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THE SINGULARITY STRUCTURE OF THE YANG–MILLS CONFIGURATION SPACE

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Abstract. The geometric description of Yang–Mills theories and their configuration space $\mathcal{M}$ is reviewed. The presence of singularities in $\mathcal{M}$ is explained and some of their properties are described. The singularity structure is analysed in detail for structure group SU(2). This review is based on [28].

1. Yang–Mills theory and geometry. Even more than forty years after the seminal paper of Yang and Mills on 'isotopic gauge invariance' [48], a complete characterization of those Yang–Mills theories that are relevant to particle physics is still lacking. Some uncertainties are in fact already present at the classical level; they include e.g. the specific choice of space-time manifold and the proper treatment of 'constant' gauge transformations.

I will discuss some of these subtleties in Section 8. For the moment, however, let me stick to one specific formulation of Yang–Mills theory, which is based on the geometric picture of gauge symmetries. In short, from a geometric point of view Yang–Mills theory is essentially the *theory of connections on some principal fibre bundle*. I will describe this approach in some detail, and then proceed to investigate its consequences for the structure of the configuration space.

Let me start by recalling that a principal fibre bundle $\mathcal{P}$ is a fibre bundle with total space P and base space M, together with the *structure group* G, a Lie group of dimension $\dim G = \dim P - \dim M$, which satisfies the following properties. There is an action $\mu : P \times G \to P$, $\mu(p,\gamma) \equiv p\gamma$, of G on P which is smooth and free and is transitive on the fibres $G_x \equiv \pi^{-1}(\{x\}) \cong G$, $x \in M$, and the representation of G on the fibres is isomorphic to the representation of G on itself that is given by right multiplication. An *automorphism* of a principal bundle $\mathcal{P}$ consists of a diffeomorphism $f : P \to P$ and an

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automorphism $\sigma : G \to G$ such that $\mu \circ (f \times \sigma) = f \circ \mu$; these induce a diffeomorphism f_π of the base space M. An automorphism of $\mathcal{P}$ is said to be *vertical*, or to *cover the identity* iff $\sigma = \mathrm{id}_G$ and $f_\pi = \mathrm{id}_M$, i.e. iff the diagram

$$\begin{array}{ccc} P \times G & \stackrel{f\times\mathrm{id}}{\longrightarrow} & P \times G \\ {\scriptstyle\mu} \downarrow & & \downarrow {\scriptstyle\mu} \\ P & \stackrel{f}{\longrightarrow} & P \\ {\scriptstyle\pi} \downarrow & & \downarrow {\scriptstyle\pi} \\ M & \stackrel{\mathrm{id}}{\longrightarrow} & M \end{array} \tag{1}$$

commutes. The vertical automorphisms of $\mathcal{P}$ form a group, called the *gauge group* $\mathcal{G} \equiv \mathcal{G}_\mathcal{P}$ of $\mathcal{P}$.

An equivalent definition of $\mathcal{G}$, which is tailored to the application in Yang–Mills theories, is in terms of the *adjoint bundle* $\mathcal{P}_{\mathrm{Ad}}$. $\mathcal{P}_{\mathrm{Ad}}$ is defined as the bundle over M associated to $\mathcal{P}$ that has total space $P \times_G G$ (the elements of $P \times_G G$ are classes $[p,\gamma] := \{(p\tilde\gamma, \tilde\gamma^{-1}\gamma\tilde\gamma \mid \tilde\gamma \in G\})$ with the projection defined as $\pi_{\mathrm{Ad}}([p,\gamma]) := \pi(p)$. The gauge group $\mathcal{G}$ is then the set of sections s of $\mathcal{P}_{\mathrm{Ad}}$, with composition law $s \cdot s'(x) = [p, \gamma\gamma']$ for $s(x) = [p,\gamma]$ and $s'(x) = [p,\gamma']$, for all $x \in M$. In this description, the Lie algebra $\mathcal{L}$ of $\mathcal{G}$ is the space of sections of an analogous vector bundle $\mathcal{P}_{\mathrm{ad}}$ with total space $P \times_G L$, where L is the Lie algebra of G. According to this definition the elements of $\mathcal{G}$ can be interpreted locally as G-valued smooth functions. If the bundle $\mathcal{P}$ is trivial (i.e. $P \cong M \times G$), then this interpretation is valid globally. It must be stressed, however, that even in the latter situation there does not exist any canonical embedding of the structure group G into the gauge group $\mathcal{G}$.

A natural additional structure on a principal fibre bundle is provided by the notion of a *connection*, which allows for the definition of parallel transport and of covariant derivatives of sections in associated vector bundles. In physical terms, these are important ingredients of the dynamics; first, the connection provides the basic dynamical variables, the gauge fields, and second, the covariant derivatives allow for the consistent coupling of matter to the gauge fields.

Geometrically, a connection A consists in a particular decomposition of the tangential space T_pP of P (at any point $p \in P$) into a direct sum $H_p \oplus V_p$ of orthogonal 'horizontal' and 'vertical' spaces. Algebraically, A is a differential one-form over P with values in L which under the action of G on P transforms according to the adjoint representation of G, such that the horizontal space H_p is the kernel of A_p. The most convenient characterization of A for the application to Yang–Mills theory is as a set $\{A_\mathcal{U}\}$ of L-valued one-forms over $\mathcal{U}$, indexed by an atlas $\{\mathcal{U}\}$ of P with appropriate transition functions. These give rise locally to one-forms over $\mathcal{U}_M$, with $\{\mathcal{U}_M\}$ an atlas of the base space M, and hence in the case of trivial bundles to a one-form over M. The space of all connections on the bundle $\mathcal{P}$ will be denoted by $\mathcal{A}$. The space $\mathcal{A}$ (together with an action of the vector space $\bigwedge^1 \otimes L$ of equivariant one-forms over P with values in L) is an affine space; after the choice of a base point $\bar A$, $\mathcal{A}$ can be interpreted as a (real, infinite-dimensional) vector space according to $\mathcal{A} = \{\bar A + a \mid a \in \bigwedge^1 \otimes L\}$.

