}\cdot\frac{\partial r}{\partial x_1}\cdot\sum_{j=1}^m\frac{(x_j-y_j)}{|x-y|^m}e_j - \frac{1}{r^{m-1}}\frac{\partial}{\partial\theta_1}(\sum_{j=1}^m\frac{(x_j-y_j)}{|x-y|^m}e_j)\frac{\partial\theta_1}{\partial x_1} =$$

$$= -\frac{1}{r^m}\{r\frac{\partial}{\partial\theta_1}(\sum_{j=1}^m\frac{(x_j-y_j)}{|x-y|^m}e_j)\frac{\partial}{\partial x_1} - (m-1)(\sum_{j=1}^m\frac{(x_j-y_j)}{|x-y|^m}e_j)\frac{\partial r}{\partial x_1}\}.$$

Let

$$\phi(\theta_1,\theta_2,\ldots,\theta_{m-1}) := -\sum_{j=1}^m\frac{(x_j-y_j)}{r}e_j,$$

then

$$\frac{\partial}{\partial x_1}(\frac{x}{|x|^m}) = \frac{1}{r^m}\{r\frac{\partial\phi}{\partial\theta_1}\frac{\partial\theta_1}{\partial x_1} - (m-1)\phi\frac{\partial r}{\partial x_1}\} = \frac{f(\phi,r)}{r^m}.$$

We get

$$\frac{\partial}{\partial x_1}E_a(x-y) =$$
$$-\underline{a}E_a(x-y) - e^{-<\underline{a},x-y>}\frac{a_0}{(2\pi)^{m/2}}\cdot\frac{x_k-y_k}{|x-y|^m}(a_0|x-y|)^{m/2}K_{m/2}(a_0|x-y|)+$$
$$e^{-<\underline{a},x-y>}\frac{a_0}{(2\pi)^{m/2}}\cdot\sum_{j=1}^{m}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^m}\cdot(a_0|x-y|)^{m/2-1}K_{m/2-1}(a_0|x-y|)+$$
$$e^{-<\underline{a},x-y>}\frac{1}{a_0(2\pi)^{m/2}}\cdot(a_0|x-y|)^{m/2}K_{m/2}(a_0|x-y|)\cdot\frac{f(\phi,|x-y|)}{|x-y|^m}.$$

We have to prove that $\int_{S_1} f(\phi,1)dS_1 = 0$, where S_1 is the unit sphere in $\mathbb{R}^m$ with center in x, and that $\int_{S_1}|f(\phi,1)|^{p'}dS_1 = \text{const}$, $1 < p' < \infty$. Because of

$$dS_1 = \sin^{m-2}\theta_1\sin^{m-3}\theta_2\ldots\sin\theta_{m-2}d\theta_1 d\theta_2\ldots d\theta_{m-2}d\theta_{m-1},$$

we obtain

$$\int_{S_1} f(\phi,1)dS_1 = \int_{-\pi}^{\pi}d\theta_{m-1}\int_0^{\pi}\sin\theta_{m-2}d\theta_{m-2}\ldots\int_0^{\pi}\sin^{m-3}\theta_2 d\theta_2$$
$$\int_0^{\pi}f(\phi,1)\sin^{m-2}\theta_1 d\theta_1.$$

We have

$$f(\phi,1) = C(a_0)\{\frac{\partial\phi}{\partial\theta_1}\sin\theta_1 + (m-1)\phi\cos\theta_1\},$$

where $C(a_0)$ is a constant only depending on a_0. Thus the inner integral is equal to

$$C(a_0)\int_0^{\pi}[(m-1)\phi\cos\theta_1 + \frac{\partial\phi}{\partial\theta_1}\sin\theta_1\sin^{m-2}\theta_1 d\theta_1 =$$
$$C(a_0)\int_0^{\pi}\frac{\partial}{\partial\theta_1}[\sin^{m-1}\theta_1\cdot\phi(\theta_1,\ldots,\theta_{m-1})]d\theta_1 = 0$$

and thus

$$\int_{S_1} f(\phi,1)dS_1 = 0.$$

Furthermore, we have

$$\frac{\partial\phi}{\partial\theta_1}\sin\theta_1 = -\sin^2\theta_1 e_1 + \sin\theta_1\cos\theta_1\cos\theta_2 e_2 + \ldots +$$
$$+\sin\theta_1\cos\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\cos\theta_{m-1}e_{m-1}+$$
$$+\sin\theta_1\cos\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\sin\theta_{m-1}e_m$$

