QUANTUM GROUPS AND QUANTUM SPACES
BANACH CENTER PUBLICATIONS, VOLUME 40
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

ON *-REPRESENTATIONS OF $U_q(sl(2))$: MORE REAL FORMS

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Dedicated to M.P.

Abstract. The main goal of this paper is to do the representation-theoretic groundwork for two new candidates for locally compact (nondiscrete) quantum groups. These objects are real forms of the quantized universal enveloping algebra $U_q(sl(2))$ and do not have real Lie algebras as classical limits. Surprisingly, their representations are naturally described using only bounded (in one case only two-dimensional) operators. That removes the problem of describing their Hopf structure "on the Hilbert space level"([W]).

1. Real forms of $U_q(sl(2))$ - algebraic preliminaries. There are several Hopf algebras over $\mathbb{C}$ known by the same name $U_q(sl(2))$ (here we deal with a complex $q \neq -1, 0, 1$). The first one is given by the simply-connected rational form of Drinfeld's "Poisson-Lie deformation algebra" $U_h(sl(2))$ (see e.g. [CP, sec. 9.1]); it was introduced by Jimbo in [J] as $U_q^{(1)} = \langle k, k^{-1}, e, f\rangle$ with the relations

$$kk^{-1} = k^{-1}k = 1$$

$$ke = qek; \quad kf = q^{-1}fk$$

$$ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}}$$

$$\Delta(k) = k \otimes k; \Delta(e) = e \otimes 1 + k^2 \otimes e; \Delta(f) = f \otimes k^{-2} + 1 \otimes f$$

$$\varepsilon(k^{\pm 1}) = 1; \quad \varepsilon(e) = \varepsilon(f) = 0$$

$$S(k) = k^{-1};\ S(e) = -k^{-2}e;\ S(f) = -fk^2.$$

The other one is associated to the adjoint form of $U_h(sl(2))$ and is defined (see e.g. [L])

1991 *Mathematics Subject Classification*: Primary 17B37; Secondary 16W30.
The paper is in final form and no version of it will be published elsewhere.

as $U_q^{(2)} = \langle K, K^{-1}, E, F\rangle$ with the relations

$$KK^{-1} = K^{-1}K = 1$$

$$KE = q^2 EK; \quad KF = q^{-2} FK \tag{1}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$\Delta(K) = K \otimes K; \Delta(E) = E \otimes 1 + K \otimes E; \Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

$$\varepsilon(K^{\pm 1}) = 1;\ \varepsilon(E) = \varepsilon(F) = 0$$

$$S(K) = K^{-1};\ S(E) = -K^{-1}E;\ S(F) = -FK.$$

For a fixed q we see that $U_q^{(2)}$ is a Hopf subalgebra of $U_q^{(1)}$ generated by $k^2 = K, k^{-2} = K^{-1}, e = E, f = F$. As explained in [CP, sec. 9.1] these two Hopf algebras are in some sense the only rational forms of $U_h(sl(2))$.

Definition 1. A *real form* or a *Hopf* **-algebraic structure* of a Hopf algebra A is a conjugate-linear map on A: $a \to a^*$ such that

(i) $1^* = 1$, $(ab)^* = b^* a^*$, $(a^*)^* = a$ for all $a, b \in A$ (in other words $(A, *)$ is a *-algebra);

(ii) $\varepsilon(a^*) = \overline{\varepsilon(a)}$, $\Delta(a^*) = ((* \otimes *)\Delta)(a)$ for all $a \in A$ (i.e. the counit ε and comultiplication Δ are *-homomorphisms).

Two *-algebras $(A_1, *_1)$ and $(A_2, *_2)$ are *equivalent* if there is an algebraic isomorphism $\phi : A_1 \to A_2$ such that $\phi \circ *_1 = *_2 \circ \phi$. If ϕ is also a coalgebraic isomorphism we say that $(A_1, *_1)$ and $(A_2, *_2)$ are *equivalent Hopf* **-algebras*.

