MATHEMATICS OF GRAVITATION
PART II, GRAVITATIONAL WAVE DETECTION
BANACH CENTER PUBLICATIONS, VOLUME 41
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1997

POST-NEWTONIAN APPROXIMATION IN THE TEST PARTICLE LIMIT

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Abstract. Gravitational radiation from a small mass particle orbiting a massive black hole can be analytically studied to a very high order in the post-Newtonian expansion. Thus it gives us useful information on the evolution of a coalescing compact binary star. In this talk, I report on recent progress made in the black-hole perturbation approach.

1. Introduction. Black holes are most relativistic astronomical objects and we have exact solutions of the Einstein equations that describe black holes. Hence black holes are often used to examine various relativistic astrophysical phenomena as well as to test various theories. As for gravitational wave physics, studies of the perturbations of black-hole spacetimes will give us insight into the nature of gravitational waves in the case of strong gravity such as the effect of curvature scattering in fully non-linear regime. Furthermore, the existence of the so-called quasi-normal modes of black holes [1] enables us to study the intrinsic physical properties of black holes in detail and perhaps has profound implications to the radiation reaction problem. Thus it is hoped that the black-hole perturbation approach can (at least partially) solve some of the unsolved problems in gravitational wave physics.

Here, however, we do not consider such issues but confine our attention to a topic of gravitational radiation from coalescing compact binaries. Recently this has became a subject of great interest because one of the most promising targets of the planned future gravitational wave detectors such as LIGO [2, 3]/VIRGO [4] is the gravitational radiation from compact binaries just before their coalescence, the so-called “last three minutes” [5].

Gravitational radiation from coalescing compact binaries at its inspiral stage have a characteristic waveform, called a “chirp” signal, with both the frequency and amplitude increasing rapidly until the final coalescing stage begins [3]. The last few milliseconds of coalescence are the stage which is not yet well-understood and is an issue of numeri-

1991 *Mathematics Subject Classification*: 83C25, 83C35.
The paper is in final form and no version of it will be published elsewhere.

cal relativity. Turning back to the inspiral state, since how the chirp signal develops in time depends on the rate of gravitational radiation, which further depends on orbital parameters of a binary, it brings us rich information about mass, spin and other physical quantities of the binary stars. In addition, together with the detected amplitude of gravitational waves, we will have an accurate measurement of the distance to the source, by which we may be able to determine cosmological parameters of the universe. Furthermore, provided we have an accurate theoretical prediction of the evolutionary behavior of inspiraling binaries based on general relativity, the observed data can be used to test the validity of general relativity or constrain alternative theories of gravity. Thus, much effort has been recently made to construct accurate theoretical templates [6].

To construct theoretical templates, the post-Newtonian approximations are usually employed to solve the Einstein equations. Up to now, the gravitational wave luminosity from a binary in quasi-circular orbits has been calculated up through 2.5PN order, i.e, $O(v^5)$ beyond Newtonian quadrupole formula where v is the orbital velocity of the binary [7, 8]. However, based on numerical calculations of the gravitational radiation from a particle in circular orbit around a non-rotating black hole, Cutler et al. [9] showed that evaluation of the gravitational wave luminosity to a post-Newtonian order much higher than presently achieved level might be required. Then in order to find out the necessary post-Newtonian order, the same problem was investigated by Tagoshi and Nakamura [10] with much higher accuracy, to 4PN order and concluded that the accuracy to at least 3PN order is required for the construction of effective theoretical templates.

These previous studies using the perturbation equation of a black hole show its powerfulness to the problem. Although restricted by the condition that $\mu \ll M$, where μ and M are the reduced mass and total mass, respectively, of the system, because the black hole perturbation approach takes full account of relativistic effects by nature, we can spell out the relevant post-Newtonian effects from perturbation calculations in a rather straightforward manner. Hence it plays a complementary role to the standard post-Newtonian approach and provides a useful guideline for higher post-Newtonian calculations.

