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LISSAJOUS KNOTS AND BILLIARD KNOTS

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Abstract. We show that Lissajous knots are equivalent to billiard knots in a cube. We consider also knots in general 3-dimensional billiard tables. We analyse symmetry of knots in billiard tables and show in particular that the Alexander polynomial of a Lissajous knot is a square modulo 2.

0. Introduction. A Lissajous knot K is a knot in R^3 given by the parametric equations

$$x = \cos(\eta_x t + \phi_x), \quad y = \cos(\eta_y t + \phi_y), \quad z = \cos(\eta_z t + \phi_z),$$

for integers η_x, η_y, η_z. A *Lissajous link* is a collection of disjoint Lissajous knots.

The fundamental question was asked in [BHJS94]: which knots are Lissajous?

One defines a *billiard knot* (or *racquetball knot*) as the trajectory inside a cube of a ball which leaves a wall at rational angles with respect to the natural frame, and travels in a straight line except for reflecting perfectly off the walls; generically it will miss the corners and edges, and will form a knot. We will show that these knots are precisely the same as the Lissajous knots. We will also speculate about more general billiard knots, e.g. taking another polyhedron instead of the ball, considering a non-Euclidean metric, or considering the trajectory of a ball in the configuration space of a flat billiard. We will illustrate these by various examples. For instance, the trefoil knot is not a Lissajous knot

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but we can easily realize it as a billiard knot in a room with a regular triangular floor ([1]); Fig. 0.1.

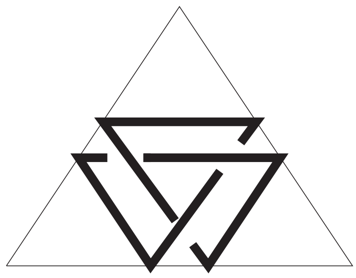


Fig. 0.1. The left handed trefoil knot in a room with a regular triangular floor ("Odin's triangle")

In the third part of the paper we discuss the symmetry of billiard knots. We sharpen the observation of [BHJS94] that a Lissajous knot is either strongly + amphicheiral or Z_2-periodic. We show that in the Z_2-periodic case the linking number of an axis of the Z_2 action with the knot is equal to ± 1. We use this to show that any Lissajous knot has an Alexander polynomial congruent to a square modulo 2. We also study other billiard knots which exibit symmetry (e.g. knots in a cylinder). Finally, in the fourth part, we consider knots given by trajectories which are not time-reversible. We are motivated by a movement in ponds of a unicellular slipper-shaped organism *Paramecium Caudatum*.

1. Deformation. We will consider here a family of curves which generalize Lissajous and billiard curves. We will show how to deform Lissajous knots to billiard knots inside this family.

Consider the family $\mathcal{F}$ of continuous fuctions $f : R \to R$ which satisfy the following properties:

(i) $f(t+1) = -f(t)$ (skew-period 1),
(ii) $f(-t) = f(t)$ (even function),
(iii) $f(0) = 1$ (initial condition),
(iv) f is strictly decreasing on the interval $(0, \frac{1}{2})$.

As a consequence of our conditions we immediately have the following:

(v) f has period 2; $f(k) = (-1)^k$ for any integer k.
(vi) $f(\frac{1}{2}+t) = -f(\frac{1}{2}-t)$; i.e. $f(\frac{1}{2}+t))$ is an odd function, in particular $f(\frac{1}{2}) = 0$.
(vii) $\mathcal{F}$ is in bijection with strictly decreasing functions from $[0, \frac{1}{2}]$ onto $[1, 0]$.
(viii) $\mathcal{F}$ is a convex space; that is, if $f, g \in \mathcal{F}$, then $(1-s)f + sg$ is in $\mathcal{F}$ for any $s \in [0, 1]$.

We show (vi) as an example: $f(\frac{1}{2}+t) = -f(-\frac{1}{2}+t)$ by (i) and $-f(-\frac{1}{2}+t) = -f(\frac{1}{2}-t)$ by (ii).

([1]) This same figure was, according to [Cro95], used by the Norse people of Scandinavia and known as "Odin's triangle" or "Walknot".

The simplest examples of our fuctions are $\cos(\pi t)$ and the piecewise linear "sawtooth" function $p(t) = 2||t|-2E[\frac{1}{2}|t|]-1|-1$, where $E[x]$ is the greatest integer part of x; Fig. 1.1.