The list of all Hopf *-algebraic structures of $U_q^{(1)}$ was given in [MM], they exist only for $q \in \mathbb{R}$ or $|q| = 1$ and are the following:

$$su_q^{(1)}(2) : k^* = k, e^* = fk^2, f^* = k^{-2}e; \quad q \in \mathbb{R}$$

$$su_q^{(1)}(1,1) : k^* = k, e^* = -fk^2, f^* = -k^{-2}e; \quad q \in \mathbb{R},$$

$$sl_q^{(1)}(2, \mathbb{R}) : k^* = k, e^* = e, f^* = f; \quad |q| = 1.$$

Remark 1. As an associative algebra A has two more *-structures on which the condition (i) of definition 1 is satisfied but the comultiplication fails to be *-homomorphic. These *-algebras and their interesting representation theory are discussed in [V1].

The list of real forms of $U_q^{(2)}$ is given by Twietmeyer in [T] (in fact he describes the real forms for all $U_q(\mathcal{G})$ where $\mathcal{G}$ is a simple Lie algebra); it contains five Hopf *-algebras (see also [CP, p.310]):

$$su_q^{(2)}(2) : K^* = K, E^* = FK, F^* = K^{-1}E; \quad q \in \mathbb{R};$$

$$su_q^{(2)}(1,1) : K^* = K, E^* = -FK, F^* = -K^{-1}E; \quad q \in \mathbb{R};$$

$$sl_q^{(2)}(2, \mathbb{R}) : K^* = K, E^* = E, F^* = F; \quad |q| = 1;$$

$$A_4(q) : K^* = K, E^* = iFK, F^* = iK^{-1}E; \quad q \in i\mathbb{R};$$

$$A_5(q) : K^* = K, E^* = -iFK, F^* = -iK^{-1}E; \quad q \in i\mathbb{R}.$$

Observation. There is a natural correspondence between the real forms of $U_q^{(1)}$ and the first three real forms of $U_q^{(2)}$, namely $su_q^{(2)}(2), su_q^{(2)}(1,1), sl_q^{(2)}(2, \mathbb{R})$ are subHopf

*-algebras of respectively $su_q^{(1)}(2)$, $su_q^{(1)}(1,1)$, $sl_q^{(1)}(2,\mathbb{R})$ each generated by $k^{\pm 2}, e, f$.

These Hopf *-algebras have the corresponding classical objects (cocommutative Hopf *-algebras built on real forms of $sl(2)$) as their limits at $q=1$ (see e.g. [CP]).

We want to study the real forms $A_4(q)$ and $A_5(q)$ of $U_q^{(2)}$ which do not have obvious classical limits because their quantization parameter q is in the domain $i\mathbb{R}$ which does not contain 1.

Let us first use some symmetries of $U_q(sl(2))$ to establish equivalences of these real forms.

PROPOSITION 1. (a) *The Hopf isomorphism* $U_q^{(2)} \to U_{-q}^{(2)}$ *sending* $K \to K$, $E \to E$, $F \to -F$ *makes* $A_4(q)$ *and* $A_5(-q)$ *equivalent Hopf* *-*algebras for all* $q \in i\mathbb{R}$.

(b) *The antipode* $S: K \to K^{-1}$, $E \to -K^{-1}E$, $F \to -FK$ *can be viewed as an algebraic isomorphism* $U_q^{(2)} \to U_{q^{-1}}^{(2)}$. *It yields the following: for all* $q \in i\mathbb{R}$

$$A_4(q) \cong A_4(q^{-1}), \quad A_5(q) \cong A_5(q^{-1}) \quad \textit{as *-algebras.}$$

(c) *An algebraic isomorphism* $U_q^{(2)} \to U_{-q}^{(2)}$ *sending*
$K \to K^{-1}$, $E \to -qF$, $F \to q^{-1}E$ *gives: for all* $q \in i\mathbb{R}$

$$A_4(q) \cong A_4(-q), \quad A_5(q) \cong A_5(-q) \quad \textit{as *-algebras.}$$

(d) *The equivalent pairs listed in (b),(c) are not equivalent as Hopf* *-*algebras.*

Proof of (d). The coalgebraic structure of $U_q^{(2)}$ does not depend on the parameter q. By [T] for any coalgebraic isomorphism $\phi: U_{q_1}^{(2)} \to U_{q_2}^{(2)}$ we must have

$$\phi(K) = K, \ \ \phi(E) = \alpha FK + \beta E, \ \ \phi(F) = \gamma F + \delta K^{-1}E.$$