To strengthen the black hole perturbation approach further, it is then much desirable to develop an analytical method in which coefficients of the post-Newtonian expansion of the luminosity, for example, are evaluated exactly so that the results can be compared with those by the standard post-Newtonian calculations without any ambiguity. Poisson [11] first developed such a method and calculated the luminosity to 2PN order from a particle in circular orbits around a non-rotating black hole. Then extending Poisson's method, a more systematic method was developed by Sasaki [12] and analytical waveforms and luminosity up through 4PN order were analytically derived by Tagoshi and Sasaki [13]. The results were in excellent agreement with those of Tagoshi and Nakamura [10]. Recently, the method has been extended to the case of a rotating black hole by Shibata et al. [14]. They have obtained the energy and angular momentum luminosities to 2.5PN order from a particle in circular orbits with small inclination angle, which hence clarified the next leading order effects of spin-orbit coupling. For slightly eccentric orbits around a Kerr black hole, Tagoshi has done the calculation to 2.5PN order [15]. For circular orbits around a Kerr black hole, the calculation to 4PN order has been done by Tagoshi et al. [16]. Furthermore, extention of the method to the case of a spinning particle has been done by Tanaka et al. [17] and the luminosity to 2.5PN order has been obtained for circular orbits, which includes the effect of spin-spin coupling.

In the next section, I review the black-hole perturbation approach based on the Teukolsky [18] and Regge-Wheeler equations [19]. (Due to limitation of space, however, I focus on the case of a non-rotating black hole.) Then I summarize the recent results and discuss future issues. For notational simplicity, we set $c = G = 1$ in the following.

1.1. *Regge-Wheeler-Teukolsky formalism.* Let us consider the case when a particle of mass μ is in a circular orbit around a Schwarzschild black hole of mass $M \gg \mu$. The gravitational waves radiated out to infinity from the system is then described by the fourth Newman-Penrose quantity ψ_4 [18], which is related to the two independent modes of gravitational waves h_+ and $h_\times$ at infinity as

$$\psi_4 = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times). \tag{1}$$

On the Schwarzschild background, it can be decomposed as

$$\psi_4 = \frac{1}{r^4} \sum_{\ell m \omega} R_{\ell m \omega}(r)\, {}_{-2}Y_{\ell m}(\theta, \varphi) e^{-i\omega t}, \tag{2}$$

where ${}_{-2}Y_{\ell m}$ are the $s = -2$ spin-weighted spherical harmonics. The radial function $R_{\ell m\omega}$ satisfies the inhomogeneous Teukolsky equation [18],

$$\left[\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{d}{dr}\right) - U(r)\right] R_{\ell m \omega}(r) = T_{\ell m \omega}(r), \tag{3}$$

where

$$U(r) = \frac{r^2}{\Delta}[\omega^2 r^2 - 4i\omega(r - 3M)] - (\ell - 1)(\ell + 2), \quad \Delta = r(r - 2M), \tag{4}$$

and $T_{\ell m\omega}$ is the source term determined by the energy momentum tensor of the particle.

To solve Eq. (3), we employ the Green function method. Then $R_{\ell m\omega}$ at $r \to \infty$ takes the form,

$$\begin{aligned} R_{\ell m \omega}(r \to \infty) &= \frac{r^3 e^{i\omega r^*}}{2i\omega B^{in}_{\ell\omega}} \int_{2M}^{\infty} dr R^{in}_{\ell\omega}(r) T_{\ell m\omega}(r) \Delta^{-2} \\ &\equiv r^3 e^{i\omega r^*} \widetilde{Z}_{\ell m \omega}, \end{aligned} \tag{5}$$

where $R^{in}_{\ell\omega}(r)$ is a homogeneous solution which satisfies the boundary condition,

$$R^{in}_{\ell\omega}(r) = \begin{cases} D_{\ell\omega}\Delta^2 e^{-i\omega r^*} & \text{for} \quad r^* \to -\infty, \\ r^3 B^{out}_{\ell\omega} e^{i\omega r^*} + r^{-1} B^{in}_{\ell\omega} e^{-i\omega r^*} & \text{for} \quad r^* \to +\infty, \end{cases} \tag{6}$$

where $r^* = r + 2M \ln(r/2M - 1)$. In the case of a circular orbit with radius $r = r_0$, $T_{\ell m\omega}(r)$ takes the form,

$$T_{\ell m \omega} \sim [a_0 \delta(r - r_0) + a_1 \delta'(r - r_0) + a_2 \delta''(r - r_0)]\, \delta(\omega - m\Omega), \tag{7}$$

where Ω is the orbital angular frequency. Hence what we need to know are the behavior of $R_{\ell m\omega}$ around $r = r_0$ and its incident amplitude $B^{in}_{\ell\omega}$. Also because of Eq. (7), the amplitude $\widetilde{Z}_{\ell m\omega}$ takes the form,

$$\widetilde{Z}_{\ell m \omega} = Z_{\ell m} \delta(\omega - m\Omega). \tag{8}$$

In terms of $Z_{\ell m}$, the gravitational wave form at infinity is given by

$$h_+ - ih_\times = -\frac{2}{r}\sum_{\ell m}\frac{1}{\omega^2} Z_{\ell m\ -2}Y_{\ell m}(\theta,\varphi)e^{-i\omega(t-r^*)}, \tag{9}$$

and the luminosity is given by

$$\frac{dE}{dt} = \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell} \mid Z_{\ell m}\mid^2 /2\pi\omega^2, \tag{10}$$

where $\omega = m\Omega$.