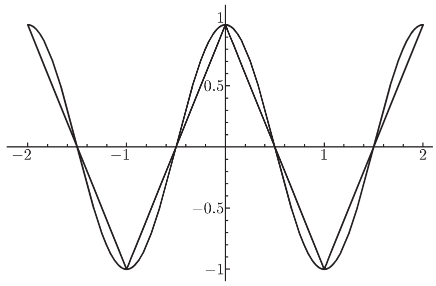


Fig. 1.1

Consider now any $f \in \mathcal{F}$.

LEMMA 1.1. *Let $f_1(t) = f(n_1 t + c_1)$, $f_2(t) = f(n_2 t + c_2)$, where n_1, n_2 are co-prime integers. Then the closed curve $(f_1(t), f_2(t)) : R \to R^2$ has period 2 and one of the following conditions is satisfied*:

(a) *For some integers k_1, k_2,*

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{n_1 k_2 - n_2 k_1}{n_1 n_2}$$

and our curve reaches two corners of the square $[-1,1] \times [-1,1]$ at $t_0 = -\frac{c_1}{n_1} + \frac{k_1}{n_1}$ and $t_0 + 1$ (bounces off at these corners). The curve restricted to $[t_0, t_0+1]$ is locally embedded with $(n_1-1)(n_2-1)/2$ double points.

(b) *For $t \in [0,2]$ it describes a closed locally embedded curve with $2n_1n_2 - n_1 - n_2$ double points for parameters t', t'' given by*:

$$t' = \left(\frac{k_2}{n_2} + \frac{k_1}{n_1}\right) - \frac{c_2}{n_2} \quad \textit{and} \quad t'' = \left(\frac{k_2}{n_2} - \frac{k_1}{n_1}\right) - \frac{c_2}{n_2} \quad \textit{and}$$

$$t' = \left(\frac{k_2}{n_2} + \frac{k_1}{n_1}\right) - \frac{c_1}{n_1} \quad \textit{and} \quad t'' = \left(-\frac{k_2}{n_2} + \frac{k_1}{n_1}\right) - \frac{c_1}{n_1}.$$

P r o o f. $f(t_1) = f(t_2)$ iff $t_1 = \pm t_2 + 2k$. This follows from the fact that $f(t)$ is strictly increasing on the interval $[-1,0]$ and strictly decreasing on the interval $[0,1]$, symmetric with respect to the maximum and periodic. Therefore

$$f_1(t') = f_1(t'') \quad \text{and} \quad f_2(t') = f_2(t'')$$

reduces to

$$n_1 t' + c_1 = \pm n_1(t'' + c_1) + 2k_1 \quad \text{and} \quad n_2 t' + c_2 = \pm n_2(t'' + c_2) + 2k_2.$$

Thus one has to consider four possibilities:

$$(++) \qquad n_1 t' + c_1 = n_1 t'' + c_1 + 2k_1, \quad n_2 t' + c_2 = n_2 t'' + c_2 + 2k_2.$$

Therefore $t' - t'' = \frac{2k_1}{n_1} = \frac{2k_2}{n_2}$. Thus $2k_1 n_2 = 2k_2 n_1$ and because n_1 is relatively prime with respect to n_2, we have k_1 as a multiple of n_1 and k_2 of n_2. Finally $t' - t'' = 2m$ for some integer m, so our closed curve has no double points related to the equalities $(++)$.

$$(+-) \qquad n_1t' + c_1 = n_1t'' + c_1 + 2k_1, \quad n_2t' + c_2 = -n_2t'' - c_2 + 2k_2.$$

Thus $t' = (\frac{k_2}{n_2} + \frac{k_1}{n_1}) - \frac{c_2}{n_2}$ and $t'' = (\frac{k_2}{n_2} - \frac{k_1}{n_1}) - \frac{c_2}{n_2}$.

If k_1 is a multiple of n_1 then $t' \equiv t''$ mod 2 so we do not deal with a crossing. Thus to count all crossings we have to consider $k_1 \in [1, n_1 - 1]$, $k_2 \in [0, n_2 - 1]$. These describe $(n_1 - 1)n_2$ crossings, and the crossings are all different since for pairs parametrizing crossings (t_1', t_1'') and (t_2', t_2''), one has to have $|t_1' - t_2'| < 2$ and $t_1' \neq t_2''$.

$$(-+) \qquad n_1t' + c_1 = -n_1t'' - c_1 + 2k_1, \quad n_2t' + c_2 = n_2t'' + c_2 + 2k_2.$$

Thus $t' = (\frac{k_2}{n_2} + \frac{k_1}{n_1}) - \frac{c_1}{n_1}$ and $t'' = (-\frac{k_2}{n_2} + \frac{k_1}{n_1}) - \frac{c_1}{n_1}$. The consideration of $(+-)$ remains valid, only the roles of f_1 and f_2 are interchanged. Thus we get here $(n_2 - 1)n_1$ different crossings obtained for t' and t'' such that $k_1 \in [0, n_1 - 1]$, $k_2 \in [1, n_2 - 1]$.