and

$$\phi\cos\theta_1 = \cos^2\theta_1 e_1 + \sin\theta_1\cos\theta_1\cos\theta_2 e_2 + \ldots +$$
$$+\sin\theta_1\cos\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\cos\theta_{m-1}e_{m-1}+$$
$$+\sin\theta_1\cos\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\sin\theta_{m-1}e_m$$

and thus

$$\frac{\partial\phi}{\partial\theta_1}\sin\theta_1 + (m-1)\phi\cos\theta_1 = -1\cdot e_1 + m\cdot\cos^2\theta_1 e_1 +$$
$$+m\cdot\sin\theta_1\cos\theta_1\cos\theta_2 e_2 + \ldots + m\cdot\sin\theta_1\cos\theta_1\sin\theta_2\ldots\sin\theta_{m-2}\sin\theta_{m-1}e_m$$

and

$$|\frac{\partial\phi}{\partial\theta_1}\sin\theta_1 + (m-1)\phi\cos\theta_1| \le 1+m^2$$

and we obtain finally

$$\int_{S_1}|f(\phi,1)|^{p'}dS_1 \le (1+m^2)^{p'}\cdot A_m, \quad 1<p'<\infty.$$

If $k\neq 1$, we choose spherical coordinates such that the same situation arises.We obtain that $\int_G \frac{\partial}{\partial x_k}E_a(x-y)u(y)dy$ consists of weakly singular integral operators and a Calderon-Zygmund operator (singular integral operator). This completes the proof.

Lemma. *Let $u\in C^d_C(G)$ then we have*

$$(D_x+a)T_a u = \begin{cases} u(x) & \textit{in } G\\ 0 & \textit{in } \mathbb{R}^m\setminus\overline{G}.\end{cases}$$

This follows immediately from the construction of the operator T_a.

An important connection between the operators D_x+a, T_a and F_a is given by the Cauchy-Green formula.

Theorem (Cauchy-Green formula). *Let $u\in C^1_C(G)$ then we have*

$$F_a u + T_a(D_x+a)u = \begin{cases} u(x) & \text{in } G\\ 0 & \text{in } \mathbb{R}^m\setminus\overline{G}.\end{cases}$$

Proof. We have

$$(D_y-a)\{e^{-<\underline{a},x-y>}K_{a_0}(x-y)u(y)\} + e^{-<\underline{a},x-y>}K_{a_0}(x-y)au(y) =$$
$$= \underline{a}e^{-<\underline{a},x-y>}K_{a_0}(x-y)u(y) - e^{-<\underline{a},x-y>}D_xK_{a_0}(x-y)u(y) +$$
$$+e^{-<\underline{a},x-y>}K_{a_0}(x-y)D_yu(y) - ae^{-<\underline{a},x-y>}K_{a_0}(x-y)u(y) +$$
$$+e^{-<\underline{a},x-y>}K_{a_0}(x-y)au(y) =$$
$$-e^{-<\underline{a},x-y>}(D_x+a_0)K_{a_0}(x-y)u(y) + e^{-<\underline{a},x-y>}K_{a_0}(x-y)(D_y+a_0)u(y).$$

Let $G_\epsilon = \{y\in G : |x-y|>\epsilon\}$ then

$$\int_{G_\epsilon}(D_y-a)\{e^{-<\underline{a},x-y>}K_{a_0}(x-y)u(y)\}dy =$$
$$-\int_{G_\epsilon}e^{-<\underline{a},x-y>}(D_x+a_0)K_{a_0}(x-y)u(y)dy +$$
$$+\int_{G_\epsilon}e^{-<\underline{a},x-y>}K_{a_0}(x-y)(D_y+a)u(y)dya\int_{G_\epsilon}e^{-<\underline{a},x-y>}K_{a_0}(x-y)u(y)dy =$$
$$-\int_{G_\epsilon}e^{-<\underline{a},x-y>}(D_x+a_0)K_{a_0}(x-y)u(y)dy + \int_\Gamma e^{-<\underline{a},x-y>}K_{a_0}(x-y)n(y)u(y)dy +$$
$$-\int_{S_\epsilon}e^{-<\underline{a},x-y>}K_{a_0}(x-y)n(y)u(y)dy - \underline{a}\int_{G_\epsilon}e^{-<\underline{a},x-y>}K_{a_0}(x-y)u(y)dy +$$

$$+\int_{G_\epsilon} e^{-<\underline{a},x-y>} D_x K_{a_0}(x-y)u(y)dy = a\int_{G_\epsilon} e^{-<\underline{a},x-y>} K_{a_0}(x-y)u(y)dy+$$
$$+\int_\Gamma e^{-<\underline{a},x-y>} K_{a_0}(x-y)n(y)u(y)dy - \int_{S_\epsilon} e^{-<\underline{a},x-y>} K_{a_0}(x-y)n(y)u(y)dy.$$

