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HYPERBOLIC-LIKE MANIFOLDS, GEOMETRICAL PROPERTIES AND HOLOMORPHIC MAPPINGS

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Abstract. The authors are dealing with the Dirichlet integral-type biholomorphic-invariant pseudodistance $\rho_X^\alpha(z_0, z)[\mathcal{U}]$ introduced by Dolbeault and Ławrynowicz (1989) in connection with bordered holomorphic chains of dimension one. Several properties of the related hyperbolic-like manifolds are considered remarking the analogies with and differences from the familiar hyperbolic and Stein manifolds. Likewise several examples are treated in detail.

1. Introduction and outline of results. We are going to recall, after [5], the definition of a hyperbolic-like manifold:

Let X be a complex manifold of complex dimension n and let γ be a C^1-cycle of real dimension one with support relatively compact in X. Let Γ be a complex analytic subvariety of complex dimension one of $U = X\backslash \mathrm{spt}\gamma$, such that the integration current $[\Gamma]$ defined by Γ admits a simple extension $[\tilde{\Gamma}]$ having compact support in X and satisfies $d[\tilde{\Gamma}] = \gamma$.

By an *elementary* (*bordered holomorphic*) *chain* we will understand the integration current $[\Gamma]$ defined by Γ and to simplify notation we will denote it also by Γ. By Reg Γ we will understand the regular part of Γ. Reg Γ is the image of a connected Riemann surface S by biholomorphic mapping f; Γ is also the image of a Riemann surface $\sum$ by a holomorphic mapping $\psi : \Sigma \to X\backslash \mathrm{spt}\gamma$ such that $S \subset \Sigma$, $\psi/S = f$ and $\Sigma\backslash S$ is a discrete

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set of points of Σ.

Let $\mathcal{U} = \{U_j : j \in I\}$ be a locally finite open covering of X. Denote by $F[\mathcal{U}] \equiv \mathrm{adm}(X,\mathcal{U})$ the family of all *admissible* pluriharmonic C^2-functions u_j in U_j, defined in each member of the covering, which satisfy the conditions

(a) the oscillation of u_j in U_j is less than one,
(b) $d\,u_j = d\,u_k$ in $U_j \cap U_k \neq \emptyset$.

Now we equip X with an hermitian metric h so that, in local coordinates, we have the following associated geodesic distance:

$$ds^2 = h_{j\bar{k}} dz_j \otimes d\bar{z}_{\bar{k}} \text{ (with the Einstein summation convention).}$$

The induced hermitian metric on a connected Riemann surface S of X can be expressed, via the associated geodesic distance, as $f^*ds^2 = g dz \otimes d\bar{z}$. Let $g(d^cv, d^cv) \equiv ||\Delta v||^2_{ds} := g^{1\bar{1}}(\partial/\partial z)v(\partial/\partial \bar{z})v$ and consider an integral of the form

$$2i \int_s \hat{g}^\alpha |(\partial/\partial z)v|^2 dz \wedge d\bar{z}, \quad \hat{g} := g(d^cv, d^cv), \quad \alpha \neq 0. \tag{1}$$

Within Γ there exist compact connected C^1-cycles of dimension one. Let γ' be one of them. We assume that the length $|\gamma'|$ of the border γ' of Γ' $\quad(\gamma' = d\Gamma')$ is uniformly bounded in Γ, so $\Delta' := \psi^{-1}[\Gamma']$ is an elementary chain with the border $\delta' := \psi^{-1}[\gamma']$, lying on Σ. Now we can consider an elementary chain as the image under ψ of an elementary chain with border on Σ. Given a bordered holomorphic chain passing through distinct points z_0, z of $\mathcal{U}$, we can consider it the sum of elementary chains passing through distinct points z_{j-1}, z_j of $U := X\backslash \mathrm{spt}\gamma$, $j = 1, \ldots, p$, so that z_0 is the first given point, while z_p is the last one: $z_p = z$.

For each elementary chain Γ'_j passing through the points z_{j-1}, z_j with Γ'_j contained in a fixed elementary chain Γ_j, consider a holomorphic mapping $\psi_j : \Sigma_j \to \Gamma_j \subset X\backslash \mathrm{spt}\gamma_j$. It is biholomorphic except perhaps for a discrete set of points, so we get

$$\inf_{\Gamma'_j \subset \Gamma_j} \{\frac{|\gamma'_j|}{|\Gamma'_j|} | \int_{\Gamma'_j} (\psi^*_j \hat{g})^\alpha du \wedge d^cu|\} = \inf_{\Delta'_j \subset \Delta_j} \{\frac{|\delta'_j|}{|\Delta'_j|} | \int_{\Delta'_j} \hat{g}^\alpha dv \wedge d^cv|\} \tag{2}$$

with $\hat{g}^\alpha$ as in (1), where $|\Gamma'_j|$ denotes the volume of Γ'_j, $\Delta'_j := \psi_j^{-1}[\Gamma'_j]$ and $\Delta_j := \psi_j^{-1}[\Gamma_j]$. Thus with any bordered holomorphic chain passing through points z_0, z of X, such that $|\gamma'|$ is uniformly bounded in Γ, we may associate the expression

$$\mu^\alpha_\Gamma(z_0, z)[u] := \sum_{j\in I} \inf_{\Gamma'_j} \mu^\alpha_{\Gamma'_j}[u],$$

where $\mu^\alpha_{\Gamma'_j}[u]$ is defined as the expression from which the infimum is taken at the right-hand side of (2). The expression

$$\rho^\alpha_X(z_0, z)[u, \mathcal{U}] : \inf\{\mu^\alpha_\Gamma(z_0, z)[u] : \Gamma \text{ passing through } z_0, z \in U\}$$

appears to be well defined as well.

Without loss of generality, let z_0, z_1 be points of the same coordinate neighbourhood U identified through a chart with an open set in $\mathbb{C}^n$. Suppose that z_0, z_1 are sufficiently near to each other so that the segment $[z_0; z_1]$ is contained in U. Consider the set $\Gamma_\epsilon := \{z \in L :$

dist $(z, z_0; z_1) < \epsilon\}$, $\epsilon > 0$, where L is the complex line in $\mathbb{C}^n$ passing through z_0, z_1. If ϵ is so small that the closure of Γ_ϵ with respect to X is contained in U, then the expression

$$\rho_X^\alpha(z_0, z)[\mathcal{U}] := \sup\{\mu_X^\alpha(z_0, z)[u, \mathcal{U}] : u \in \mathcal{F}[\mathcal{U}]\}$$

appears to be well defined for every $z_0, z \in X$ and, as a function of z_0, z, to be a continuous pseudodistance. It will be called an $(\alpha, \mathcal{U})$-*Dolbeault-Lawrynowicz pseudodistance.* If $\rho_X^\alpha(z_0, z) > 0$ for $z_0 \neq z$, it will be called an $(\alpha, \mathcal{U})-$*Dolbeault-Ławrynowicz distance.* If $\rho_X^\alpha(\ ,\ )[\mathcal{U}]$ is a distance, X is called, following [5], an $(\alpha, \mathcal{U})$- *hyperbolic-like manifold.* An $(\alpha, \mathcal{U})-$hyperbolic-like manifold is said to be *complete* if it is complete with respect to the corresponding distance $\rho_X^\alpha(\ ,\ )[\mathcal{U}]$. With a minor modification the definitions are still valid for the closure of an arbitrary bounded domain in $\mathbb{C}^n$.