It is easy to check that no such map can be a *-algebraic isomorphism between $A_4(q)$ and $A_4(q^{-1})$ or between $A_4(q)$ and $A_4(-q)$. ■

Thus the real forms $A_4(q)$ and $A_5(q)$ are in fact only one Hopf *-algebra. We will choose to consider it as $A_5(q)$ and denote this real form $su_{q,i}(2)$. So we assume from now on:

$$K^* = K, \quad E^* = -iFK, \quad F = iE^*K^{-1}.$$

2. *-Representations of $su_{q,i}(2)$. Let us use the real parameter $p = iq^{-1}$. By Proposition 1 it is enough to consider the case $p \in (0,1]$.Then $su_{q,i}(2)$ is the *-algebra generated by K, K^{-1}, E with the relations: $KK^{-1} = K^{-1}K = 1$ and

$$KE = -p^{-2}EK; \quad KE^* = -p^2E^*K; \tag{2}$$

$$EE^* + p^2E^*E = \frac{p}{1+p^2}(I - K^2). \tag{3}$$

We see from (3) that in the sense of the usual *-algebraic ordering (i.e. $a^*a \ge 0$ for all a):

$$0 \le K^2 \le I, \ \ 0 \le EE^* \le \frac{p}{1+p^2}I, \ \ 0 \le E^*E \le \frac{1}{p(1+p^2)}I.$$

In order to be able to avoid unbounded operators (for some time at least) let us take the following definition:

Definition 2. By a *representation* of the *-algebra $su_{q,i}(2)$ we understand a pair of bounded operators $K = K^*$ and E on a Hilbert space $\mathcal{H}$ such that: (i) the operators K, E, E^* satisfy the relations (2) and (3); (ii) the operator K has an (unbounded) inverse K^{-1}, i.e. $Ker K = 0$.

Let us start with the "quasiclassical" situation when $q = i$ ($p = 1$). In this case the relations (2), (3) transform into:

$$KE = -EK; \quad KE^* = -E^*K; \tag{4}$$

$$EE^* + E^*E = \frac{1}{2}(I - K^2). \tag{5}$$

Proposition 2. *For $q = i$ the *-algebra $su_{q,i}(2)$ has the following irreducible representations*:

1) *one-dimensional*: $K = \pm 1,\ E = E^* = 0$;

2) *two-dimensional*;

a) *degenerate*;

$$K = \pm s \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \sqrt{\frac{1-s^2}{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $s \in (0, 1)$;

b) *nondegenerate*

$$K = \pm s \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \sqrt{\frac{1-s^2}{2}} \begin{pmatrix} 0 & t\zeta \\ \sqrt{1-t^2} & 0 \end{pmatrix},$$

where $s \in (0, 1), t \in (0, 1), |\zeta| = 1$.

Observe at this point that in every irreducible representation the operators K^{-1} and $F = iE^*K^{-1}$ are bounded - so there will be no problem to consider the comultiplication on the representation level. Recall that such problems do arise for the quantum groups $E_q(2)$ and $SU_q(1,1)$ ([W]).

Proof. It follows from (4) that K^2 commutes with everything, so irreducibility implies

$$K^2 = const I.$$

Since $0 \le K^2 \le I$ and $Ker K^2 = Ker K = 0$ we can write

$$K^2 = s^2 I, \qquad s \in (0, 1].$$

If $K^2 = I$ (i.e. $s = 1$) the relation (5) gives

$$EE^* + E^*E = 0 \Rightarrow E = E^* = 0.$$

Otherwise for $\widetilde{K} = \widetilde{K}^* = \frac{1}{s}K$, $\widetilde{E} = \sqrt{\frac{2}{1-s^2}}E$ we have

$$\widetilde{K}^2 = I; \quad \widetilde{K}\widetilde{E} + \widetilde{E}\widetilde{K} = 0; \quad \widetilde{K}\widetilde{E}^* + \widetilde{E}^*\widetilde{K} = 0; \quad \widetilde{E}\widetilde{E}^* + \widetilde{E}^*\widetilde{E} = I,$$

which is very close to the canonical anticommutation relations. Now we use the standard CAR representation technique: Let $\widetilde{K}, \widetilde{E}, \widetilde{E}^*$ act on a Hilbert space $\mathcal{H}$. Consider the orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that $\widetilde{K}$ acts as $\pm I$ on $\mathcal{H}_\pm$. Then the relations imply that $\widetilde{E}(\mathcal{H}_\pm) = \mathcal{H}_\mp$, $\widetilde{E}^*(\mathcal{H}_\pm) = \mathcal{H}_\mp$, and besides $\widetilde{E}^2$, $(\widetilde{E}^*)^2$ commute with everything and so are central. This means every irreducible representation has dimension 2. The rest is just computation. ■