$$(--) \qquad n_1t' + c_1 = -n_1t'' - c_1 + 2k_1, \quad n_2t' + c_2 = -n_2t'' - c_2 + 2k_2.$$

Thus $n_1c_2 - n_2c_1 = n_1k_2 - n_2k_1$, or equivalently $\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k_2}{n_2} - \frac{k_1}{n_1}$. This corresponds to case (a) of Lemma 1.1. One can comment on this condition:

First of all, it is independent of t; thus for any t' there is a unique (up to period 2) t'' ($t'' = -t' - \frac{2c_1}{n_1} + \frac{2k_1}{n_1}$) such that $f_1(t') = f_1(t'')$, and we can say that our curve "bends on itself" (see Fig. 1.2). To see this clearly, notice that our curve reaches two corners of

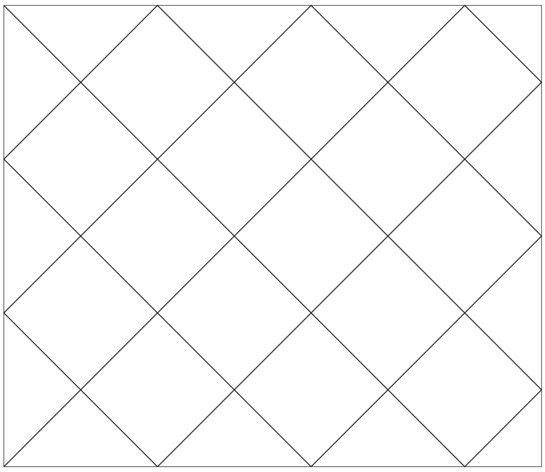


Fig. 1.2. The curve $(f(6t + 1), f(7))$ bounces at corners and bends on itself.

the square $[-1, 1] \times [-1, 1]$, namely reparametrize our curve $(f_1(t), f_2(t))$ by changing t to $t - \frac{c_1}{n_1} + \frac{k_1}{n_1}$ (which by above is the same as changing t to $t - \frac{c_2}{n_2} + \frac{k_2}{n_2}$). In the new parametrization we get the curve $(f(n_1t + k_1), f(n_2t + k_2))$, so it reaches corners at $t = 0$ and $t = 1$, then bounces at these corners and "bends on itself" (by condition (ii) and the fact that $f(1 + t) = f(1 - t)$). In particular if n_1 and n_2 are odd then the curve goes through 0.

To finish the proof of Lemma 1.1 we have to analyse when crossings described in the $(+-)$ and $(-+)$ cases are different.

Let

$$(t_1', t_1'') = \left(\left(\frac{k_2}{n_2} + \frac{k_1}{n_1}\right) - \frac{c_2}{n_2}, \left(\frac{k_2}{n_2} - \frac{k_1}{n_1}\right) - \frac{c_2}{n_2}\right)$$

and

$$(t_2', t_2'') = \left(\left(\frac{k_2'}{n_2} + \frac{k_1'}{n_1}\right) - \frac{c_1}{n_1}, \left(-\frac{k_2'}{n_2} + \frac{k_1'}{n_1}\right) - \frac{c_1}{n_1}\right)$$

be parameters giving crossings in $(+-)$ and $(-+)$, respectively. Thus $t_1' = t_2'$ or $t_1'' = t_2'$ if and only if

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k_2 - k_2'}{n_2} - \frac{k_1' \pm k_1}{n_1}.$$

This is exactly the (a) case of Lemma 1.1 and it was analysed in case $(--)$; that is, a pair of crossings coincide because our curve "bends on itself".

If the crossings of cases $(+-)$ and $(-+)$ are different, we have $(n_1-1)n_2+(n_2-1)n_1 = 2n_1n_2 - n_1 - n_2$ of crossings. This completes the proof of Lemma 1.1. ∎

Let $f \in \mathcal{F}$ and consider a closed curve in a 3-dimensional cube, parametrized by $F(t) = (f_1(t), f_2(t), f_3(t))$, where $f_i(t) = f(n_i t + c_i)$. Of course, our curve has period 2.