Thus

$$\int_{G_\epsilon} e^{-<\underline{a},x-y>} K_{a_0}(x-y)(D_y+a)u(y)dy - \int_{G_\epsilon} e^{-<\underline{a},x-y>}(D_x+a_0)K_{a_0}(x-y)u(y)dy =$$
$$= \int_\Gamma e^{-<\underline{a},x-y>} K_{a_0}(x-y)n(y)u(y)dy - \int_{S_\epsilon} e^{-<\underline{a},x-y>} K_{a_0}(x-y)n(y)u(y)dy.$$

Now, ϵ tends to zero, than G_ϵ tends to G and the integral over S_ϵ tends to zero, thus

$$-T_{-\overline{a}}u + \int_{G_\epsilon} e^{-<\underline{a},x-y>} K_{a_0}(x-y)(D_y+a)u(y)dy =$$
$$\int_\Gamma e^{-<\underline{a},x-y>} K_{a_0}(x-y)n(y)u(y)dy.$$

Application of $(D_x - \overline{a})$ from the left leads to

$$\int_\Gamma e^{-<\underline{a},x-y>}(D_x-a_0)K_{a_0}(x-y)n(y)u(y)dy+$$
$$\int_{G_\epsilon} e^{-<\underline{a},x-y>}(D_x-a_0)K_{a_0}(x-y)(D_y+a))u(y)dy = \begin{cases} u(x) & \text{in } G \\ 0 & \text{in } \mathbb{R}^m \setminus \overline{G} \end{cases}$$

or

$$F_a u + T_a(D_x+a)u = \begin{cases} u(x) & \text{in } G \\ 0 & \text{in } \mathbb{R}^m \setminus \overline{G} \end{cases}.$$

Because the operators

$$D + a : W^1_{p,C}(G) \to L_{p,C}(G), \qquad 1 < p < \infty,$$
$$T_a : L_{p,C}(G) \to W^1_{p,C}(G), \qquad 1 < p < \infty$$

are continuous, the operator

$$F_a : W^{1-1/p}_{p,C}(\Gamma) \to W^1_{p,C}(G)$$

is also continuous. Thus we are able to extend the lemma and the Cauchy-Green formula:

LEMMA. *Let* $u \in W^1_{p,C}(G)$, $1 < p < \infty$, *then we have*

$$(D_x+a)T_a u = \begin{cases} u(x) & \textit{in } G \\ 0 & \textit{in } \mathbb{R}^m \setminus \overline{G} \end{cases}.$$

THEOREM (Cauchy-Green formula). *Let* $u \in W^1_{p,C}(G)$, $1 < p < \infty$, *then we have*

$$F_a u + T_a(D_x+a)u = \begin{cases} u(x) & \textit{in } G \\ 0 & \textit{in } \mathbb{R}^m \setminus \overline{G} \end{cases}.$$

4. A note on elementary functions. If X is an arbitrary element of the Clifford-algebra C, then

$$e^X = \sum_{n=0}^\infty \frac{X^n}{n!} = 1 + X + \frac{x^2}{2!} + \frac{X^3}{3!} + \ldots$$

and

$$\sinh x = \frac{e^X - e^{-X}}{2} = \sum_{n=0}^{\infty} \frac{X^{2n+1}}{(2n+1)!}, \qquad \cosh X = \frac{e^X + e^{-X}}{2} = \sum_{n=0}^{\infty} \frac{X^{2n}}{(2n)!},$$

thus

$$e^X = \cosh X + \sinh X.$$

Furthermore, we have

$$\sin X = \sum_{n=0}^{\infty} (-1)^n \frac{X^{2n+1}}{(2n+1)!}, \qquad \cos X = \sum_{n=0}^{\infty} (-1)^n \frac{X^{2n}}{(2n)!}.$$

Lemma. *If $JX = XJ$ for all X and $J^2 = -1$, then we have* $\cosh JX = \cos X$ *and* $\sinh JX = J \sin X$ *and* $e^{JX} = \cos X + J \sin X$.

R e m a r k. If we denote by $I_m = e_1 e_2 \ldots e_m$ the pseudoscalar of C the only possible J are the scalars $\pm i$ and the elements $\pm i I_{4p+1}$ and $\pm I_{4p+3}$, where $p = 0, 1, 2, \ldots$.