THEOREM 1. *For* $p = iq^{-1} \in (0,1)$ *the* **-algebra* $su_{q,i}(2)$ *has the following irreducible representations*:

1) *one-dimensional*: $K = \pm 1,\ E = E^* = 0$;

2) *infinite-dimensional degenerate*:

$$K = \pm s\ diag\ (1, -p^2, p^4, -p^6, \ldots)$$

$$E = \begin{pmatrix} 0 & \sqrt{\mu_1} & 0 & \ldots\ldots & \ldots & \\ \ldots & 0 & \sqrt{\mu_2} & 0 & \ldots & \ldots \\ \ldots & \ldots & 0 & \sqrt{\mu_3} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ddots & \ddots\ddots & \ldots \end{pmatrix},$$

where $s \in (0,1)$, $\mu_n = \frac{p}{(1+p^2)^2}\{1-(-p^2)^n\}[1-s^2(-p^2)^{n-1}]$, $n \geq 1$.

We will give a self-contained ad hoc argument. A more general technique for the representations (possibly unbounded) of *-algebras of the type (2),(3) is given in [V2] and used in [V1]. It is closely related to the Mackey imprimitivity systems.

Proof of Theorem 1. Denote $C = E^*E \geq 0$. Consider the polar decomposition $E = U|E|$ of operator E , where a nonnegative $|E|$ and a partial isometry U are such that

$$|E|^2 = E^*E = C, \quad Ker|E| = KerU = KerE.$$

Then from (2) we have

$$KC = KE^*E = -p^2E^*KE = E^*EK = CK,$$

so K and C are commuting selfadjoint operators. Also from (2):

$$KU|E| = -p^{-2}U|E|K = -p^{-2}UK|E|.$$

Since U and $|E|$ have the same (K-invariant) nullspace this relation is equivalent to

$$KU = -p^{-2}UK, \quad KU^* = -p^2KU^*. \tag{6}$$

Claim: The partial isometry U must have a nullspace. Suppose it does not, then $U^*U = I$. In this case (6) gives

$$U^*K^2U = p^{-4}K^2 \Longrightarrow Spec(p^{-4}K^2) \subseteq Spec(K^2).$$

But since $p^{-4} > 1$ and $K^2 > 0$ we see that K^2 is unbounded. This cannot be since (3) means $K^2 \leq I$.

Next we want to show that, unless we have the trivial case $E = E^* = 0$, operator U^* must be an isometry. Since $E^* = |E|U^*$ the relation (3) in polar coordinates becomes:

$$UCU^* = \frac{p}{1+p^2}(I - K^2) - p^2C. \tag{7}$$

Consider the subspace $\mathcal{K} = KerU \cap KerU^*$. Then (6) shows it is K-invariant. Besides, $KerE = KerU$ implies $E = 0$ and also we have $E^* = |E|U^* = 0$ on this subspace. So $\mathcal{K}$ is an invariant subspace and (7) shows that $K^2|_{\mathcal{K}} = I$ - this gives us one-dimensional irreducible representations.

Claim: Let $\xi \in KerU^* \cap (KerU)^\perp$ then $\xi = 0$. To prove it take $\eta = U\xi$, then we have $\|\eta\| = \|\xi\|$ and $U^*\eta = \xi,\ U^*\xi = 0$. Now (7) shows:

$$UCU^*\xi = 0 = \frac{p}{1+p^2}(I - K^2)\xi - p^2 C\xi \Longrightarrow\ C\xi = \frac{p^{-1}}{1+p^2}(I-K^2)\xi,$$

so we compute:

$$\frac{p}{1+p^2}(I-K^2)\eta - p^2C\eta \overset{(7)}{=} UCU^*\eta = UC\xi = U\frac{p^{-1}}{1+p^2}(I-K^2)\xi \overset{(6)}{=}$$
$$= \frac{p^{-1}}{1+p^2}(I - p^4K^2)U\xi = \frac{p^{-1}}{1+p^2}(I-p^4K^2)\eta.$$

But then $p^2C\eta = -\frac{1-p^2}{1+p^2}(p^{-1}I + pK^2)\eta$. Now recall that $0 < p < 1$ and C and K^2 are nonnegative operators. We have a contradiction, unless $\eta = 0 \Rightarrow \xi = U^*\eta = 0$.