Any gauge transformation $g \in \mathcal{G}$ can be pulled back from $\mathcal{P}$ to $\mathcal{A}$. The pull-back g_*

acts locally as

$$g_*(A) = A^g := g^{-1}Ag + g^{-1}\mathrm{d}g. \tag{2}$$

On the vector space associated to $\mathcal{A}$, $\mathcal{G}$ acts homogeneously, $\bigwedge^1 \otimes L \ni a \mapsto g^{-1}ag$.

2. The configuration space $\mathcal{M}$. Having introduced the geometric concepts above, I can now present the definition of Yang–Mills theory, or more precisely, of its 'kinematical part'. This is described by the space $\mathcal{A}$ of connections on a principal fibre bundle $\mathcal{P}$ and the action (2) of the gauge group $\mathcal{G}$ on $\mathcal{A}$, where the structure group G is required to be a simply connected (semi-)simple compact matrix Lie group, and the base space M to be a 'space-time'. The latter term just refers to the requirement that M should be of potential interest to applications in physics, but otherwise is by no means precise. In the following I take $M = S^4$, a choice to be justified later on. The *configuration space* of the Yang–Mills theory is the space

$$\mathcal{M} := \mathcal{A}/\mathcal{G} \tag{3}$$

of connections modulo gauge transformations, or more explicitly, the *orbit space* $\mathcal{M} = \{\mathcal{O}_A \mid A \in \mathcal{A}\}$, whose points are the *gauge orbits* $\mathcal{O}_A := \{B \in \mathcal{A} \mid B = A^g \text{ for some } g \in \mathcal{G}\}$.

For $M = S^4$, the isomorphism classes of principal G bundles $\mathcal{P}$ are labelled by an integer k, the instanton number or Pontryagin class of $\mathcal{P}$(1). For the geometric description of Yang–Mills theory one must choose(2) a representative of a fixed isomorphism class, and hence fix the instanton number to a definite value k. Actually, for many of the aspects I will mention below, the relevant value will be $k = 0$, so that $\mathcal{P}$ is a trivial bundle. It should be noted that for any fixed k there is a separate space $\mathcal{A}_{(k)}$ of connections, each being acted on by a separate gauge group $\mathcal{G}_{(k)}$, and hence a separate configuration space $\mathcal{M}_{(k)} = \mathcal{A}_{(k)}/\mathcal{G}_{(k)}$. For some applications in physics it is actually necessary to allow for arbitrary $k \in \mathbb{Z}$; doing so, one deals with a 'total configuration space' which is the disjoint union over all $\mathcal{M}_{(k)}$,

$$\mathcal{M}_{\mathrm{tot}} = \dot{\bigcup_{k\in\mathbb{Z}}} \mathcal{A}_{(k)}/\mathcal{G}_{(k)}. \tag{4}$$

From now on, I will always refer to a definite bundle at fixed instanton number, and correspondingly will suppress the label k.

Let me remark that the *infinite-dimensional* geometry of the spaces $\mathcal{A}$ and $\mathcal{M}$ is conceptually rather different from the finite-dimensional geometry of a specific gauge field configuration, i.e. of a single fixed connection (it is often the latter which is referred to as 'the geometry of Yang–Mills theory'). However, under favourable circumstances the local geometry of $\mathcal{M}$ is closely related to global properties of the base manifold. For instance, if one restricts to the *finite-dimensional* solution set of suitable differential equations, for $G = \mathrm{SU}(2)$ and M a compact four-manifold there are the relations described by Donaldson theory [22, 23] and Seiberg–Witten theory [24, 47].

(1) Similarly, for $M = S^2 \times \mathbb{R}$ and structure group U(1), the isomorphism classes of principal bundles are again labelled by an integer, the monopole number.

(2) There may actually be subtleties to this choice, as the family of isomorphic bundles need not be a set.

To complete the definition on Yang–Mills theory, I still have to prescribe its 'dynamical part'. This is done by specifying an action functional $S_{\mathrm{YM}} : \mathcal{A} \to \mathbb{R}$. For definiteness, this will be taken to be the ordinary Yang–Mills action

$$S_{\mathrm{YM}}[A] = \frac{1}{4e^2}\,|F|^2, \tag{5}$$

where e is a coupling constant, $F = \mathrm{d}A + \frac{1}{2}\,[A, A]$ is the curvature of the connection A, and $|\cdot|$ denotes the L^2 norm on equivariant p-forms on M with values in L that is induced by the L^2 scalar product

$$(B, C) := \int_M \mathrm{tr}(B \wedge *C). \tag{6}$$

(Thus in local coordinates, $4e^2\, S_{\mathrm{YM}}[A] = \int_M \mathrm{d}^4x\, \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ with $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]$ and $A = \sum_{\mu=1}^{4} A_\mu(x)\mathrm{d}x^\mu$.)

In the analysis below the specific form of the action functional will not play any particular rôle. Rather, the only really relevant property of S_{YM} is that it is gauge invariant in the sense that

$$S_{\mathrm{YM}}[A^g] = S_{\mathrm{YM}}[A] \tag{7}$$

for all $g \in \mathcal{G}$. For most of the considerations below, one could therefore also have in mind any other action functional sharing this crucial property, such as in three dimensions the Chern–Simons action or [7] the combination of Chern–Simons and Yang–Mills actions. The gauge invariance (7) implies that the action is in fact a well-defined functional $S_{\mathrm{YM}} : \mathcal{M} \to \mathbb{R}$ on the configuration space $\mathcal{M}$ rather than just on the 'pre-configuration space' $\mathcal{A}$.