If a is a vector, i.e. $a = \sum_{j=1}^{m} a_j e_j$, then

$$a^n = \begin{cases} |a|^n, & n = 2k \\ |a|^{n-1} a, & n = 2k+1 \end{cases}.$$

In this case we obtain

$$e^a = \cosh a + \sinh a = \cosh |a| + \frac{a}{|a|} \sinh |a|$$

$$e^{Ja} = e^{aJ} = \cosh Ja + \sinh Ja = \cos a + J \sin a = \cos |a| + J \frac{a}{|a|} \sinh |a|.$$

5. The Dirac-type operator $\frac{\partial}{\partial t} D$. We consider the equation

$$\left(\frac{\partial}{\partial t} D\right) E = \delta(x) \otimes \delta(t)$$

in distributional sense. We use the partial Fourier transform to compute the fundamental solution. Thus

$$\left(\frac{\partial}{\partial t} \hat{E} + D(iy) \hat{E}\right) = l(y) \otimes \delta(t).$$

The solution of the problem is

$$\hat{E}(t) = \exp(-tD(iy)).$$

Theorem. *The fundamental solution for the wave operator $\frac{\partial}{\partial t} + D$, $t \geq 0$ is*

$$E(x,t) = \left(\frac{\partial}{\partial t} - D_x\right) \mathcal{F}^{-1}\left(\frac{\sin |y| t}{|y|}\right),$$

where $\mathcal{F}^{-1}(\frac{\sin |y| t}{|y|})$ is the fundamental solution for the wave operator for $t \geq 0$.

P r o o f. We have seen that the partial Fourier transform of the fundamental solution is

$$\hat{E}(t) = \exp(-tD(iy)).$$

We use the results of section 3 to get an explicit formula. We have

$$\exp(-tD(iy)) = \exp(-i\sum_{j=1}^{m} e_j y_j t) = \cos(|y|t) - i\frac{y}{|y|}\sin(|y|t) =$$
$$\cos(|y|t) - \frac{D)iy)}{|y|}\sin(|Y|t) = (\frac{\partial}{\partial t} - D(iy))\frac{\sin|y|t}{|y|}.$$

Thus

$$E(x,t) = (\frac{\partial}{\partial t} - D_x)\mathcal{F}^{-1}\frac{\sin|y|t}{|y|}.$$

Remark. The fundamental solution for the wave operator depends on the dimension m. In general it is a distributional derivative of a measure.

A similar problem is the following operator

$$\frac{\partial}{\partial t} + \beta_0(D+a),$$

where β_0 is a complex non-zero constant and $a = a_0e_0 + \sum_{j=1}^m a_je_j = a_0e_0 + \underline{a}$.

THEOREM. *The fundamental solution of*

$$\frac{\partial}{\partial t} + \beta_0(D+a),$$

for $t \geq 0$ *is*

$$e^{-a_0t<\underline{a},x>} \cdot e^{\beta_0} \cdot (\frac{\partial}{\partial t} - D_x)\mathcal{F}^{-1}(\frac{\sin|y|t}{|y|}).$$

The proof is obvious.

6. The Cauchy problem for the operator $\frac{\partial}{\partial t} + D$. An important problem for this Dirac-type operator is the Cauchy problem because this problem for hyperbolic operators is well-posed.

THEOREM. *The Cauchy problem for the Dirac-type operator*

$$\frac{\partial u}{\partial t} + Du = 0, \qquad u(x,0) = u_0$$

has a unique solution in D' *for* $t \geq 0$

$$u(x,t) = \mathcal{F}^{-1}(\cos(|y|t))\langle *, x\rangle u_0 - D_x\mathcal{F}^{-1}(\frac{\sin|y|t}{|y|})\langle *, x\rangle u_0,$$

where $\langle *, x\rangle$ *denotes the convolution only with respect to* x. *If* $u_0 \in L_{2,C}(\mathbb{R}^m)$ *then* $(u,x) \in L_{2,C}[0,T;\mathbb{R}^m]$.

Proof. The proof follows from the fact that $\cos(|y|t)$ and $D(iy)\frac{\sin|y|t}{|y|}$ if $t \geq 0$ are multipliers in $L_{2,C}(\mathbb{R}^m)$; see [DL].