LEMMA 1.2. *$F(t)$, for $t \in [0,2]$, parametrizes a simple closed curve (a knot) unless*

(i) $\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{m}{n_1 n_2}$, *or*
(ii) $\frac{c_3}{n_3} - \frac{c_1}{n_1} = \frac{m'}{n_1 n_3}$, *or*
(iii) $\frac{c_3}{n_3} - \frac{c_2}{n_2} = \frac{m''}{n_2 n_3}$ *for some integer m (or m' or m'').*

Furthermore if (i) (*resp.* (ii) *or* (iii)) *holds but not all three, then we deal with a singular knot with $n_3 - 1$ (resp. $n_2 - 1$ or $n_1 - 1$) double points.*

Proof. We want to find all possible self-crossings of the curve F (on $[-1,1]$), so we have to solve the system of equations

$$f_1(t') = f_1(t''), \quad f_2(t') = f_2(t''), \quad f_3(t') = f_3(t''),$$

or equivalently

$$n_1t'+c_1 = \pm n_1t''+c_1+2k_1, \quad n_2t'+c_2 = \pm n_2t''+c_2+2k_2, \quad n_3t'+c_3 = \pm n_3t''+c_3+2k_3.$$

We have to consider 8 possibilities, which we can denote succinctly as $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ where $\varepsilon_i = \pm$. We base our analysis on Lemma 1.1 and its proof.

$(++\varepsilon)$ If two ε's are $+$, say $\varepsilon_1, \varepsilon_2$, then by case $(++)$ of the proof of Lemma 1.1, we will not produce a self-crossing.

$(---)$ From case $(--)$ of the proof of Lemma 1.1 applied to any pair f_i, f_j, we get the system of equations

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k_2}{n_2} - \frac{k_1}{n_1}, \quad \frac{c_3}{n_3} - \frac{c_2}{n_2} = \frac{k_3}{n_3} - \frac{k_2}{n_2}, \quad \frac{c_1}{n_1} - \frac{c_3}{n_3} = \frac{k_1}{n_1} - \frac{k_3}{n_3};$$

of course, only two of these are independent. Observe, as in case $(--)$, that by reparametrizing the curve by a shift $t \to (t - \frac{c_1}{n_1} - \frac{k_1}{n_1})$, one gets the curve $(f(n_1t - k_1), f(n_2t - k_2), f(n_3t - k_3))$ and that this curve hits corners at $t = 0$ and $t = 1$ then bounces at these corners and "bends on itself".

$(--+)$ If two ε's are $-$, say $\varepsilon_1, \varepsilon_2$, then by case $(--)$ of the proof of Lemma 1.1,

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k_2}{n_2} - \frac{k_1}{n_1}, \qquad t' = -t'' - \frac{2c_1}{n_1} + \frac{2k_1}{n_1}$$

and by case $(-+)$ applied to (f_1, f_3) and (f_2, f_3),

$$t' = \left(\frac{k_3}{n_3} + \frac{k_1}{n_1}\right) - \frac{c_1}{n_1} \quad \text{and} \quad t'' = \left(-\frac{k_3}{n_3} + \frac{k_1}{n_1}\right) - \frac{c_1}{n_1}, \quad \text{and}$$

$$t' = \left(\frac{k_3}{n_3} + \frac{k_2}{n_2}\right) - \frac{c_2}{n_2} \quad \text{and} \quad t'' = \left(-\frac{k_3}{n_3} + \frac{k_2}{n_2}\right) - \frac{c_2}{n_2}.$$

From the first and the second conditions, one gets

$$t' = \frac{k_3}{n_3} + \frac{k_1}{n_1} - \frac{c_1}{n_1}.$$

k_1 and k_2 are determined, so the only choice we have is for k_3, thus we get $k_3 - 1$ double points. Assuming that $(---)$ cannot be satisfied, these are the only double points of our closed curve.

This completes the proof of Lemma 1.2. ∎

THEOREM 1.3. *For given integers n_1, n_2, n_3 and real numbers c_1, c_2, c_3 the knot* (*up to ambient isotopy*) $F(t) = (f_1(t), f_2(t), f_3(t))$, *where $f_i(t) = f(n_i t + c_i)$, does not depend on the choice of $f \in \mathcal{F}$. This theorem also holds for the case of a singular knot.*

Proof. Let $f_0, f_1 \in \mathcal{F}$. Consider the homotopy between f_0 and f_1: $f_s = (1-s)f_0 + sf_1$. By convexity of $\mathcal{F}$, $f_s \in \mathcal{F}$. Thus we have also the homotopy $F_s = (1-s)F_0 + sF_1$ between $F_0(t) = ((f_0)_1(t), (f_0)_2(t), (f_0)_3(t))$ and $F_1(t) = ((f_1)_1(t), (f_1)_2(t), (f_1)_3(t))$. Furthermore if F_0 is a knot, F_t is also a knot and thus represents the same topological type. Similarly, if F_0 is a singular knot, then F_s has double points for the same parameters t as F_0 for any s (by Lemma 1.1). The proof of Theorem 1.3 is complete. ∎

COROLLARY 1.4. *Lissajous and billiard knots are the same up to ambient isotopy.*

2. General billiards. There is extensive literature devoted to billiards, including 3-dimensional billiard tables [GKT94, Ta95]. The problem of closed trajectories has also been widely studied [BGKT94, GSV92, KMS86, Ta95]. Apparently, however, the problem of knots as trajectories of a ball in a billiard table has not been studied before ([2]).