3. Gauge fixing. While $\mathcal{A}$ is an affine space and hence easy to handle (e.g. it is contractible), the structure of the configuration space $\mathcal{M}$ is much more complicated. For instance, $\mathcal{M}$ is not a manifold (see below), and even when restricting to manifold points, at least one homotopy group is non-trivial. It is therefore most desirable to describe $\mathcal{M}$ as concretely as possible in terms of the simple space $\mathcal{A}$. (Often it is in fact advantageous to allow in intermediate steps for quantities which are not well-defined on $\mathcal{M}$, e.g. this can help to formulate the theory in a 'less non-linear' way. This option is in fact one of the basic reasons for dealing with gauge theories in the redundant description in terms of connections. In the present context, one should avoid the use of gauge-noninvariant objects whenever possible.)

One of the key ideas in approaching the space $\mathcal{M}$ via $\mathcal{A}$ is to identify a subset of $\mathcal{A}$ that is isomorphic to $\mathcal{M}$ modulo boundary identifications. Such a subset is called a *fundamental modular domain* and will generically be denoted by Λ. A prescription for determining a modular domain Λ (referred to as *fixing a gauge*) consists in picking in a continuous manner a representative out of each gauge orbit $\mathcal{O}_A$. Technically, one tries to achieve this by considering the set $\{A \in \mathcal{A} \mid \mathcal{C}[A] = 0\}$ for a suitable functional $\mathcal{C}$ on $\mathcal{A}$. This set is called a *gauge slice* associated to the gauge condition $\mathcal{C}$ and will be denoted by Γ. In a geometrical context the natural gauge condition is the *background gauge*

$$\mathcal{C}[A] \equiv \mathcal{C}_{\bar{A}}[A] := \nabla^*_{\bar{A}}(A - \bar{A}), \tag{8}$$

where $\bar{A}$, called the background connection, is an arbitrary element of $\mathcal{A}$. (In the special case of $k = 0$ and background $\bar{A} = 0$ in which (8) reduces to $\mathcal{C}[A] = \mathrm{d}^*A$, this is also known as the Lorentz, or Landau gauge, or [46] as the Hodge gauge.) Thus the gauge slice is the subset

$$\Gamma \equiv \Gamma_{\bar{A}} = \{A \in \mathcal{A} \mid \nabla^*_{\bar{A}}(A - \bar{A}) = 0\} \tag{9}$$

of $\mathcal{A}$. As already indicated by the use of a different symbol Γ in place of Λ, (8) does in fact not provide a complete gauge fixing. This failure is not an artifact of the background gauge, but (at least for $M = S^4$, and presumably for any compact M) happens for any continuous gauge condition $\mathcal{C}$ [44]. (There do exist continuous gauge conditions which avoid this problem on a large subset of $\mathcal{A}$, though not globally on $\mathcal{A}$, such as the axial gauge on $M = \mathbb{R}^4$ [19] and axial-like gauges on tori [37].)

Thus for any $\bar{A} \in \mathcal{A}$ there exist orbits $\mathcal{O}$ containing distinct connections A and B such that both A and B lie in $\Gamma_{\bar{A}}$; A and B are then called *gauge copies* or *Gribov copies* [31] of each other. Gauge copies appear at least outside a subset $\Omega \equiv \Omega_{\bar{A}}$ of Γ (that is, any $A \in \Gamma \setminus \Omega$ has a gauge copy within Ω) which can be described as the set of those connections for which $g = e$ (the unit element of $\mathcal{G}$) is a minimum of the functional

$$\Phi_A \equiv \Phi_{\bar{A};A}[g] := |A^g - \bar{A}|^2 \tag{10}$$

on $\mathcal{G}$. The functionals Φ_A also contain the (unphysical) information about the topology of the orbit $\mathcal{O}_A$ and correspondingly are sometimes referred to as Morse functionals. The subset Ω is known as the *Gribov region*. Any orbit $\mathcal{O}$ intersects Ω at least once, and Ω is convex and bounded. The rôle of the Morse functionals Φ_A is best understood by observing that

$$\begin{aligned} \left.\frac{\delta \Phi_A}{\delta g}\right|_{g=e} &= -2\,\nabla^*_{A}(A - \bar{A}) = -2\,\nabla^*_{\bar{A}}(A - \bar{A}), \\ \left.\frac{\delta^2 \Phi_A}{\delta g^2}\right|_{g=e} &= 2(\nabla_{\bar{A}} w, \nabla_A w) = -2(w, \nabla^*_{\bar{A}} \nabla_A w). \end{aligned} \tag{11}$$

Thus the vanishing of the first variation of Φ_A yields the gauge condition, so that for all $A \in \Gamma$, $e \in \mathcal{G}$ is a stationary point of Φ_A, while the Hessian of the variation is given by the Faddeev–Popov operator $\Delta_{\mathrm{FP}} = -\nabla^*_{\bar{A}} \nabla_A$, implying that $A \in \Omega$ for all $A \in \Gamma$ for which this operator is positive.