In Section 2 the basic properties of the above notions are studied. A special attention is paid to the properties analogous to those known for the Kobayashi pseudodistance and the hyperbolic manifolds. It seems to be important that, under suitable hypotheses (Theorem 1), the family of distance-decreasing mappings with respect to a Dolbeault-Ławrynowicz pseudodistance is locally compact with respect to the compact-open topology.

In Section 3 we prove four lemmas which enable us to establish in Section 4 Theorems 2-4 and Corollaries 4-8 providing examples of hyperbolic-like manifolds of four kinds: (a) hyperbolic-like and hyperbolic, (b) hyperbolic-like: (b_1) Stein but not hyperbolic, and (b_2) neither hyperbolic nor Stein.

The final Section 5 is devoted to a generalisation of the theorems of Dolbeault and Ławrynowicz [5] on extendability of holomorphic mappings.

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2. Basic properties. We start with quoting, after [5], two properties (Propositions 1-2). The proof of the first property requires the following.

LEMMA 1. *Suppose that X and Y are complex manifolds, and $f : X \to Y$ is a proper holomorphic mapping. Then the image of any elementary chain, passing through points z_0, z of X, is a bordered holomorphic chain passing through the points $f(z_0), f(z)$ of Y.*

Proof. Suppose that γ is the border of the elementary chain Γ in question, $f[\gamma]$ a C^1-cycle of dimension one with compact support, $f_\star[\tilde{\Gamma}]$ a current of bidimension (1,1) with compact support and the border $f[\gamma]$, and $f_*[\tilde{\Gamma}]Y\backslash \mathrm{spt} f[\Gamma]$ a d-closed positive rectifiable current of bidimension (1, 1). By the structural theorem of J. King. [6], it is a current of integration, defined by a holomorphic one-chain with positive coefficients in $Y\backslash f[\gamma]$. Therefore $f_*[\tilde{\Gamma}]$ is a holomorphic chain with the border in Y and, clearly, its support contains $f(z_0)$ and $f(z)$, as desired.

Dolbeault showed us the proof of the following result:

PROPOSITION 1. *Let $(X, h, \mathcal{U}), (Y, \tilde{h}, \tilde{\mathcal{U}})$ be two complex analytic manifolds with hermitian metrics $h, \tilde{h}$ and coverings $\mathcal{U}, \tilde{\mathcal{U}}$ and admissible families $F[\mathcal{U}]$, $F[\tilde{\mathcal{U}}]$ of pluriharmonic functions. Let $f : X \to Y$ be a proper holomorphic mapping such that $f^{-1}(\tilde{\mathcal{U}}) \subset \mathcal{U}$ and $f^*\tilde{h} = h$, then*

$$\rho_X^\alpha(z_0, z)[\mathcal{U}] \geq \rho_Y^\alpha(f(z_0), f(z))[\tilde{\mathcal{U}}]$$

P r o o f. Let $u \in \mathcal{F}[\tilde{\mathcal{U}}]$; then $f^*u \in F[\mathcal{U}]$.

For each elementary chain Γ_j either $f(\Gamma_j)$ is one point or $f(\Gamma_j)$ is a one-dimensional complex variety and the restriction $f|_{\Gamma_j} : \Gamma_j \to f(\Gamma_j)$ is a finite ramified covering. Then we have the next diagram

$$\begin{tikzcd}
\Delta'_j \arrow[r] \arrow[d, hook] & \Gamma'_j = \Phi_j(\Sigma_j) \subset \Gamma_j \subset X \arrow[r, "f"] & Y \supset \hat{\Gamma}_j = f(\Gamma_j) \supset \hat{\Gamma}'_j = f(\Gamma'_j) & \hat{\Delta}'_j \arrow[l] \arrow[d, hook] \\
\Sigma_j \arrow[ur, "\Phi_j"'] \arrow[rrr, "f_j = \Phi_j^{-1} \circ f \circ \Phi_j"'] & & & \hat{\Sigma}_j \arrow[ul, "\hat{\Phi}_j"]
\end{tikzcd}$$

By the definition Φ_j is a holomorphic map which is locally biholomorphic except on a discrete set of Σ_j and each $f_j : \Sigma_j \to \tilde{\Sigma}_j$ is locally biholomorphic except on a discrete set of Σ_j, thus

$$|\int_{\Delta'_j} (\widehat{f^*\tilde{g}_j})^\alpha dv_j \wedge d^c v_j| = \deg f \int_{\tilde{\Delta}'_j} \hat{\tilde{g}}_j{}^\alpha d\tilde{v}_j \wedge d^c \tilde{v}_j$$

with

$$v_j = \Phi_j^* f_j^* \tilde{\Phi}_j^{-1^*} \tilde{v}_j \quad , \text{ as } \frac{|\tilde{\delta}'_j|}{|\tilde{\Delta}'_j|} = \frac{|\delta_j|}{|\Delta'_j|} \text{ so } \mu^\alpha_{\Gamma'_j}[f^*\tilde{u}] \geq \mu^\alpha_{\tilde{\Gamma}'_j}[\tilde{u}].$$

Then taking the supremum for $\tilde{u} \in F[\tilde{u}]$ and $u \in F[\mathcal{U}]$ we obtain the desired inequality.

PROPOSITION 2. *A complex submanifold X' of a complete hyperbolic-like manifold X is hyperbolic-like. If, in addition, X' is closed, it is also complete.*

P r o o f. It is sufficient to apply Proposition 1 to the holomorphic inclusion mapping $f : X' \to X$.

PROPOSITION 3. *Suppose that X and $X_j, j \in I$, are complex manifolds such that $X = \Pi_{j\in I} X_j$. If all X_j are hyperbolic-like, then also X is hyperbolic-like.*

P r o o f. Follows directly from the definition of the set-theoretical product.

PROPOSITION 4 ([5], Proposition 4). *Let X be a complex manifold, M a covering manifold of X with covering projection $\pi : M \to X$, and α a positive number. Let $z_0, z \in X$ and $s_0, s \in M$ so that $\pi(s_0) = z_0$ and $\pi(s) = z$. Then*

$$\rho^\alpha_X(z_0, z) = \inf\{\rho^\alpha_M(s_0, s) : s \in M, \pi(s_0) = z_0\ \pi(s) = z\}.$$

PROPOSITION 5. *Suppose that X is a complex manifold and M its covering manifold such that, for every $z \in X$, the preimage of x under the covering projection consists of one point only. Then, if X is hyperbolic-like, so is M.*

P r o o f. Proposition 5 is a direct consequence of Proposition 4.