We give, in this section, some examples of knots as billiard trajectories and formulate a few conjectures, but the field is widely open for future research; and as a new field, it has plenty of possible directions and problems (from easy to very difficult ones).

We start from the general definition.

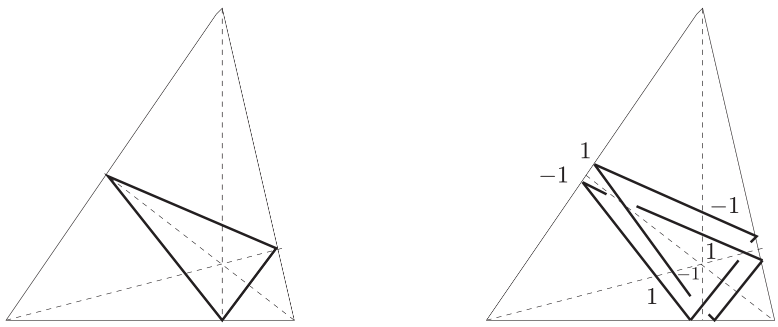
DEFINITION 2.1 ([Ta95]). A *billiard table* (or *racquetball room*) is a Riemannian manifold M with a piecewise smooth boundary. The *billiard dynamical system* in M is generated by the free motion of a mass-point (called a *billiard ball*) subject to elastic reflection off the boundary. This means that the point moves along a geodesic line in M with a constant speed until it hits the boundary. At a smooth boundary point the billiard

([2]) For us the initial motivation was the possiblity of the hand drawing of Lissajous knots.

ball reflects so that the tangential component of its velocity remains the same, while the normal component changes its sign. If the billiard ball hits a corner, its further motion is usually not defined (for us the important exception is a right dihedral angle at an edge and a cubic corner in the polyhedron in R^3).

A billiard knot (or link) is a simple closed trajectory (trajectories) of a ball in a 3-dimensional billiard table. ([3]) The simplest billiards to consider would be polytopes (finite convex polyhedra in R^3). But even for Platonian bodies we know nothing of the knots they support except in the case of the cube. It seems that polytopes which are the products of polygons and the interval ($[-1,1]$) (i.e. polygonal prisms) are more accessible. This is the case because diagrams of knots are billiard trajectories in 2-dimensional tables. We will list some examples below.

Example 2.2. (i) The trivial knot and the trefoil knot are the trajectories of a ball in a room (prism) with an acute triangular floor. In Fig. 2.1(a), the diagram of the trivial knot is an inscribed triangle Δ_I whose vertices are the feet of the triangle's altitudes. If we move the first vertex of Δ_I slightly, each of its edges splits into two and we get the diagram of the trefoil. We should be careful with the altitude of the trajectory: We start from level 1 at the vertex close to the vertex of Δ_I and opposite to the shortest edge of Δ_I. Then we choose the vertical parameter so that the trajectory has 3 maxima and three minima (Fig. 2.1(b)).



(a) (b) The trefoil knot as a trajectory of a ball in a room whose floor is an acute triangle

Fig. 2.1

(ii) The trivial knot is a trajectory of a ball in a room with a right triangular floor, Fig. 2.2.

(iii) If the floor of a room is a general obtuse triangle, it is an open problem whether any knot can be realized as the trajectory of a ball in it. However we have the general theorem that periodic points are dense (in the phase space of the billiard flow) in a rational polygon (that is, all polygonal angles are rational with respect to π) [BGKT94].

([3]) One can also consider closed trajectories in the phase space of a 2-dimensional billiard table but we will not pursue this possibility here.