The Gribov region is not yet the modular domain Λ, as is indicated by the choice of the different symbol Ω. However for *generic* background $\bar{A}$ it already comes rather close to Λ. Namely, in the definition of Ω both absolute and relative minima of the Morse functionals Φ_A contribute, and Λ is obtained by just restricting to the absolute minima,

$$\Lambda \equiv \Lambda_{\bar{A}} = \{A \in \Gamma \mid \Phi_A[g] \geq \Phi_A[e] \text{ for all } g \in \mathcal{G}\} \subseteq \Omega_{\bar{A}}. \tag{12}$$

The set (12) contains at least one representative of each gauge orbit, provided that one considers the gauge group $\mathcal{G}$ as completed in the L^2 norm. (It is not known whether this remains true when completing with respect to an arbitrary Sobolev norm.) Also, the interior of $\Lambda_{\bar{A}}$ contains (for generic $\bar{A}$) at most one representative, i.e. gauge copies only occur on the boundary $\partial\Lambda_{\bar{A}}$. Thus for generic background, $\Lambda_{\bar{A}}$ is a fundamental modular domain. Furthermore, just as the Gribov region Ω, its subset Λ is convex and bounded, and (for any compact space-time with $H^2(M, \mathbb{Z}) = 0$) it is properly contained in Ω.

The action of the gauge group $\mathcal{G}$ on $\mathcal{A}$ is not free, i.e. the *stabilizer* (or isotropy subgroup) $\mathcal{S}_A := \{g \in \mathcal{G} \mid A^g = A\}$ of a connection A may be non-trivial. Indeed, for all $A \in \mathcal{A}$ the stabilizer contains the group $\mathcal{Z}$ of constant gauge transformations with values in the centre Z of G. In the statements made above, the qualification of the background $\bar{A}$ as 'generic' means that $\mathcal{S}_{\bar{A}} \cong \mathcal{Z}$. Connections with this property are also referred to as *irreducible*, while connections A for which $\mathcal{S}_A$ contains $\mathcal{Z}$ as a proper subset are called *reducible*.

Some properties of the stabilizer groups are the following. Any stabilizer $\mathcal{S}_A$ is isomorphic to the centralizer $\mathcal{C}(H_A(p))$ of the holonomy group of the connection A, and accordingly is isomorphic to a closed Lie subgroup of the structure group G, and hence in particular finite-dimensional. Within any fixed orbit $\mathcal{O}$, all stabilizers $\mathcal{S}_A$, $A \in \mathcal{O}$, are conjugate subgroups of $\mathcal{G}$, i.e. for any $A, B \in \mathcal{O}$ there exists an element $g \in \mathcal{G}$ such that $\mathcal{S}_B = g^{-1}\mathcal{S}_A g$. For any space-time M and any compact G, the set of all such conjugacy classes is countable [35]. Finally, $\mathcal{S}_A \cong G$ iff $\bar{A}$ is a pure gauge, i.e. iff $A = g^{-1}\mathrm{d}g$.

The gauge invariance of the L^2 norm $|\cdot|$ implies that $\Phi_A[gh] = |A^{gh} - \bar{A}|^2 = |A^{gh} - \bar{A}^h|^2 = |(A^g - \bar{A})^h|^2 = \Phi_A[g]$ for any $h \in \mathcal{S}_{\bar{A}}$. Because of this systematic degeneracy, in particular the absolute minima are degenerate, which means that in fact not $\Lambda_{\bar{A}}$, but rather the quotient

$$\breve{\Lambda}_{\bar{A}} := \Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}} \tag{13}$$

is a fundamental modular domain. For irreducible $\bar{A}$ this is, however, irrelevant, since then $\mathcal{S}_{\bar{A}} = \mathcal{Z}$ and $\mathcal{Z}$ acts trivially, so that $\breve{\Lambda}_{\bar{A}} = \Lambda_{\bar{A}}/\mathcal{Z} = \Lambda_{\bar{A}}$.

The distinction between reducible and irreducible backgrounds also appears in various other circumstances. For instance, if $\bar{A}$ is irreducible, then the Gribov region Ω can be described as the set $\{A \in \Gamma \mid (A, \Delta_{\mathrm{FP}} A) \geq 0\}$ in which the Faddeev–Popov operator is positive, whereas for reducible $\bar{A}$, $\det(-\nabla_{\bar{A}}^* \nabla_A)$ vanishes identically (but even then Ω can be described in terms of the Morse functionals Φ_A as above).

4. The stratification of $\mathcal{M}$. For any subgroup $\mathcal{S}$ of $\mathcal{G}$, let $[\mathcal{S}] = \{\mathcal{S}' \subseteq \mathcal{G} \mid \mathcal{S}' = g^{-1}\mathcal{S}g,\ g \in \mathcal{G}\}$ denote its conjugacy class in $\mathcal{G}$. The set of all orbits $\mathcal{O}_A \in \mathcal{M}$ which have fixed stabilizer type $[\mathcal{S}]$, i.e. whose elements $A \in \mathcal{O}_A$ have stabilizers $\mathcal{S}_A \in [\mathcal{S}]$, is a *Hilbert manifold*, i.e. an infinite-dimensional C^∞ manifold modelled on a Hilbert space. However, the full configuration space $\mathcal{M}$ is not a manifold, but rather it has singularities, which are at the orbits of reducible connections. More precisely [35], $\mathcal{M}$ is a *stratified variety*. Thus as a set $\mathcal{M}$ is the disjoint union of (countably many) smooth manifolds, the strata of $\mathcal{M}$. Any stratum with stabilizer type $[\mathcal{S}]$ is dense in the union of all strata that have stabilizers containing some $\mathcal{S}' \in [\mathcal{S}]$. In particular, the *main stratum*, i.e. the stratum of orbits of irreducible connections, is dense in $\mathcal{M}$. Also, each stratum can be described as the main stratum of another configuration space that is obtained from the space of connections on some subbundle of $\mathcal{P}$ [34].

Let me pause to point out that almost all results I described so far can be found, though widely scattered, in the literature. Some of the main references are [8–10, 12–15, 20, 21, 31, 33–36, 38, 40, 43–45, 49]. However, a few of the details I presented were found quite recently [28]; the results reported below are again based on [8–10, 12–15, 20, 21, 31, 33–36, 38, 40, 43–45, 49], but to a large extent have been obtained in [28].