To solve the inhomogeneous Cauchy problem we use again partial Fourier transform and obtain the problem

$$\frac{\partial \hat{u}}{\partial t} + D(iy)\hat{u} = \hat{f},$$
$$\hat{u}(0) = u_0.$$

Now, we set $\hat{u} = e^{-D(iy)\cdot t}v(t)$ and obtain a problem in v.

$$e^{-D(iy)\cdot t}\frac{\partial v}{\partial t} = \hat{f}, \qquad v(0) = \hat{u}(0)$$

and the solution is

$$v(t) = \hat{u}_0 + \int_0^t e^{-D(iy)\cdot t}v(t)\hat{f}(r)dr$$

and thus

$$\begin{aligned}\hat{u}(t) = e^{-D(iy)\cdot t}\hat{u}_0 + \int_0^t e^{-D(iy)(t-r)}\hat{f}(r)dr = \\ = e^{-D(iy)\cdot t}\hat{u}_0 + Y(t)\cdot e^{-D(iy)\cdot t}\langle *, t\rangle \hat{f}(t),\end{aligned}$$

where $Y(t)$ denotes the Heaviside function. To summarize we state

THEOREM. *The Cauchy problem*

$$\frac{\partial u}{\partial t} + Du = f(x,t), \qquad u(x,0) = u_0$$

has for $u_0 \in L_{2,C}(\mathbb{R}^m)$ *and* $f \in L_{2,C}[0,T; L_{2,C}(\mathbb{R}^m)]$ *a unique solution*

$$u \in L_{2,C}[0,T; L_{2,C}(\mathbb{R}^m)]$$

and

$$\hat{u}(t) = e^{-D(iy)\cdot t}\hat{u}_0 + Y(t)\cdot e^{-D(iy)\cdot t}\langle *, t\rangle \hat{f}(t).$$

7. Examples of problems with Dirac-type operators. 1) A relativistic particle with spin 1/2 in an electromagnetic field with vector potential $\vec{A}$:

$$\sum_{k=1}^{3} e_k(i\frac{\partial}{\partial x_t} + b_k) + e_0 m_0$$

with rest-mass m_0 and $(b_1, b_2, b_3) = \vec{b} = -Q\vec{A}$, where Q is the charge.

2) The Dirac "Hamilton"-operator for a free particle

$$H = e_0 - i\sum_{j=1} e_j\frac{\partial}{\partial x_j}$$

and the corresponding Cauchy problem:

$$\begin{gathered}\frac{\partial \psi}{\partial t} + (\sum_{j=1}^{3} e_j\frac{\partial}{\partial x_j} + ie_0)\psi = 0, \\ \psi(0) = \psi(0).\end{gathered}$$

3) The equation of small perturbations for an irrotational perfect compressible gas:

$$\begin{gathered}\frac{\partial p}{\partial t} + \rho_0 \mathrm{div} v = 0, \qquad \rho_0\frac{\partial v}{\partial t} + \mathrm{grad} p = 0 \\ v(x,0) = v^0(x), \qquad p(x,0) = p^0(x),\end{gathered}$$

is equivalent to the system

$$(\frac{\partial}{\partial t} + D)\begin{pmatrix} p^0 \\ p_0 v\end{pmatrix} = 0,$$

$$\begin{pmatrix} p^0 \\ p_0 v \end{pmatrix}(x,0) = \begin{pmatrix} p^0 \\ p_0 v^0 \end{pmatrix}.$$

4) Stationary Maxwell equations:

We set $E(x,t) = E_0(x)e^{i\omega t}B(x,t) = B_0(x)e^{i\omega t}$, where $E(x,t)$ is the electrical field and $B(x,t)$ is the magnetic inductivity and then

$$\begin{aligned} \mathrm{div}E_0 &= \rho_0, \\ i\omega E_0 - \mathrm{curl}B_0 &= -j_0, \\ i\omega B_0 + \mathrm{curl}E_0 &= 0, \\ \mathrm{div}B_0 &= 0 \end{aligned}$$

in the domain G. If we set $U = (0, E_0, B_0, 0)$, then the system above is equivalent to

$$(\sum_{k=1}^{3} e_k \frac{\partial}{\partial x_t} + i\omega e_0)U = (\rho_0, -j_0\ 0, 0).$$

5) Time-dependent Maxwell equations in vacuum

$$\begin{aligned} \mathrm{div}E_0 - \rho &= 0, \\ \frac{\partial E}{\partial t} - \mathrm{curl}B + j &= 0, \\ \frac{\partial B}{\partial t} + \mathrm{curl}E &= 0, \\ \mathrm{div}B &= 0. \end{aligned}$$

If we set $V = (0, E, B, 0)$ the system above is equivalent to

$$(\frac{\partial}{\partial t} + \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_t})V = (\rho, -j, 0, 0).$$

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