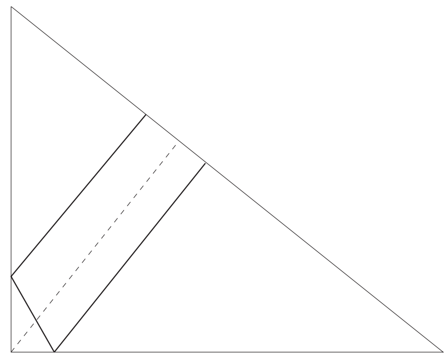
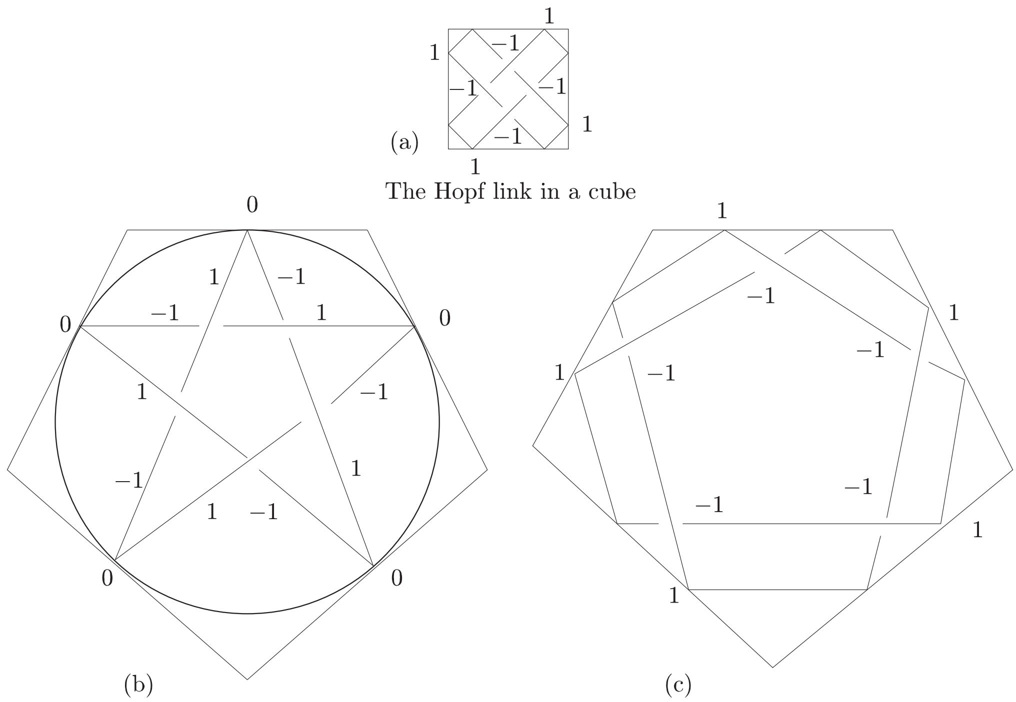


Fig. 2.2. The trivial knot can be realized as a trajectory of a ball in any room with a right triangular floor.

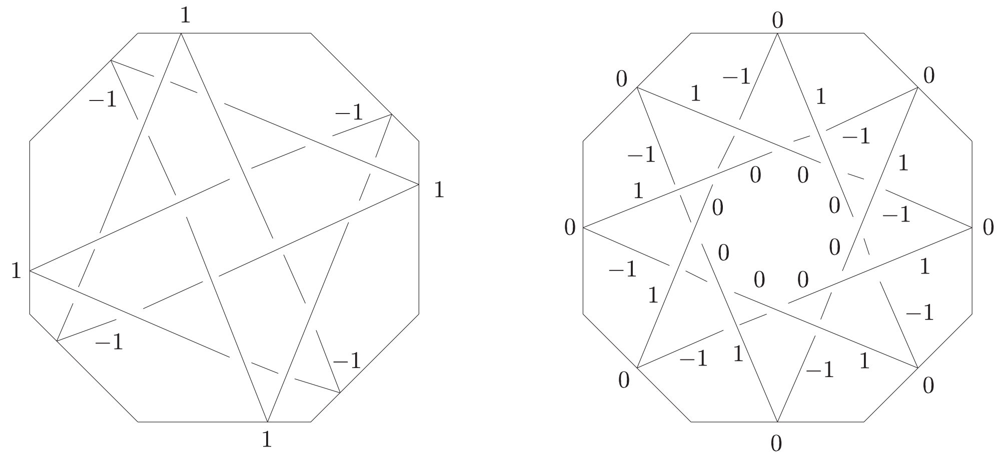
Example 2.2(i) is of interest because it was shown in [BHJS94] that the trefoil knot is not a Lissajous knot and thus it is not a trajectory of a ball in a room with a rectangular floor. More generally we show in Section 3 that no nontrivial torus knot is a Lissajous knot. However, we can construct infinitely many torus knots in prisms and in the cylinder.