I will first describe the stabilizers of the connections within the domains $\Lambda_{\bar{A}}$ in more detail. (This completes the proof of the claim that $\breve{\Lambda}_{\bar{A}} = \Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$ is a fundamental modular domain, or in other words, that the degeneracy of the functionals Φ_A is already exhausted by the systematic degeneracy associated to $\mathcal{S}_{\bar{A}}$.) Given the domain Λ, define $\tilde{\Lambda}$ as the subset of connections in Λ that only have the systematic degeneracy. The functional $\Phi_{\bar{A};\bar{A}}[g] = |\bar{A}^g - \bar{A}|^2 = |g^{-1}\nabla^*_{\bar{A}}g|^2$ attains its absolute minimum, namely 0, iff $g \in \mathcal{S}_{\bar{A}}$. Thus $\bar{A} \in \tilde{\Lambda}$, hence in particular $\breve{\Lambda}$ is not empty. Similarly, for any $B \in \Lambda \backslash \tilde{\Lambda}$ one can show that the straight line between B and $\bar{A}$ (except for the point B itself) is contained in $\breve{\Lambda}$, so that $\Lambda\backslash\tilde{\Lambda} \subset \partial\Lambda$. On the other hand, if for some $A \in \Lambda$ the stabilizer $\mathcal{S}_A$ is not contained in $\mathcal{S}_{\bar{A}}$, then there is an additional degeneracy, so that $A \in \partial\Lambda$. In particular, if for some $A \in \Lambda \backslash \partial\Lambda$, $\mathcal{O}_A$ belongs to the same stratum as $\mathcal{O}_{\bar{A}}$, then the stabilizers are not just conjugated, but in fact coincide, $\mathcal{S}_A = \mathcal{S}_{\bar{A}}$. Thus the interior of Λ contains only connections whose stabilizer is either identical or strictly contained in that of $\bar{A}$. In particular, for irreducible $\bar{A}$, all reducible connections in $\Lambda_{\bar{A}}$ lie on the boundary $\partial\Lambda_{\bar{A}}$.

5. Boundary identification and geodesic convexity. The fundamental modular domain $\Lambda_{\bar{A}}$ is isomorphic to $\mathcal{M}$ only modulo boundary identifications, which account for the topologically non-trivial features of $\mathcal{M}$. To describe some aspects of the required boundary identifications, consider the modular domain $\Lambda_{\bar{A}}$ for an irreducible background $\bar{A}$, and a point B on the boundary $\partial\Lambda_{\bar{A}}$ that is irreducible as well. Next regard B instead as an element of the modular domain Λ_B which is isomorphic to $\mathcal{M}$ modulo boundary identifications, too. B is an inner point of Λ_B and hence corresponds to a smooth inner point of $\mathcal{M}$. This implies that upon boundary identification of $\Lambda_{\bar{A}}$, a neighbourhood of B on $\partial\Lambda_{\bar{A}}$ (consisting of irreducible connections only, since reducible connections are nowhere dense) gets identified with another neighbourhood on $\partial\Lambda_{\bar{A}}$.

In contrast, if B is reducible, then upon proceeding from Λ_B to the modular domain $\Lambda_B/\mathcal{S}_B$, B becomes a singular point, and this remains true upon boundary identification. Thus when $B \in \partial\Lambda_{\bar{A}}$ for irreducible $\bar{A}$, then in the boundary identification process B must again become a singular point; as a consequence, there are 'less' boundary identifications for reducible elements of $\partial\Lambda_{\bar{A}}$ than for irreducible elements. As reducible connections cannot be boundary points of $\Lambda_{\bar{A}}|_{\text{bound. id.}}$, and hence of the configuration space $\mathcal{M}$ (the codimension of the reducible strata is infinite), it follows in particular that $\mathcal{M}$ does not possess any boundary points.

As, contrary to $\Lambda_{\bar{A}}$, the set $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}|_{\text{bound.id.}}$ is no longer a subset of an affine space, there is no notion of convexity any more. But as each of the strata is a Hilbert manifold, there is still the notion of *geodesic convexity*, meaning that any two non-singular points can be joined by a geodesic which only consists of non-singular points (in singular points geodesics cannot be defined). Now the *main* stratum of $\mathcal{M}$ is geodesically convex. (But it is not known whether the non-main strata are geodesically convex.) To see this, take $P_A, P_B \in \Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$ non-singular and let B be a representative of P_B in $\Lambda_{\bar{A}}$ and C a representative of P_C in Λ_B. As Λ_B is convex, the straight line from B to C is contained in Λ_B and (since B is irreducible) contains no reducible connection. This line gets projected to a geodesic in $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$ [15]. This is still true if C is an irreducible connection on the boundary $\partial\Lambda_B$, and hence also after boundary identification.

6. SU(2) Yang–Mills theory. In this section I specialize to $G = \mathrm{SU}(2)$ (and $M = S^4$), which allows to produce several rather concrete results. In this situation, the instanton number of any reducible connection vanishes [44], so that $\mathcal{M}$ has singularities only for $k = 0$. Correspondingly I will only consider $\mathcal{M} \equiv \mathcal{M}_{k=0}$; the connections and gauge transformations can then be described as smooth functions on M.

For structure group SU(2), there are only three possible stabilizer types, namely either $\mathcal{S}_A = \mathcal{Z} \equiv Z(\mathrm{SU}(2)) = \{\pm 1\!\!1\}$, in which case A is irreducible; or $\mathcal{S} \cong \mathrm{SU}(2)$, then A is a pure gauge; or else $\mathcal{S} \cong \mathrm{U}(1)$. An obvious task is to classify the conjugacy classes with respect to $\mathcal{G}$ which correspond to each of the three isomorphism classes of stabilizers. It turns out that in the SU(2) case there is a unique stratum for each of the three stabilizer types.