Now since the radius of the orbit and the angular velocity are related as

$$\frac{M}{r_0} = (r_0\Omega)^2 \equiv v^2, \tag{11}$$

where v is the orbital velocity, we have the small non-dimensional parameters of the problem, $r_0\omega = O(v)$ and $M\omega = O(v^3)$. Originally the parameter $r_0\omega$ represents the slowness of the particle motion, hence plays a role of the post-Newtonian expansion parameter. The parameter $M\omega$ represents the strength of the gravity, hence plays a role of the post-Minkowski parameter. In the present case, since the particle is in bound orbits, these parameters are related to each other as shown above through the orbital velocity of the particle.

Thus our task reduces to calculating the ingoing-wave Teukolsky function $R^{in}_{\ell\omega}$ at $r\omega = O(v) \ll 1$ as well as to extracting out its incident amplitude $B^{in}_{\ell m\omega}$ to a required order of $M\omega = O(v^3)$.

For technical reasons, however, it is much easier to deal with the Regge-Wheeler equation, rather than the Teukolsky equation, by the transformation [20, 21],

$$R^{in}_{\ell\omega} \to R^{in}_{\ell\omega} = \Delta\left(\frac{d}{dr^*} + i\omega\right)\frac{r^2}{\Delta}\left(\frac{d}{dr^*} + i\omega\right) rX_{\ell\omega}. \tag{12}$$

Then $X_{\ell\omega}$ satisfies the homogeneous Regge-Wheeler equation [19],

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V_\ell(r)\right] X_{\ell\omega}(r) = 0, \tag{13}$$

where

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right). \tag{14}$$

Corresponding to Eq. (6), we have the asymptotic forms of $X^{in}_{\ell\omega}$ as

$$X^{in}_{\ell\omega}(r) = \begin{cases} C_{\ell\omega}e^{-i\omega r^*}, & r^* \to -\infty \\ A^{out}_{\ell\omega}e^{i\omega r^*} + A^{in}_{\ell\omega}e^{-i\omega r^*}, & r^* \to +\infty. \end{cases} \tag{15}$$

where $A^{in}_{\ell\omega}$, $A^{out}_{\ell\omega}$ and $C_{\ell\omega}$ are respectively related to $B^{in}_{\ell\omega}$, $B^{out}_{\ell\omega}$ and $D_{\ell\omega}$ defined in Eq. (6) as

$$\begin{aligned} B^{in}_{\ell\omega} &= -\frac{c_0}{4\omega^2}A^{in}_{\ell\omega}, \\ B^{out}_{\ell\omega} &= -4\omega^2 A^{out}_{\ell\omega}, \\ D_{\ell\omega} &= \frac{c_0}{16(1-2iM\omega)(1-4iM\omega)M^3}C_{\ell\omega}, \end{aligned} \tag{16}$$

where $c_0 = (\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega$.

To make the structure of the equation more transparent, we rescale the independent variable r to $z = r\omega$ and rewrite the homogeneous Regge-Wheeler equation as

$$\left[\frac{d^2}{dz^{*2}} + 1 - \left(1 - \frac{\epsilon}{z}\right)\left(\frac{\ell(\ell+1)}{z^2} - \frac{3\epsilon}{z^3}\right)\right] X_\ell^{in} = 0, \tag{17}$$

where $z^* = z + \epsilon \ln(z - \epsilon) = r^*\omega + \epsilon \ln \epsilon$ with $\epsilon \equiv 2M\omega$, and we have suppressed the index ω since it is trivially absorbed in ϵ and z. As noted previously, the post-Newtonian expansion corresponds to expanding X_ℓ^{in} with respect to ϵ and evaluating X_ℓ^{in} at $z \ll 1$ as well as A_ℓ^{in} to required orders in ϵ. This procedure to the first order of ϵ was carried out by Poisson [11]. However, as can be seen from the structure of Eq. (17), the equation would become very complicated if we go beyond the first order in this procedure. Further, it becomes rather unclear how to impose the correct boundary condition that $X_\ell^{in} \propto e^{-iz^*}$ at horizon ($z \to \epsilon$) if we expand the equation naively in powers of ϵ, since z^* involves ϵ in itself.