The torus knot of type $(5, 2)$ as a trajectory of a ball in a room with a regular pentagonal floor

Fig. 2.3

EXAMPLE 2.3. (i) Any torus knot (or link) of type $(n, 2)$ can be realized as a trajectory of a ball in a room whose floor is a regular n-gon ($n \geq 3$). Fig. 0.1 shows the $(3, 2)$ torus knot (trefoil) in the regular triangular prism; Fig. 2.3(a) depicts the $(4, 2)$ torus link in the cube; and Fig. 2.3(b)(c) illustrates the $(5, 2)$ torus knot in a room with a regular pentagonal floor.



(a) The torus knot of type $(4, 3)$
as a trajectory of a ball
in a room with a regular octagonal floor

(b) The torus knot of type $(8, 3)$
as a trajectory of a ball
in a room with a regular octagonal floor

Fig. 2.4

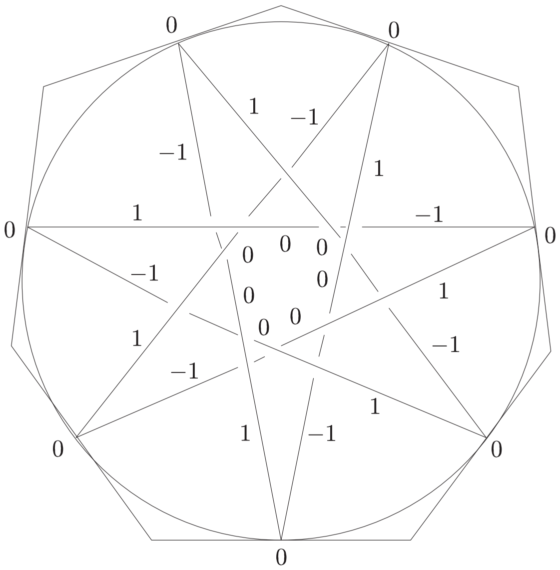


Fig. 2.5. The torus knot of type $(7, 3)$ realized as a trajectory of a ball in a room with a regular heptagonal floor.

(ii) The $(4,3)$ torus knot is a trajectory of a ball in a room with regular octagonal floor; Fig. 2.4(a).

(iii) Figures 2.4(b) and 2.5 illustrate how to construct a torus knot (or link) of type $(n,3)$ in a room with a regular n-gonal floor for $n \geq 7$.

(iv) Any torus knot (or link) of type (n,k), where $n \geq 2k+1$, can be realized as a trajectory of a ball in a room with a regular n-gonal floor. The pattern generalizes that of Figures 2.3(b), 2.4(b) and 2.5. Edges of the diagram go from the center of the ith edge to the center of the $(i+k)$th edge of the n-gon. The ball bounces from walls at altitude 0 and its trajectory has n maxima and n minima. The whole knot (or link) is Z_n-periodic.

Example 2.4. Let D be a closed billiard trajectory on a 2-dimensional polygonal table. If D is composed of an odd number of segments, then we can always find the "double cover" closed trajectory $D^{(2)}$ in the neighborhood of D (each segment will be replaced by two parallel segments on the opposite sides of the initial segment). This idea can be used to construct, for a given billiard knot K in a polygonal prism (the projection D of K having an odd number of segments), a 2-cable $K^{(2)}$ of K as a billiard trajectory (with projection $D^{(2)}$). This idea is illustrated in Fig. 2.1 and 2.3(c) (the (5,2) torus knot as a 2-cable of a trivial one). Starting from Example 2.3(iv) we can construct a 2-cable of a torus knot of the type (n,k) in a regular n-gonal prism, for n odd and $n \geq 2k+1$.

It follows from [BHJS94] that 3-braid alternating knots of the form $(\sigma_1\sigma_2^{-1})^{2k}$ are not Lissajous knots as they have a non-zero Arf invariant (see Section 3). For $k=1$ we have the figure eight knot and for $k=2$ the 8_{18} knot [Ro76].

Example 2.5. (i) The Listing knot (figure eight knot) can be realized as a trajectory of a ball in a room with a regular octagonal floor, Fig. 2.6 [4].