PROPOSITION 6. *Let X be a hyperbolic-like manifold and f a holomorphic function on X. Then the submanifold $X' = \{z \in X : f(z) \neq 0\}$ of X can be made also hyperbolic-like.*

P r o o f. The set $D = \{z \in X : f(z) = 0\}$ is closed, so X' is an open submanifold of X. Hence, by Proposition 2, X' is hyperbolic-like.

R e m a r k 3. Propositions 2, 3, 4, and 6 have their counterparts for the Kobayashi (pseudo)distance; cf. [7], pp. 48 and 57.

THEOREM 1. *Suppose that M is a locally compact, complex manifold with a Dolbeault-Lawrynowicz pseudodistance ρ_M^α, and N a locally compact, complete manifold with respect to Dolbeault-Lawrynowicz pseudodistance ρ_N^α. Then the family of distance-decreasing mappings $f : M \to N$ is locally compact with respect to the compact-open topology. If $p \in M$ and K is a compact subset of N, then the subset $F(p, K) := \{f \in F : f(p) \in K\}$ of F is compact as well.*

Proof. Is analogous to that of Theorem V.3.1. in [7], pp. 73-74.

3. Lemmas. Through the next set of lemmas we will obtain information about the $(\alpha,\mathcal{U})$-pseudometric in a manifold respect to the open covering $\mathcal{U}$ and the family of pluriharmonic functions $F[\mathcal{U}]$.

LEMMA 2. *Suppose that a collection $\{U_1, \dots U_k\}$ of open sets forms a covering of a connected open set U and let $x, y \in U$. Then there is a sequence of sets $U^1, \dots, U^\ell, \ell < k$, of that covering with the property*

$$x \in U^1, y \in U^\ell, U^j \cap U^{j+1} \neq \emptyset, \quad j = 1, \dots, \ell - 1.$$

Proof. Denote by U^1 the set of $\{U_1, \dots U_k\}$ containing x. If $y \in U^1$, the lemma is proved. Otherwise we take into account all those remaining sets which have nonempty intersections with U^1; denote them by $U_1^2, \dots, U_n^2$. If $y \in U_j^2$, $j \in \{1, \dots, n_2\}$, then U^1, U_j^2 form the required sequence. Otherwise consider all those remaining sets which have nonempty intersections with $U_1^2, \dots U_{n_2}^2$, and denote them by $U_1^3, \dots U_{n_3}^3$. There exists at least one such set since otherwise the union of already chosen sets and the union of all the remaining sets would be disjoint open sets covering $\mathcal{U}$, and this would contradict its connectivity.

Next we are continuing the above described procedure until we find a set U_j^ℓ containing y. This is the only case since the sets are chosen only once and therefore the procedure is finite. The way of choosing the sets implies that the sought sequence can be chosen by taking, subsequently, sequences $U_j^\ell, U_{j,\ell-1,\dots,}^{\ell-1} U_{j,2}^2, U^1$ with the property

$$U_j^\ell \cap U_{j,\ell-1}^{\ell-1} \neq \emptyset, \dots, U_{j,2}^2 \cap U^1 \neq \emptyset.$$

LEMMA 3. *Suppose that $\mathcal{U} = \{U_j : j \in I\}$ and $\mathcal{V} = \{V_k : k \in J\}$ are distinct coverings of the same complex manifold, while $F[\mathcal{U}]$ and $F[\mathcal{V}]$ denote the corresponding families of all admissible pluriharmonic C^2-functions. Consider the covering $\mathcal{W} = \{W_\ell : \ell \in K\}$, where each W_ℓ is a component of some set $U_j \cap V_k$, where $j \in I$ and $k \in J$. If there exists a positive integer n such that for each $j \in I$ the set U_j is the union of at most n sets of $\mathcal{W}$, then for every $u \in F[\mathcal{W}]$ there exists an element $\tilde{u} \in F[\mathcal{U}]$ with the property*

$$du = n d\tilde{u} \quad \textit{and} \quad d^c u = n d^c \tilde{u}. \tag{4}$$

Proof. Take $u \in F[\mathcal{W}]$. For any fixed $U_{j^*} \in \mathcal{U}$, the components of $U_{j^*} \cap V_k$, $k \in J$, form a covering of U_{j^*} and the functions u_ℓ, defined on those components, generate the form du_{j^*k} on U_{j^*}. Since U_{j^*} is a simply connected set, there is a $C^\infty-$function f_{j^*}, defined on U_{j^*}, with the property

$$df_{j^*} = du \quad \textit{and} \quad d^c f_{j^*} = d^c u.$$

Since, on each component W_ℓ of $U_{j^*} \cap V_k$, the function f_{j^*} differs from u_ℓ by a constant, and the oscillation of u_ℓ on W_ℓ does not exceed one, so is the oscillation of f_{j^*} on W_ℓ.

If $x, y \in U_{j^*}$, then by Lemma 2 there exists a sequence $W^1, \ldots, W^\ell$, $\ell \le n$, of components of U_{j^*} such that

$$x \in W^1,\ y \in W^2,\ W^1 \cap W^2 \neq \emptyset, \ldots, W^{\ell-1} \cap W^\ell \neq \emptyset.$$

Let $x_1 \in W^1 \cap W^2, \ldots x_{\ell-1} \in W^{\ell-1} \cap W^\ell$. Hence

$$\begin{aligned} |f_{j^*}(x) - f_{j^*}(y)| &\le |f_{j^*}(x) - f_{j^*}(x_1)| + |f_{j^*}(x_1) - f_{j^*}(x_2)| \\ &\quad + \cdots + |f_{j^*}(x_{\ell-1}) - f_{j^*}(y)| \le n, \end{aligned}$$

and thus

$$\operatorname{osc}_{U_j^*} f_{j^*} = \sup_{x,y \in U_{j^*}} |f_{j^*}(x) - f_{j^*}(y)| \le n.$$

Therefore $\tilde{u}_{j^*} := (1/n) f_{j^*}$ is a $C^\infty-$ function on U_{j^*} with oscillation less than one. After repeating the construction for each $j \in I$, we get

$$\tilde{u} := \{\tilde{u}_j : j \in I\} \in F[\mathcal{U}].$$

In consequence we arrive at the formula (4), as desired.