Each orbit of connections with stabilizers $\mathcal{S}_A$ of type [U(1)] contains one representative A' for which $\mathcal{S}_{A'}$ consists of constant gauge transformations with values in U(1). Namely, in any fibre of $\mathcal{P}$ one has the freedom to multiply by an element of SU(2), and the only question is whether this can be extended globally in the appropriate way. Now $g \in \mathcal{S}_A$ implies $\nabla_A g = 0$. By considering g close to the identity of $\mathcal{S}_A$, it follows that $\nabla_A \sigma = 0$ for some element σ of the Lie algebra $\mathcal{L}$ of $\mathcal{G}$, and hence $\ell := \mathrm{tr}(\sigma(x)\sigma(x)^+)$ is constant on M. SU(2) acts transitively on elements $\sigma(x) \in \mathrm{su}(2)$ that have a fixed value of ℓ, and hence one can fix a τ in su(2) with $\mathrm{tr}(\tau\tau^+) = \ell$ such that for any $x \in M$ there is an element $\gamma_x \in \mathrm{SU}(2)$ with $\sigma(x) = \gamma_x \tau \gamma_x^{-1}$. Provided that setting $\tilde{g}(x) := \gamma_x$ for all x yields a well-defined gauge transformation, it follows that τ lies in the Lie algebra $\mathcal{S}_{A'}$ of the stabilizer $\mathcal{S}_{A'}$ of $A' := A^{\tilde{g}^{-1}}$, and hence $\mathcal{S}_{A'}$ is the U(1) generated by τ, which consists of constant gauge transformations. Thus the representative whose existence I claimed is the $\tilde{g}$-transform A' of A. Now for $M = S^4$ the above prescription does provide a well-defined $\tilde{g} \in \mathcal{G}$. The only ambiguity in γ_x, and hence the only potential obstruction, is in the stabilizer U(1) of τ, and as one can smoothly continue $\tilde{g}(x)$ as long as the topological non-triviality of M is irrelevant, the obstruction is in fact parametrized by the set of maps from $S^3 \subset S^4$ to $U(1)$, i.e. the homotopy group $\pi_3(\mathrm{U}(1))$, which however vanishes. (More generally, for $M = S^n$ the obstruction is parametrized by $\pi_{n-1}(\mathrm{U}(1)) = \delta_{n,2}\mathbb{Z}$.)

To summarize, for any orbit of connections with U(1) stabilizer type there is a representative A' of the form

$$A'_\mu(x) = \tau a_\mu(x) \tag{14}$$

with ordinary functions a_μ. In particular, the theory has only one *single* U(1) stratum. For arbitrary semi-simple structure group G, analogous considerations apply to the particular stratum whose stabilizer type is the maximal torus of G. (In contrast, one must expect an infinite number of strata whose stabilizer type is a proper non-Abelian Lie subgroup of G.)

One may also analyse which of the specific connections A^g of the form (14) lie in the gauge slice. Choosing for simplicity the background $\bar{A} = 0$, the gauge condition then reads explicitly $\tau\partial^\mu a_\mu = [\partial_\mu g\, g^{-1}, \tau] a_\mu - \partial^\mu(\partial_\mu g g^{-1})$, which as a special class of solutions has $g(x) = \exp(\mathrm{i}\tau\gamma(x))$, where $\gamma(x) = -\mathrm{i}\int_M \mathrm{d}^4y H(x,y)\partial^\mu a_\mu(y) + \text{const.}$ with $H(x,y)$ the Green function of the Laplacian on M. But the *general* solution of the differential equation for g is less trivial; in particular one cannot easily determine which solutions lie in the fundamental modular domain Λ, except in the special case where the connection is flat, where the only representative of the orbit of (14) that is contained in Λ is the background $A = 0$ itself. The flat case is precisely the one in which the stabilizer type gets extended to SU(2), and $A = 0$ is the only point on that orbit for which the stabilizer consists precisely of the constant gauge transformations.

In fact, the fundamental modular domain Λ is not a particularly convenient tool for the global description of the reducible connections. Fortunately, in the specific case of SU(2), another characterization is available, namely via a relation with the configuration space of electrodynamics. As any principal U(1)-bundle over S^4 is trivial, one can consider $a = 0$ as a base point in the space $\mathcal{A}_1$ of connections of the U(1)-bundle, so that $\mathcal{A}_1$ can be viewed as the vector space of (4-tuples of) smooth functions a_μ on S^4. The gauge group then acts on these functions as $a \mapsto a + \mathrm{i}\,\mathrm{d}\lambda$ with $\lambda \in C^\infty(S^4, \mathbb{R})$. Except for constant λ which leaves all a invariant, this action is free, so that the configuration space of the U(1) gauge theory is a manifold. Now fixing an element τ in the Lie algebra su(2), the map

$$\phi : \mathcal{A}_1 \to \mathcal{A}, \qquad a \mapsto A = \tau a \tag{15}$$

is well-defined on orbits because the images of a and $a+\mathrm{i}\,\mathrm{d}\lambda$ are related as $\tau(a+\mathrm{i}\,\mathrm{d}\lambda) = A^g$ with the SU(2) gauge transformation $g = \mathrm{e}^{\mathrm{i}\lambda\tau}$. However, in the U(1) theory a and $-a$ lie on distinct orbits (for $a \neq 0$), while in the SU(2) theory τa and $-\tau a$ are on the same orbit. Thus the mapping (15) is not one-to-one on orbits, but two-to-one, and hence the U(1) stratum of the configuration space $\mathcal{M}$ of SU(2) Yang–Mills theory is a $\mathbb{Z}_2$-orbifold of the configuration space $\mathcal{M}_1$ of the U(1) theory. The SU(2) stratum is the unique fixed point of the $\mathbb{Z}_2$ action.