These difficulties were then resolved by Sasaki [12] by the following choice of the dependent variable. Setting

$$X_\ell^{in} = e^{-i\epsilon \ln(z-\epsilon)} z \xi_\ell(z), \tag{18}$$

we find that Eq. (17) becomes

$$\left[\frac{d}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right)\right] \xi_\ell = \epsilon e^{-iz} \frac{d}{dz}\left[\frac{1}{z^3}\frac{d}{dz}\left(e^{iz} z^2 \xi_\ell(z)\right)\right]. \tag{19}$$

Since ϵ appears only as an overall factor on the r.h.s., this form of the equation is most suited for an iterative treatment. Thus expanding ξ_ℓ with respect to ϵ as

$$\xi_\ell(z) = \sum_{n=0}^{\infty} \epsilon^n \xi_\ell^{(n)}(z), \tag{20}$$

we obtain the recursive equations,

$$\left[\frac{d}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right)\right] \xi_\ell^{(n)}(z) = \epsilon e^{-iz} \frac{d}{dz}\left[\frac{1}{z^3}\frac{d}{dz}\left(e^{iz} z^2 \xi_\ell^{(n-1)}(z)\right)\right], \tag{21}$$

with the lowest order solution given by

$$\xi_\ell^{(0)} = \alpha^{(0)} j_\ell + \beta^{(0)} n_\ell, \tag{22}$$

where j_ℓ and n_ℓ are the usual spherical Bessel functions. The boundary condition is that $\xi_\ell^{(0)}$ be regular at $z = 0$ [11, 12]. Hence $\beta^{(0)} = 0$ and for convenience we set $\alpha_\ell^{(0)} = 1$.

Putting the Regge-Wheeler equation in the above form helped us a lot to calculate the first and second order solutions by iteration [12]. Using a similar transformation of the Teukolsky equation [22], one obtains similar iterative equations for a Kerr black hole and the first and second order solutions are expressed also in closed analytical form [14], which are sufficient to calculate the gravitational wave luminosity to 4PN order.

We were recently informed that Mano, Suzuki and Takasugi [23] have succeeded in developing a systematic method to obtain the solution by using the Coulomb wave functions and the hypergeometric functions, in which calculations to a very high order, including the effect of black hole absorption are considerably simpler than by our method.

2. Gravitational wave luminosity up to 4PN order. Here we only show the final result of the luminosity to 4PN order for circular orbits around a Kerr black hole by summarizing the results obtained in [13, 14, 16, 17]. The orbits are assumed to be on the equatorial plane with the angular frequency Ω. The black hole has mass M and the angular momentum J_{BH}. For the case of spinning particle, the z-component of the spin angular momentum is j_z, and the spin-dependent part is calculated only to 2.5PN order in the first order in the magnitude of spin [17]. Further in this case, we note the orbit precesses slightly with inclination proportional to the radial component of the spin, but it does not affect the luminosity. The result is

$$\begin{aligned}\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{dE}{dt}\right)_N \Bigg\{ & 1 - \frac{1247}{336} v'^2 + \left(4\pi - \frac{11}{4} q - \frac{5}{4}\hat{s}\right) v'^3 \\ & + \left(-\frac{44711}{9072} + \frac{33}{16} q^2 + \frac{31}{8} q\hat{s}\right) v'^4 + \left(\frac{-8191\pi}{672} - \frac{59}{16} q - \frac{13}{16}\hat{s}\right) v'^5 \\ & + \bigg(\frac{6643739519}{69854400} - \frac{1712\gamma}{105} + \frac{16\pi^2}{3} - \frac{3424 \ln 2}{105} \\ & \qquad - \frac{1712}{105} \ln v' - \frac{65\pi}{6} q + \frac{611}{504} q^2\bigg) v'^6 \\ & + \left(\frac{-16285\pi}{504} + \frac{162035}{3888} q + \frac{65\pi}{8} q^2 - \frac{71}{24} q^3\right) v'^7 \\ & + \bigg(-\frac{323105549467}{3178375200} + \frac{232597\gamma}{4410} - \frac{1369\pi^2}{126} + \frac{39931 \ln 2}{294} - \frac{47385 \ln 3}{1568} \\ & \qquad + \frac{232597}{4410} \ln v' - \frac{359\pi}{14} q + \frac{22667}{4536} q^2 + \frac{17}{16} q^4\bigg) v'^8 \Bigg\}, \end{aligned} \tag{23}$$

where

$$\left(\frac{dE}{dt}\right)_N := \frac{32}{5} \left(\frac{\mu}{M}\right)^2 v'^{10}, \tag{24}$$

and

$$v' := (M\Omega)^{1/3}, \qquad q := \frac{J_{BH}}{M^2}, \qquad \hat{s} := \frac{j_z}{\mu M}. \tag{25}$$

3. Future issues. Here I discuss several issues of the Regge-Wheeler-Teukolsky approach to be solved in order to make this approach more realistic and physically more fruitful.