LEMMA 4. *Suppose that* $\mathcal{U} = \{U_j : j \in I\}$ *and* $\mathcal{U}' = \{U'_k : k \in J\}$ *are open coverings of a complex manifold* X, *such that for each* $j \in I$ *there is an element* $k \in J$ *such that* $U_j \subset U'_k$. *Next, let* $F[\mathcal{U}]$ *and* $F[\mathcal{U}']$ *denote the corresponding families of all admissible pluriharmonic* C^2-*functions. Then*

$$\rho_X^\alpha(z_0, z)[\mathcal{U}'] \le \rho_X^\alpha(z_0, z)[\mathcal{U}] \quad \text{for any} \quad z_0, z \in X \quad \text{and} \quad \alpha \ge 0.$$

Proof. By the definition of U and U' for any $u' \in F[\mathcal{U}']$ there is a function $u \in F[\mathcal{U}]$ such that $u_j = u_k$ on $U_j \subset U_k$ for $j \in I$: in order to see this it is sufficient to take $u_j := u'_j | U_j, j \in I$. Hence $F[\mathcal{U}'] \subset F[\mathcal{U}]$ and, consequently,

$$\begin{aligned} \rho_X^\alpha(z_0, z)[\mathcal{U}'] &= \sup\{\mu_X^\alpha(z_0, z)[u', \mathcal{U}'] : u' \in F[\mathcal{U}']\} \\ &\le \sup\{\mu_X^\alpha(z_0, z)[u, \mathcal{U}] : u \in F[\mathcal{U}]\} = \rho_X^\alpha(z_0, z)[\mathcal{U}]. \end{aligned}$$

LEMMA 5. *Suppose that* $\mathcal{U} = \{U_1, \ldots, U_n\}$ *and* $\mathcal{V} = \{V_k; k \in J\}$ *are distinct open coverings of a complex manifold* X, *such that each* $U_j \cap V_k$ *is either connected or empty. Next, let* $F[\mathcal{U}]$ *and* $F[\mathcal{V}]$ *denote the corresponding families of all admissible pluriharmonic* C^2-*functions. Then there exists a positive number* A *with the property*

$$\rho_X^\alpha(z_0, z)[\mathcal{U}] \le A \rho_X^\alpha(z_0, z)[\mathcal{V}] \quad \text{for any } z_0, z \in X \text{ and } \alpha \ge 0. \tag{5}$$

Proof. Let $\mathcal{W} = \{W_\ell = U_j \cap V_k : j = 1, \ldots, n, k \in J\}$. The coverings $\mathcal{U}$ and $\mathcal{V}$ satisfy the hypotheses of Lemma 4. Hence

$$\rho_X^\alpha(z_0, z)[\mathcal{U}] \le \rho_X^\alpha(z_0, z)[\mathcal{W}] \text{ for any } z_0, z \in X \text{ and } \alpha \ge 0 \tag{6}$$

and, moreover, for every chain $\Gamma \subset X$ we have, by Lemma 3,

$$\int_\Gamma \hat{g}^\alpha du \wedge d^c u = \int_\Gamma \hat{g}^\alpha n^2 d\tilde{u} \wedge d^c \tilde{u} \text{ for } u \in F[\mathcal{W}], \tilde{u} \in F[\mathcal{V}],$$

so

$$\mu_X^\alpha(z_0, z)[u, \mathcal{W}] \le \mu_X^\alpha(z_0, z)[\tilde{u}, \mathcal{V}] \text{ for any } z_0, z \in X \text{ and } \alpha \ge 0$$

Therefore

$$\rho_X^\alpha(z_0,z)[\mathcal{W}] \le A\rho_X^\alpha(z_0,z)[\mathcal{V}] \textit{ for any } z_0, z \in X \textit{ and } \alpha \ge 0. \tag{7}$$

and an analoguos estimate holds for $\rho_X^\alpha(z_0,z)[\mathcal{V}]$. Thus, by (6) and (7), we arrive at (5) indeed.

Corollary 3. *Given a complex manifold X, if there is a finite open covering $\mathcal{U}$ with the corresponding family of admissible pluriharmonic C^2-functions, such that for some $\alpha > 0$, X is a $(\alpha,\mathcal{U})$-hyperbolic-like manifold, then X is $(\alpha,\mathcal{V})$-hyperbolic-like for every open covering $\mathcal{V}$ of X satisfying the condition of Lemma* 5 *with respect to the covering $\mathcal{U}$.*

4. Existence theorems. With the use of Lemmas 2-5 we are going to establish several existence theorems for hyperbolic-like manifolds of a special kind, being at the same time examples showing their range relations to hyperbolic and Stein manifolds. In all the cases we repeat the phrase "can be made hyperbolic-like" because in all the cases, the hyperbolic-like property depends on the considered covering and the corresponding family of pluriharmonic functions.

Theorem 2. *An arbitrary bounded domain in $\mathbb{C}^n$ can can be made hyperbolic-like and is hyperbolic, but, for $n > 1$, it is not, in general, Stein.*

Proof. At first, let $n = 1$. Denote by X the domain in question and take a positive number r such that $|z| \le r$ for every $z \in X$; let $\mathcal{U} = \{X\}$ be a one-element covering of X. Define, globally on X, a harmonic function, e.g.,

$$u(x,y) = \frac{1}{2r}(x+y), x = \mathrm{re}z, y = \mathrm{im}z, z \in \mathbb{C}. \tag{8}$$

Evidently, $|u| < 1$. Let Γ be an elementary chain with border γ passing through arbitrary points $z_0, z \in X$, and let $\alpha \ge 0$. Hence (cf. (1) and (2)):

$$\mu_\Gamma^\alpha(z_0,z)[u,\mathcal{U}] = \frac{|\gamma|}{|\Gamma|}\Big|\int_\Gamma \hat{g}^\alpha du \wedge d^c u\Big| = \frac{|\gamma|}{|\Gamma|}\Big|\int_\Gamma \hat{g}^\alpha \Big|\frac{\partial}{\partial z}u\Big|^2 dz \wedge d\bar{z}\Big|.$$

Since the derivatives $(\partial/\partial x)u$ and $(\partial/\partial y)u$ are constant, the functions $g^\alpha|(\partial/\partial z)u|^2$ are uniformly bounded from below by a number $A > 0$. Therefore

$$\mu_\Gamma^\alpha(z_0,z)[u,\mathcal{U}] \ge (|\gamma|/|\Gamma|)A|\Gamma| = A|\gamma| > 0.$$

Since the elementary chain Γ has been chosen arbitrarily, then

$$\mu_\Gamma^\alpha(z_0,z)[u,\mathcal{U}] = \inf_\Gamma \mu_\Gamma^\alpha(z_0,z)[u,\mathcal{U}] > 0 \text{ for any } z_0, z \in X \textit{ and } \alpha \ge 0.$$

Consequently, by the definition of ρ_X^α, we get $\rho_X^\alpha(z_0,z)[\mathcal{U}] > 0$ and, by Lemma 4, $\rho_X^\alpha(z_0,z)[\mathcal{U}'] > 0$ for an arbitrary locally finite open covering $\mathcal{U}'$ of X. This concludes the proof for $n = 1$.

If $n > 1$, it is sufficient to replace the function (8) by

$$u(x,y) = \frac{1}{2nr}\sum_{j=1}^n (x_j + y_j), x_j = \mathrm{re}z_j, y_j = \mathrm{im}z_j, z = (z_j) \in \mathbb{C}^n.$$

The remaining statements of the theorem are well-known. In particular, if $n > 1$, X is not, in general, holomorphically convex and also is not, in general, a Stein manifold; cf. [7], pp. 55-57.