Further analysis shows that $\mathcal{M}$ has a 'cone over cones' structure. The U(1) connections form an infinite-dimensional cone with tip $A = 0$, and any orbit with U(1) stabilizer is itself the tip of an infinite-dimensional cone whose base consists of irreducible connections.

7. The pointed gauge group. The *pointed gauge group*, defined by

$$\mathcal{G}_0 := \{g \in \mathcal{G} \mid g(x_0) = e \in G\} \tag{16}$$

with $x_0 \in M$ fixed, acts freely on the space $\mathcal{A}$ of connections. Thus the associated orbit space $\mathcal{M}_0 := \mathcal{A}/\mathcal{G}_0$ is a manifold (and accordingly is popular among mathematicians). For instanton number $k = 0$, any $g \in \mathcal{G}$ can be written as $g = g_\mathrm{c} g_0$ with $g_0 \in \mathcal{G}_0$ and g_c the constant gauge transformation with $g_\mathrm{c}(x) = g(x_0)$ for all $x \in M$. One can show that the map

$$\varphi : \breve{\Lambda}_\circ|_{\text{bound. id.}} \to \mathcal{M}_0, \qquad \breve{\Lambda}_\circ \ni A \mapsto [A \bmod \mathcal{G}_0] \in \mathcal{M}_0 \tag{17}$$

(with $A \in \breve{\Lambda}_\circ|_{\text{bound. id.}}$ considered as an element of $\breve{\Lambda}_\circ := \breve{\Lambda}_{\bar{A}=0}$) is a diffeomorphism between $\breve{\Lambda}_\circ|_{\text{bound. id.}}$ and $\mathcal{M}_0$ and intertwines the group action of G on these spaces. (On both spaces, G acts via constant gauge transformations g_c, namely on $\breve{\Lambda}_\circ$ as $A \mapsto A^{g_\mathrm{c}}$, which is well-defined because $g_\mathrm{c} \in \mathcal{S}_\circ$, and on $\mathcal{M}_0$ as $[A \bmod \mathcal{G}_0] \mapsto [A^{g_\mathrm{c}} \bmod \mathcal{G}_0]$, which is well-defined because $\mathcal{G}_0$ is a normal subgroup of $\mathcal{G}$ so that $B^{g_c} = A^{g_0 g_c} = A^{g_c g_0'}$ with $g_0' = g_c{}^{-1} g_0 g_c$ for $B = A^{g_0}$.) The intertwining property of φ means that $\varphi(A^{g_c}) = [A^{g_c} \bmod \mathcal{G}_0]$, which immediately follows from the definition of φ.

8. Motivations. Let me come back to the issue of the relevance of defining Yang–Mills theories in the way described above. Thus I must explain the various choices I made, and also discuss whether there are any effects which distinguish between these and different choices. Concerning the first task, I have to justify the specific choice of space-time and the definition of the gauge group. While I cannot offer any fully convincing arguments, the following aspects certainly play an important rôle.

The reasoning leading to $M = S^4$ goes as follows. First, space-time is taken to be Euclidean on the following grounds.

▷ I have in mind a Lagrangian framework, and ultimately a description of the *quantum* theory in terms of path integrals, which have a better chance of being well-defined if M is Euclidean.

▷ The four-dimensional treatment appears to depend to a smaller extent on the classical dynamics (i.e. choice of the action). In the $3+1$-dimensional approach the constraints which enforce the reduction of the naive 'pre-phase space' (spanned by three-dimensional gauge fields and their conjugate momenta, the electric field strengths) to a constraint surface are part of the four-dimensional equations of motion, and hence the classical dynamics enters already at an early stage.

Also, I feel a bit uneasy about the Schrödinger picture employed in the $3+1$-dimensional approach to approach the quantum theory; as it is so close to quantum mechanics, one potentially misses such significant features of quantum field theory as superselection rules.

Second, some reasons for compactifying $\mathbb{R}^4$ to $S^4 \cong \mathbb{R}^4 \cup \{\infty\}$, the topological one-point compactification of $\mathbb{R}^4$ (and to consider S^4 as endowed with its natural metric so that it has finite volume), are the following.

▷ Requiring the action, and hence $|F^2|$, to be finite, one needs $F \to 0$ for $|x| \to \infty$, and hence $A \to g^{-1}\mathrm{d}g$ (pure gauge), i.e. $\mathcal{O}_A \to \mathcal{O}_0$. Thus the compactification to S^4 corresponds to natural boundary conditions at infinity.

▷ Instanton sectors (i.e. bundles with $k \neq 0$) only appear for compact M. Even though still poorly understood, topologically non-trivial gauge field configurations are expected to play a significant rôle for various non-perturbative features of quantum field theory, ranging from the structure of the QCD vacuum to baryon number violation in the electroweak interactions.

▷ Having finite volume has technical advantages, e.g. concerning the normalizability of constant gauge transformations. (This is important whenever one needs on $\mathcal{G}$ not only the group structure, but also a topological structure, such that $\mathcal{G}$ becomes an infinite-dimensional *Lie* group.)

▷ Finite volume automatically provides an infrared regularization. This is often convenient, as it is to be expected that any detailed study of the quantum theory involves (at intermediate steps) the introduction of an infrared cutoff.

Finally one has to specify which transformations are to be counted as gauge transformations in the sense of *redundancy* transformations, i.e. which configurations of the basic variables (here the connections A) are to be considered as equivalent and hence as describing the same physical state. Thus in particular I must argue why to take the full gauge group $\mathcal{G}$ (rather than e.g. the pointed gauge group $\mathcal{G}_0$), and the full configuration space (rather than e.g. restricting to the main stratum). An obvious reason for this choice is that it is just the most straightforward thing to do. Other arguments include:

▷ $\mathcal{G}_0$ depends on the choice of base point x_0. Even though one stays within a definite isomorphism class, in principle the choice of $\mathcal{G}_0$ might introduce an unwanted dependence of the physics on the space-time point x_0. (In physics isomorphic structures can still lead to distinct predictions, e.g. sometimes the specific choice of a basis is significant.)