Now we know that there is a nonzero subspace $\mathcal{H}_0 = KerU$, and U^* is an isometry. Then we have an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_0 \oplus U^*\mathcal{H}_0 \oplus (U^*)^2\mathcal{H}_0 \oplus \dots$$

Again (6) shows that $K : \mathcal{H}_0 \to \mathcal{H}_0$; denote $K_0 = K|_{\mathcal{H}_0}$. Then each $\mathcal{H}_n = (U^*)^n\mathcal{H}_0$ is also K-invariant and $K_n = K|_{\mathcal{H}_n} = (-p^2)^nK_0$. Besides, $C_0 = C|_{\mathcal{H}_0} = 0$, and (7) means:

$$C_{n+1} = C|_{\mathcal{H}_{n+1}} = \frac{p}{1+p^2}(I - K_n^2) - p^2C_n.$$

If there is a nontrivial projection P_0 on $\mathcal{H}_0$ that commutes with K_0, then $P_1 = U^*P_0U : \mathcal{H}_1 \to \mathcal{H}_1$ commutes with K_1 and C_1, also $P_2 = (U^*)^2P_0U^2 : \mathcal{H}_2 \to \mathcal{H}_2$ commutes with K_2 and C_2, and so on. This would produce a nontrivial projection $P_0 \oplus P_1 \oplus P_2 \oplus \dots$ on $\mathcal{H}$ commuting with everything. So in the irreducible situation $\mathcal{H}_0$ must be a one- dimensional eigenspace $\langle \xi_0 \rangle$ for K with an eigenvalue $\kappa_0 \neq 0$ (since $KerK = 0$). Then every $\mathcal{H}_n = \langle \xi_n = (U^*)^n\xi_0 \rangle$ is a one-dimensional eigenspace for K and C with the eigenvalues determined by the formulas:

$$\kappa_{n+1} = -p^2\kappa_n, \quad c_{n+1} = \frac{p}{1+p^2}(1-\kappa_n^2) - p^2c_n.$$

This gives all irreducible representations of the relations (6),(7) and we have to pick those for which $C \geq 0$. It is equivalent to the condition: $c_n > 0$ for all $n \geq 1$ (since the corresponding $\xi_n \perp \mathcal{H}_0 = KerC$) or $\kappa^2 < 1$ - so we parametrize $\kappa = \pm s,\ s \in (0,1)$. ∎

Note that if we want we could represent the relations (2),(3) with no extra conditions on K. The relations (2):

$$KE = -p^{-2}EK; \quad KE^* = -p^2E^*K$$

by themselves mean that the nullspace $KerK$ is an invariant subspace. So we have some irreducible representations of (2),(3) with $K = 0$ and

$$EE^* + p^2E^*E = \frac{p}{1+p^2}I.$$

These representations correspond to a boundary degenerate form of our quantum *-algebra $su_{q,i}(2)$. ∎

PROPOSITION 3. *Besides the representations listed in Theorem* 1 *the relations* (2), (3) *have the following irreducible representations*:

1) *one-dimensional*: $K = 0$, $E = \frac{\sqrt{p}}{1+p^2}\zeta$; *where* $|\zeta| = 1$;

2) *infinite-dimensional* : $K = 0$,

$$E = \begin{pmatrix} 0 & \sqrt{\mu_1} & 0 & \dots\dots & \dots & \\ \dots & 0 & \sqrt{\mu_2} & 0 & \dots & \dots \\ \dots & \dots & 0 & \sqrt{\mu_3} & 0 & \dots \\ \dots & \dots & \dots & \ddots & \ddots\ddots & \dots \end{pmatrix},$$

where $\mu_n = \frac{p}{(1+p^2)^2}\{1 - (-p^2)^n\}$, $n \geq 1$.

The proof is a simpler version of the argument in the proof of Theorem 1.

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