COROLLARY 4. *A polydisc domain in* $\mathbb{C}^n$ *can be made hyperbolic-like and as we know is hyperbolic and Stein.*

Suppose that X is a closed manifold $\mathbb{C}^n$, ϵ an arbitrary positive number and $D=\{z\in \mathbb{C}^n : \operatorname{dist}(z,X) < \epsilon\}$. Consider an arbitrary locally finite open covering $\mathcal{U}$ of D and the corresponding family $F[\mathcal{U}]$ of admissible pluriharmonic C^2-functions. Let Γ, $\Gamma \subset D$, be an arbitrary bordered holomorphic chain containing points $z_0, z \in X$ and let $\Gamma' = \Gamma|X$. In analogy to Section 1, we define a holomorphic mapping $\Phi'_j : D'_j \to \Gamma'_j$ such that $\Phi'_j = \Phi_j|\Delta'_j$, $\Delta'_j \subset \Delta_j$, and the quantities $\mu^\alpha_\Gamma(z_0,z)[\mathcal{U}]$ and $\mu^\alpha_X(z_0,z)[u,\mathcal{U}]$ and, finally, the $(\alpha,\mathcal{U})-$*Dolbeault-Lawrynowicz pseudodistance* $\rho^\alpha_X(z_0,z)[\mathcal{U}]$ for closed X.

COROLLARY 5. *The closure of an arbitrary bounded domain in* $\mathbb{C}^n$ *can be made hyperbolic-like.*

Proof. If, in particular, D consists of a finite set of points, say: $a_1,\dots,a_n$ it is sufficient to take the function $f(z) = (z-a_1)\dots(z-a_n)$, $z\in X$, and observe that it is holomorphic and $f(a_j) = 0$ for $j = 1,\dots,n$. Hence, by Proposition 6, $X\setminus D$ is hyperbolic-like indeed. In the general case, Proposition 6 has to be applied a finite number of times.

COROLLARY 6. *If* $[z';z'']$ *denotes the segment of line that connects* z' *with* z'', $\bar{\mathbb{C}}\setminus[z';z'']$ *can be made hyperbolic-like for any* $z', z'' \in \bar{\mathbb{C}}$.

Proof. By Theorem 2, the open unit disc $\Delta(0;1)$ hyperbolic-like. On the other side, by the Riemann mapping theorem, $\Delta(0;1)$ can be biholomorphically mapped onto $\mathbb{C}\setminus[z',z'']$, so, by Proposition 1, the result follows.

THEOREM 3. *If* X *is a bounded domain in* $\mathbb{C}^n$ *or, more generally, it is a hyperbolic-like manifold and* D *is an algebraic set in* X, *then* $X\setminus D$ *can be made hyperbolic-like as well.*

Proof. If, in particular, D consists of a finite set of points, say: $a_1,\dots,a_n$, it is sufficient to take the function $f(z) = (z-a_1)\cdots(z-a_n)$, $z \in X$, and observe that it is holomorphic and $f(a_j) = 0$ for $j = 1,\dots,n$. Hence, by Proposition 6, $X\setminus D$ can be made hyperbolic-like indeed. In the general case, Proposition 6 has to be applied a finite number of times.

COROLLARY 7. $\mathbb{C}$ *minus a closed disc cl* $\Delta(0;r)$ *can be made hyperbolic-like.*

Proof. By Theorem 3, we know that $Y := \Delta(0;1/r)\setminus\{0\}$ can be made hyperbolic-like. The function $f(z) = 1/z$, $z \in X := \mathbb{C}\setminus \operatorname{cl}\Delta(0;r)$ maps biholomorphically X onto Y, so, by Proposition 1,

$$\rho^\alpha_X(z_0,z) = \rho^\alpha_Y(f(z_0),f(z)) > 0 \quad \textit{for} \quad z_0, z \in X,$$

and this is sufficient to conclude the proof.

THEOREM 4. $(\mathbb{C},\mathcal{U})$, *where* $\mathcal{U} = \{U_j : j \in \mathbb{Z}\}$ *and* $U_j = \{(x,y) : \frac{1}{2}j < x < \frac{1}{2}j+1, y \in \mathbb{R}\}$, *can be made hyperbolic-like, but not hyperbolic.*

Proof. Consider the harmonic function $u(x, y) = x$, $(x, y) \in \mathbb{C}$. Obviously, the oscillation of u in U_j is less than one for each j, and $g := ||\Delta u||^2 = 1$. Let $z_0, z \in \mathbb{C}$ and Γ be an arbitrary holomorphic chain in $\mathbb{C}$ with border γ, containing z_0 and z. Hence (cf. (1) and (2)):

$$\mu^{\alpha}_{\Gamma}(z_0, z)[u, \mathcal{U}] = \frac{|\gamma|}{|\Gamma|} \Big| \int_{\Gamma} \hat{g}^{\alpha} du \wedge d^c u \Big| = |\gamma| > 0 \text{ whenever } z \neq z_0, \tag{9}$$

so $(\mathbb{C}, \mathcal{U})$ is hyperbolic-like, as desired.

Remark 4. Since $|\gamma| \geq 2|[z_0; z]|$, then, by (9),

$$\rho^{\alpha}_{\mathbb{C}}(z_0, z)[\mathcal{U}] \geq 2|[z_0; z]| \quad \text{for} \quad z_0, z \in \mathbb{C}.$$

Corollary 8. *From Theorems* 3 *and* 4 *it follows that the punctured plane* $\mathbb{C}\backslash\{z_0\}$ *can be made hyperbolic-like*, *but not hyperbolic. If*, *however*, *we take* $A = \{z_1, \ldots, z_k\}$, $z_1, \ldots, z_k \in \mathbb{C}$ *and* $k \geq 2$ *then* $\mathbb{C}\backslash A$ *can be made hyperbolic-like and is hyperbolic as well.*

Corollary 9. *If domain* Y *is a simply connected in* $\mathbb{C}$ *such that every holomorphic function* f *in* Y *can be approximated by polynomials uniformly on compact subsets of* Y, *then* Y *can be made hyperbolic-like. Indeed*, *this type of domains is conformally equivalent to the open unit disc in* $\mathbb{C}$.

Suppose again that X is a complex manifold. An *analytic polyhedron* P in X is a relatively compact open set in X of the form $P = \{p \in W : |f_j(p)| < r_j, j = 1, \ldots, t\}$, where W is a neighbourhood of clP and all functions f_j are holomorphic in W. By Proposition 4 (and Corollary 4) we obtain directly

Theorem 5. *Let* X *be a complex manifold and* P *an analytic polyhedron in* X *determined by* $r = (r_j)$ *and* $f = (f_j)$, $1 \leq j \leq m$. *If*, *for some* j, *the manifold* $f_j^{-1}[\Delta(0; r_j)]$ *is hyperbolic-like*, *then so is* P.