▷ The decision to take $\mathcal{G}_0$ as the gauge group somehow gives undue prominence to the 'constant gauge transformations', which do not possess a natural geometric meaning (recall that there is no canonical embedding of G into $\mathcal{G}$).

▷ The singular connection $A = 0$ is the point about which one expands in ordinary perturbation theory. The restriction to the main stratum of $\mathcal{M}$ may therefore be incompatible with naive perturbation theory.

Concerning the relevance of these choices, first note that from a mathematical point of view one may choose one's favourite structure and enjoy whatever results one is led to. There remains, however, the question whether there is any 'physical' relevance in the sense that – at least in principle – there are definite implications for experimental predictions. In order to answer this question, it is mandatory to have insight into the theory at the quantum level. (It is not possible to 'turn on' and 'switch off' quantum corrections; this is an abuse of language which is unfortunately quite common, and sometimes has disastrous effects.) As the understanding of many aspects of quantum Yang–Mills theory so far is restricted to the realm of perturbation theory, again I can present only a few isolated observations:

▷ The treatment of singular points in the *quantum* theory potentially introduces further parameters, analogous to the construction of self-adjoint extensions of an operator on a Hilbert space that is not essentially self-adjoint.

▷ Various concepts in the quantum theory involve a description in terms of fibre bundles over the configuration space $\mathcal{M}$. Examples are the determinant line bundle for chiral fermions that are coupled to the gauge fields [2, 27] (if this bundle does not have global sections, the theory has a chiral anomaly), and wave functionals in the Schrödinger picture. The very notion of fibre bundles and the like assumes, however, that $\mathcal{M}$ is a manifold.

▷ As already mentioned, the standard perturbation theory is an expansion about $A = 0$. The orbit of $A = 0$ is precisely the most singular point of $\mathcal{M}$.

▷ In 2 + 1-dimensional Chern–Simons theory, reducible connections yield non-zero contributions to the Jones polynomial and to the Witten invariant of three-manifolds [41].

▷ In a combination of Yang–Mills and Chern–Simons theory in 2+1 dimensions, reducible connections give rise to non-trivial boundary conditions of the Schrödinger picture wave functionals [7].

9. Outlook. Let me now assume again that the description of Yang–Mills theories that I gave is appropriate, and hence in particular that $\mathcal{M}$ has the stratified nature described above. Some of the issues to be addressed in the future are then as follows.

▹ An immediate task at the classical level is to classify the strata for structure groups G other than SU(2), at least for SU(3). Ideally one would like to identify some 'invariant', i.e. a machine which for any given connection A tells to which stratum it belongs. Because of the presence of stabilizers which are isomorphic to non-Abelian proper subgroups of G, this is quite difficult.

▹ Include matter fields. The singularities of the full configuration space then include those of the gauge fields tensored with the zero matter configuration. But some of these configurations may have to be excluded because they would have infinite action (e.g. in the case of scalar matter fields with a Higgs potential), while on the other hand there could exist further singularities.

▹ Clarify the relation of the 'cone over cones' structure of $\mathcal{M}$ with a similar singularity structure that arises in a Hamiltonian formulation. The constraint subset of the pre-phase space is a submanifold except in the neighbourhood of reducible connections; after cutting out the singular points, the intersection of this constraint submanifold with the gauge slice is a symplectic submanifold of the pre-phase space, while including the reducible connections gives rise to a cone over cones structure [3, 4, 30, 39].

▹ Work out the relevant algebraic geometry aspects of stratified varieties. In a finite-dimensional context, similar aspects have been addressed e.g. in [18, 25].

▹ Desingularize $\mathcal{M}$, e.g. by replacing the naive quotient $\mathcal{A}/\mathcal{G}$ by the homotopy quotient associated to the classifying space $B\mathcal{G}$ (compare [17]), or by mimicking similar proposals for the space of Riemannian geometries [26].

▹ 'Blow up' the singularities in a manner analogous to resolving orbifold singularities in complex [32] and symplectic [11, 29] geometry. (This is one procedure by which one might introduce parameters. But situations are known, such as singularities of hypersurfaces in weighted projective spaces which appear in the description of string theory vacua, where the resolution is more or less canonical as far as the interesting 'physical' quantities are concerned.) However, in real geometry, this can presumably only be achieved when the singularities can be described as the vanishing locus of some algebraic equation, which is not available here.

▹ Investigate the possibility that the proper quantum theory may smooth out the singularities automatically. As far fetched as this may sound, according to the results of [42] such a phenomenon does occur for the moduli space of vacua of some supersymmetric gauge theories.

▹ Consider toy models, in particular models with finite-dimensional configuration space. In this context it may help to realize that in mathematics often much progress is made by making use of algebraic rather than geometric tools (such as describing a manifold in terms of the algebra of smooth functions on it). A model which can be addressed in this spirit is Chern–Simons theory for which e.g. the algebra of observables has been described explicitly in [1].

▷ Develop a measure theory on $\mathcal{M}$ so as to define the quantum theory by means of path integrals. This is rather difficult. In the case of $\mathcal{M}_0$, a rigorous formulation of path integral measures has been achieved recently in [5, 6, 16] (for earlier attempts, see e.g. [8, 14]).

Finally, it is worth mentioning that in view of the complexity of the structures described in this review the remarkable success of naive perturbation theory for many aspects of Yang–Mills theory is quite mysterious.

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