We have succeeded in obtaining the luminosity to 4PN order in the post-Newtonian expansion by the Regge-Wheeler-Teukolsky approach. In the first place, it is of interest to proceed with the calculation to a much higher order, including the effect of black hole absorption. In one respect, it will clarify the convergence property of the post-Newtonian approximation much better than what we know now and will give us more insight into the problem of matching the post-Newtonian spacetime with the fully relativistic spacetime, which is necessarily solved in order to give reasonable initial data of a coalescing compact binary to be used in numerical relativity. In another respect, it will give us detailed information of how the black hole's spacetime geometry affects the luminosity and waveforms of the gravitational radiation emitted by a small mass particle, hence will helps us to understand necessary ingredients to reconstruct the geometry out of the information contained in the waves [24].

However our approach is limited by the condition that one of the binary stars has a much smaller mass than the other, i.e., $\mu \ll M$. In order to make this approach more fruitful, it is then necessary to extend it so that it can take account of finite μ/M effects. Concerning this point, it is important to include the effect of radiation reaction in this approach. At the moment, our approach relies on the adiabatic approximation of the reaction. But even if we accept this approximation, there remains one important problem; that is the backreaction to the Carter constant. In the case of a non-rotating black hole, we may assume the orbit to be in the equatorial plane without loss of generality. Then the energy and the z-component of the angular momentum completely determines the orbit. Hence knowing the energy and angular momentum fluxes averaged over several orbital periods is enough to solve the adiabatic evolution of the orbit. But this will not be the case once we consider a rotating black hole, since the inclination of the orbit away from the equatorial plane becomes meaningful, which is represented by a non-zero value of the Carter constant. The problem is that the Carter constant is not associated with a Killing vector of the geometry, hence no field quantity representing the Carter constant has been known. Thus one cannot calculate the backreaction to the Carter constant by just evaluating the gravitational waves at infinity. In this sense, a simple adiabatic approximation scheme seems to break down anyway as soon as we take the rotation into account. Furthermore, if we take into account spin of the small mass particle, the situation gets worse; the equation of motion is not integrable in this case from the beginning. In order to clarify the validity limit of the adiabatic approximation, as well as to find out the backreaction to the Carter constant, we need to formulate a method in which the non-adiabatic backreaction can be taken into account. In other words, we need to derive a backreaction force term in the equation of motion for the particle. This will be analogous to the post-Newtonian formulation which includes the back-reaction potential to a required order of accuracy.

Whether such a backreaction force term can be obtained for general orbits is very non-trivial. However, at least for a Kerr black hole, we know from the classic work by Chrzanowski [25] that the tensor Green function can be obtained in a separable form. Hence we may calculate the perturbed metric at any spacetime point. Then once we obtain a reliable method of regularizing the self-energy, we will be able to derive the backreaction force term. In this connection, for quasi-periodic orbits, it may be possible to justify the earlier work of Gal'tsov [26], in which the backreaction force term is assumed to be derivable from the radiative Green function, i.e., the retarded minus advanced Green functions. Recently, Ori has proposed another way of deriving the reaction force [27]. He has shown that the reaction force calculated solely in terms of the retarded Green function but averaged over many periods gives the same result as that of Gal'tsov as long as the energy and angular momentum are concerned. Finally, DeWitt-Brehme's covariant approach [28] suggests that one should extract out the tail part of the retarded Green function (i.e, the part that does not vanish inside the light cone) and use it to contruct the radiation reaction force term. However, it is not clear how this extraction can be actually done.

Another important but difficult problem is to take into account the effect non-linear in μ/M in the equation of motion. In the post-Newtonian approach, we know this effect appears already in 2PN order in the luminosity formula [6], well before the radiation reaction comes into play. To my knowledge, however, no serious attempt has been made in this direction so far.

Acknowledgements. I would like to thank H. Asada, Y. Mino, T. Nakamura, E. Poisson, M. Shibata, T. Tagoshi and T. Tanaka for fruitful discussions and collaboration.

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