5. Extension theorems. First of all, it is convenient to recall some definitions.

An symmetric tensor $h = 2b(z)dz\overline{dz}$ is a *hermitian pseudometric* in $\mathcal{U} \subset \mathbb{C}$ if:

a) $b(z)$ is a continuous and real-valued function and $b(z) \geq 0$,
b) $Z = \{z \in \mathcal{U}; b(z) = 0\}$ is a discrete subset of $\mathcal{U}$,
c) $b(z)$ is a C^{∞} function on $\mathcal{U}\backslash Z$.

The *Gaussian curvature* of h on $\mathcal{U}, K_n : \mathcal{U} \to [-\infty, \infty)$ is defined by

$$K_h(z) = \begin{cases} -\frac{1}{b(z)} \frac{\partial^2 \log b(z)}{\partial z \partial \bar{z}}, & z \in \mathcal{U}\backslash Z, \\ -\infty, & z \in Z, \end{cases}$$

For two pseudo-metrics $h_i = 2b_i(z)dzd\bar{z}$, $i = 1, 2$, we write $h_1 \leq h_2$ if $b_1(z) \leq b_2(z), \forall z \in \mathcal{U}$. With this definition, on the punctured disc $D^{\star}_R(0) := \{0 < |z| < R\}$ set

$$b_R(z) = \frac{2}{|z|^2(\log|z/R|^2)^2}, \quad h_R = 2b_R(z)dzd\bar{z};$$

then b_R is a hermitian metric on $D^{\star}_R(0)$ and it is called the *Poincaré-Bergman metric* on $D^{\star}_R(0)$.

After a calculation we have

$$K_{h_R} \equiv -1$$

For the case of $D_r^R(0) := \{z \in \mathbb{C} | 0 \le r < |z| < R \le 1\}$. The Poincaré Bergman metric can be given by

$$a_r^R(Z) = \frac{2}{(|z| - r)^2 (\log|z/R|^2)^2}, h_r^R = 2a_r^R dz d\bar{z}.$$

After a long but straightforward calculation it is possible to see that given $0 < \epsilon < 1$, and z_0 with $0 < |z_0| < 1$, there exists $0 < r(\epsilon)$ such that if $0 < r < r(\epsilon)$, then

$$-1 < K_{h_r^{1-r}}(z_0) < -1 + \epsilon.$$

Now we will see that the Poincaré-Bergman metric is extremal with respect to any hermitian metric in the punctured disc (for the complete disc case, see [7], p. 40).

Lemma 6. *Let $h = 2b(z) dz d\bar{z}$ be a hermitian pseudometric on the punctured disc $D_1^\star(0)$. Assume that $K_h \le -1$; then $h \le h_1$.*

Proof. As b is defined on a neighbourhood of the closure of $D_r^{1-r}(0)$ if $0 < r < 1$, then

$$\mu(z) = \log \frac{b(z)}{a_r^{1-r}(z)},$$

so we have $\mu(z) \longrightarrow -\infty$ if $z \to \partial D_r^{1-r}(0)$ then there is a point $z_0 \in D_r^{1-r}(0)$ such that

$$\mu(z_0) = \sup\{\mu(z) : z \in D_r^{1-r}(0)\} > -\infty.$$

Hence $b(z_0)$ and b are C^∞ around z_0. By the hypothesis and the definition of the Gaussian curvature, we get

$$\begin{aligned} 0 \ge \frac{\partial^2 \mu}{\partial z \partial \bar{z}}(z_0) &= \frac{\partial^2 \log b}{\partial z \partial \bar{z}}(z_0) - \frac{\partial^2 \log a_r^{1-r}}{\partial z \partial \bar{z}}(z_0) \\ &= -b(z_0) K_h(z_0) + K_{h_r^{1-r}}(z_0) a_r^{1-r}(z_0) \ge b(z_0) + K_{h_r^{1-r}}(z_0) a_r(z_0) \end{aligned}$$

and we can take r sufficiently small to assure that $-1 < K_{h_r^{1-r}}(z_0) < 0$, $z_0 \in D_r^{1-r}(0)$. Thus

$$-K_{h_r^{1-r}}(z_0) a_r^{1-r}(z_0) \ge b(z_0), \quad \log \frac{b(z_0)}{-K_{h_r^{1-r}}(z_0)}\ a_r^{1-r}(z_0) \le 0,$$

and

$$\mu(z_0) = \log \frac{b(z_0)}{a_r^{1-r}(z_0)} \le 0$$

as well. Therefore $\mu(z) \le 0$ on $D_r^{1-r}(0)$, so

$$b(z) \le a_r^{1-r}(z) \text{ on } D_r^{1-r}(0).$$

Now consider a sequence $r_n \to 0$. Hence we have that on each $D_{r_n}^{1-r_n}$

$$b(z) \le a_{r_n}^{1-r_n}(z);$$

thus in the limit

$$b(z) \le a_0^1(z) = b_1(z) \quad \forall z \in D_1^\star(0).$$

Length of curves. Let C be a curve in X and $\Gamma = \Sigma_{j\in I}\Gamma_j$, a bordered holomorphic chain in X containing C. Consider all the bordered holomorphic chains $\Gamma'_C = \Sigma_{j\in I}\Gamma'_j$ such that $\Gamma'_j \subset \Gamma_j$, $C \subset \Gamma'_C$ and the length $|\gamma'_C|$ of the border γ'_C of Γ'_C is uniformly bounded in Γ, so for any open cover $\mathcal{U}$ of X, the corresponding family of pluriharmonic functions $\mathcal{F}[\mathcal{U}]$, and $\alpha \in \mathbb{R}$ we can associate with each $u \in \mathcal{F}[\mathcal{U}]$ the expression

$$\mu^\alpha_\Gamma(C)(u) = \Sigma_{j\in I} \inf_{\Gamma^1_j \subset \Gamma_j} \mu^\alpha_{\Gamma^1_j}(u).$$

In the same way we define the expressions

$$\rho^\alpha_X(C)[u,\mathcal{U}] = \inf_X \{\mu^\alpha_\Gamma(C)(u) : C \subset \Gamma\}$$

and

$$\rho^\alpha_X(C)[\mathcal{U}] = \sup\{\rho^\alpha_X(C)[u;\mathcal{U}], u \in \mathcal{F}[\mathcal{U}]\}.$$

We will say that C is *rectifiable* with respect the $(\alpha - \mathcal{U})$ -Dolbeault - Ławrynowicz pseudodistance if $\rho^\alpha_X(C)[\mathcal{U}] < \infty$ and in that case we will say that the length of C, length $(C)(\alpha,\mathcal{U}) := \rho^\alpha_X(C)[\mathcal{U}]$.

Consider in particular the case when X is the punctured disc $D^\star_1(0)$ and C is the circle $\{|z| = r < 1\}$. If Γ_R is the elementary chain in X containing C given by $\{0 < |z| < R\}$ with $r < R < 1$, then given a finite open covering $\mathcal{U}$ of X there exists a number $A_{\Gamma_R} > 0$ such that on Γ_R for any pluriharmonic element $u \in \mathcal{F}[\mathcal{U}]$ we have [see proof of Lemma 4 of [5]]

$$\hat{g}^\alpha |(\partial/\partial z)V|^2 < A_{\Gamma_R}.$$

Thus if $\Gamma'_{R'} = \{0 < |z| < R'\}$, $r < R' < R$, with border $\gamma'_{R'} = \{|z| = R'\}$, we have

$$\frac{|\gamma'_{R'}|}{|\Gamma'_{R'}|} \int_{\Gamma'_{R'}} \hat{g}^\alpha |(\partial/\partial z)v|^2 dv \wedge dv \le \frac{|\gamma'_{R'}|}{|\Gamma'_{R'}|} A_{\Gamma_R} |\Gamma'_{R'}| = A_{\Gamma_R} |\gamma'_{R'}|$$

where $|\gamma'_{R'}|$ are the lengths and the volume of $\gamma'_{R'}$ and $\Gamma'_{R'}$ in the original hermitian metric h. Thus

$$\mu^\alpha_\Gamma(C_R)[u] \le A_{\Gamma_R} |\gamma'_{R'}|$$

and by definition of ρ^α_X we have that

$$\rho^\alpha_{D^\star_1(0)}(C_r)[\mathcal{U}] \le A_{\Gamma_R} |\gamma'_{R'}|.$$

By the preceding Lemma 6 we have, that the Poincaré-Bergman metric is extremal with respect to any hermitian metric in $D^\star_1(0)$; thus

$$\rho^\alpha_{D^\star_1(0)}(C_r)[\mathcal{U}] \le A_{\Gamma_R} |\gamma'_{R'}| \le A_{\Gamma_R} ||\gamma'_{R'}||,$$

where $||\gamma'_{R'}||$ denotes the length of $\gamma'_{R'}$ in the Poincaré-Bergman metric. Thus if $R \to 0$, $||\gamma'_{R'}|| \to 0$ and $\rho^\alpha_{D^\star_1(0)}(C_r)[\mathcal{U}]$ converges to zero too.

With the aid of Lemma 6 and with a similar proof to that of Theorem VI,3.1 in [7] we have:

THEOREM 6. *Let Δ^* denote the punctured unit disc and Y be an $(\alpha,\mathcal{U})$-hyperbolic-like manifold. Let further $f : \Delta^* \to Y$ be a holomorphic mapping such that, for a suitable sequence of points $z_k \in \Delta^*_k$ converging to the origin, $f(z_k)$ converges to a point $s_0 \in Y$. Suppose the Dolbeault-Lawrynowicz distances considered on $\Delta^\star$ and Y and the map f satisfy the conditions of Proposition* 1. *Then f extends to a holomorphic mapping of the unit disc Δ into Y.*

Obviously, we can complete Theorem 6 by

COROLLARY 10. *Let Y be a complex manifold and $M \subset Y$ an open subset such that*:

1) cl$M$ *is compact*,

2) *M is an $(\alpha,\mathcal{U})$-hyperbolic-like manifold*,

3) *f is a map such that the Dolbeault-Lawrynowicz distances on $\Delta^\star$ and Y, and f satisfy the conditions of Proposition* 1.

Then every holomorphic mapping f satisfying 3) of a punctured disc $\Delta^*(0;r)$ into M satisfies one of the following two conditions:

a) *f can be extended to a holomorphic mapping of $\Delta(0;r)$ into M*,

b) *For every neighbourhood U of the boundary $\partial M =$ cl$M\backslash M$ of M in* cl$M$ *there exists a neighbourhood W of the origin in $\Delta^*(0;r)$ such that $f(W\backslash\{0\}) \subset U$.*

The following result gives a solution to the problem of generalising the big Picard theorem to the case of an $(\alpha,\mathcal{U})-$hyperbolic-like manifold. The proofs of Theorems 7 and 9 are similar to that of Theorems 6.1 and 6.2 in [7] and it is just necessary to be careful with some details.

THEOREM 7. *Suppose that Y is a complex manifold and $M \subset Y$ an open subset such that*:

1) *M is an $(\alpha,\mathcal{U})$-hyperbolic-like manifold*,

2) cl$M$ *is compact in Y*,

3) *Given a point p in clM and a neighbourhood W of p in Y, there exists a neighbourhood V of p in Y such that* cl$V \subset W$ *and the $(\alpha,\mathcal{U})-$Dolbeault-Lawrynowicz distance between $M \cap (Y\backslash W)$ and $M \cap V$ is positive*,

4) *f is an holomorphic mapping from $\Delta^\star(0;r)$ into Y such that the Dolbeault-Lawrynowicz distances on $\Delta^\star$ and Y, and f satisfy the conditions of Proposition* 1.

Then f can be extended to a holomorphic mapping from the disc $\Delta(0;r)$ into Y.

Proof. Let $(x_m), 0 < x_m < 1$, be a sequence converging monotonously to zero. Then we can assume (by restriction if necessary) that $f(x_m)$ converges to a point $y_0 \in$ cl$M$. If $y_0 \in M$, we can apply the same argument as in the proof of Theorem 6 and the result follows. If $y_0 \in$ cl$M\backslash M$, let W be an arbitrary neighbourhood of Y_0 in Y. The result follows if we show that there exists a number $\delta > 0$ such that if $\Delta^*_\delta = \{z \in \Delta^*(0;r) : |z| < \delta\}$, then $f(\Delta^*_\delta) \subset W$. We claim that there is a number $\delta > 0$ such that if $0 < x_k < \delta$, then $f(\partial\Delta(0;x_k)) \subset W$.

In order to prove our claim, suppose the contrary case. By taking, if necessary, a subsequence, we may assume that each circle $\partial\Delta(0;x_k)$ has a point z_k such that $f(z_k) \notin W$. Taking again, if necessary, a subsequence, we may assume that $f(z_k)$ converges to $w_0 \in$ cl$M$. If $w_0 \in M$, since the $(\alpha,\mathcal{U})-$Dolbeault-Lawrynowicz length of $\partial\Delta(0;x_m)$ converges to zero and, by Proposition 1, all but a finite number of the curves $f(\partial\Delta(0;x_k))$ are in W, so the claim follows. Otherwise we can proceed as in the proof of Theorem 6. If $w_0 \in$ cl$M\backslash M$, we note that $w_0 \notin W$ and if $V \subset\subset W$ is also a neighbourhood of y_0, then $w_0 \notin$ cl$V$ and $Y\backslash V$ is an open subset of Y which contains w_0. Now, by the condition 3) of the statement, there exists a neighbourhood B of w_0 such that cl$B \cap$ cl$V = \emptyset$ and the $(\alpha,\mathcal{U})$- Dolbeault-Lawrynowicz distance δ between $M \cap V$ and $M \cap B$ is positive, so $\rho^\alpha_M(f(x_k),\ f(z_k))[\mathcal{U}] \geq \delta > 0$ for sufficiently large k. On the other hand, as we saw, the arc length L of $\partial\Delta(0;r)$ measured in the Dolbeault-Ławrynowicz distance tends to 0 as $k \to \infty$; we have $\rho_M^\alpha(f(x_k),\ f(z_k)) \le L(f[\partial\Delta(0;x_k)]) \le L[\partial\Delta(0;x_k)]$, and this contradiction proves the claim.

Consider next the set of integers k such that $f(\Delta(0;x_k)\backslash \mathrm{cl}\Delta(0;x_{k+1}))$ is not entirely contained in W. If this set is finite, then f maps a small punctured disc $\Delta^*(0;\delta)$ into W. If the set is infinite, there is a sequence of points a_k situated in $\Delta(0;x_k)\backslash \mathrm{cl}(0;x_{k+1})$ such that $f(a_k) \notin W$. Now we can proceed as in the proof of the claim to conclude that there exists a positive number δ^* such that $\rho_M^\alpha(f(a_k),\ f(x_k))[\mathcal{U}] \ge \delta^*$ for sufficiently large k. Since we have still $\rho_M^\alpha(f(a_k), f(x_k))[\mathcal{U}] \le L[\partial\Delta(0;x_k)]$, we arrive again at a contradiction.

In turn we quote Theorem 3 of [5] which reads:

THEOREM 8. *Let X be a complex manifold and A its subset which is nowhere dense in an analytic subset*, say B, of X, *with topological codimension* ≥ 2. *Let further Y be a compact or complete $(\alpha,\mathcal{U})$-hyperbolic-like manifold. Suppose that X is the completion of $X\backslash A$ with respect to the Dolbeault-Lawrynowicz distance considered on $X\backslash A$ and that f is a map such that the triple $X\backslash A$, f, Y satisfies conditions of Proposition* 1.

Then f can be extended to a holomorphic mapping from X into Y.

With this theorem it is possible to extend Theorem 7 as follows:

THEOREM 9. *Suppose that M and Y satisfy the conditions* 1)-4) *of Theorem* 7. *Let X be a complex manifold of dimension n and A a locally closed complex submanifold of X of dimension $\le n-1$. Suppose further that X is the completion of $X\backslash A$ with respect to the Dolbeault-Ławrynowicz distance defined on $X\backslash A$. Then f can be extended to a holomorphic mapping from X into Y.*

Proof. Since A has no singular points we may assume, as in the proof of Theorem 8, that $X\backslash A = \Delta^* \times \Delta^{n-1}$. If f is a holomorphic mapping of $\Delta^* \times \Delta^{n-1}$ into M, then, by Theorem 7, the restriction of f to $\Delta^* \times \{t\}$ can be extended to a holomorphic mapping from $\Delta \times \{t\}$ into Y for every fixed $t \in \Delta^{n-1}$. It remains to prove that this extended mapping $f : \Delta^* \to Y$ is continuous at every point $(0,t) \in A$. It suffices to show that f is continuous at $(0,0) \in \Delta \times \Delta^{n-1}$. Let $y_0 = f(0,0) \in Y$. If $y_0 \in M$, then we can proceed exactly as in the proof of Theorem 8. Suppose therefore that $y_0 \in \mathrm{cl}M\backslash M$ and let W be a neighbourhood of y_0 in Y defined by $|w^j| < a$, $j = 1,\dots,m$, with respect to the local co-ordinates around y_0. Let V be a neighbourhood of y_0 given by the condition 3) of the statement and let $\beta = \rho_M^\alpha(M \cap (Y\backslash W), M \cap V)[\mathcal{U}]$. Next, let r be a positive number such that $\Delta(0;r) \times \{0\}$ is mapped by f into V, which is possible by Theorem 7. Let r' be a positive number such that $\rho_{\Delta^{m-1}}^\alpha(0,t)[\mathcal{U}] < \beta$ if $|t^j| < r$ for $j = 1,\dots,n-1$.

If $0 < |z| < r$ and $|t^j| < r$ for $j = 1,\dots,n-1$, then, by Proposition 1 applied to the mappings: (i) $f : \Delta^* \times \Delta^{n-1} \to M$ and (ii) the injection $\Delta^{n-1} \to \Delta^* \times \Delta^{n-1}$ given by $t \longmapsto (z,t)$, we get

$$\rho_M^\alpha(f(z,0), f(z,t))[\mathcal{U}] \le \rho_{\Delta^* \times \Delta^{n-1}}^\alpha((z,0),(z,t))[\mathcal{U}] \le \rho_{\Delta^{n-1}}^\alpha(0,t) < \beta$$

Since $f(z,0)$ is in V and the $(\alpha,\mathcal{U})$-Dolbeault-Ławrynowicz distance between $f(z,0)$ and $f(z,t)$ is less than β, it follows that $f(z,t)$ is in W. Therefore, by the Riemann extension theorem, f is holomorphic in $\Delta(0;r) \times \{t \in \Delta^{n-1} : |t^j| < r', j = 1, \ldots, n\}$.

With the aid of the next proposition (see [1]) we can generalise Theorem 8 to the case of a covering space (W, π) of Y.

PROPOSITION 7. *Let S be a relatively closed subset of some analytic set M of a polycylinder P in $\mathbb{C}^n$. We assume that S has topological dimension $2n - (3 + s)$ and the complex dimension of M is $n - (1 + k)$, where $0 \le k \le \frac{1}{2}s$. Then the open set $P \backslash S$ is arcwise and simply connected.*

THEOREM 10. *Let $\tau : A \to Y$ be a holomorphic mapping of an open subset A of a complex manifold X into a complex space Y. We assume that Y has a covering space (W, π) such that W is a compact or complete $(\alpha, \mathcal{U})$-hyperbolic-like manifold. Suppose that $M = X \backslash A$ is a thin set and that τ, $X \backslash M$, Y, satifies the hypothesis of Proposition* 1. *Then, if M has topological codimension ≥ 3, τ can be extended to a holomorphic mapping $\tau^* : X \to Y$ with $\tau^*|A = \tau$.*

Proof. Since the theorem is of a local nature for X, we may assume, in analogy to Theorem 9, that X is a polycylinder in $\mathbb{C}^n$. Let S be the set of irregular points of τ, that is, the set of points $x \in M$ which have the property that there is no neighbourhood V of x such that τ can be holomorphically extended to $A \cup V$. Let τ^0 be the extension of τ to $X \backslash S$. Note that τ^0 is distance decreasing too. We are going to prove that $S = \emptyset$. By Proposition 7, $X \backslash S$ is simply connected. Since τ^0 can be factorized through any covering space (W, π) of Y, there exists a holomorphic mapping $\tilde{\tau}_0 : X \backslash S \to W$ such that $\tau_0 = \pi \circ \tilde{\tau}_0$ and $\tilde{\tau}_0$ is univocally determined. Owing to Theorem 8, $\tilde{\tau}_0$ can be extended to a holomorphic mapping $\tilde{\tau}^* : X \to W$, so $\tau^* = \pi \circ \tilde{\tau}^*$ is the desired extension of τ to X.

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