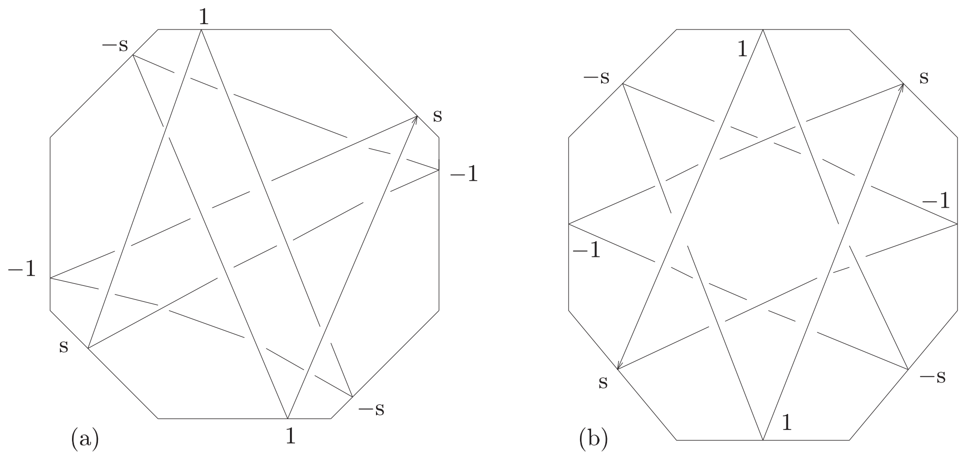


Fig. 2.6. The Listing (figure eight) knot as a trajectory of a ball in a room with a regular octagonal floor.

[4] This drawing was motivated by R. Randell's drawing at his talk at Banach Center in August 1995 and ascribed to E. Flapan, M. Meissen and J. Van Buskirk.

(ii) Fig. 2.7 describes the knot 8_{18} as a trajectory of a ball in a room with a regular octagonal floor. This pattern can be extended to obtain the knot (or link) which is the closure of the three braid $(\sigma_1\sigma_2^{-1})^{2k}$ in a regular $4k$-gonal prism ($k > 1$).

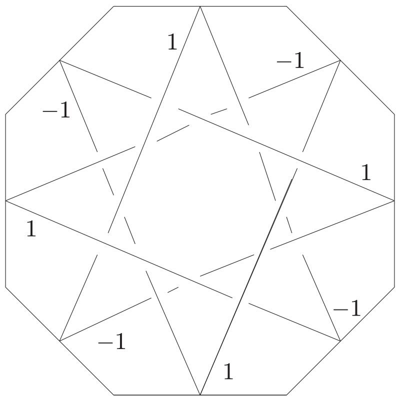


Fig. 2.7. The knot 8_{18} [Ro76], realized as a trajectory of a ball in a room with a regular octagonal floor.

We do not know if any polytope will support an infinite number of different knot types, however such is the case for the cylinder $D^2 \times [-1,1]$.

Example 2.6. (i) Any torus knot (or link) of type (n,k), where $n \geq 2k+1$, can be realized as a trajectory of a ball in the cylinder; compare Fig. 2.3(b), Fig. 2.4(b) and Fig. 2.5.

(ii) Every knot (or link) which is the closure of the three braid $(\sigma_1\sigma_2^{-1})^{2k}$ can be realized as the trajectory of a ball in the cylinder. See Fig. 2.6(b) for the case of $k = 1$ (Listing knot) and Fig. 2.7 for the case of $k = 2$ and the general pattern.

Any type of knot can be obtained as a trajectory of a ball in some polyhedral billiard (possibly very complicated). To see this, consider a polygonal knot in R^3 and place "mirrors" (walls) at any vertex, in such a way that the polygon is a "light ray" (ball) trajectory.

Conjecture 2.7. *Any knot type can be realized as the trajectory of a ball in a polytope.*

Conjecture 2.8. *Any polytope supports an infinite number of different knot types.*

Problem 2.9. 1. *Is there a convex polyhedral billiard in which any knot type can be realized as the trajectory of a ball?*

2. *Can any knot type be realized as the trajectory of a ball in a room with a regular polygonal floor?*

3. *Which knot types can be realized as trajectories of a ball in a cylinder* ($D^2 \times [-1,1]$)?

3. Symmetry of billiard knots. Until now we have been unable to classify all billiard knots in any nontrivial 3-dimensional billiard (trivial being for example D^3 which has only the trivial knot as a trajectory). We can, however, exclude some knot types as closed trajectories in some billiards, due to symmetry principles. This was first observed for Lissajous knots in [BHJS94]. Namely, let $f \in \mathcal{F}$ (see Section 1), and $f_i(t) = f(n_i t + c_i)$, $i = 1, 2, 3$. We can assume that n_1, n_2 are odd. We call the knot described by (f_1, f_2, f_3) *even* if n_3 is even and *odd* otherwise.

DEFINITION 3.1. (i) A knot in R^3 is called *strongly+ amphicheiral* if it has a realization in R^3 which is preserved by a central symmetry $((x, y, z) \to (-x, -y, -z))$ (this symmetry changes orientation of R^3). "Plus" means that the involution preserves the orientation of the knot.

(ii) A knot is called *n-periodic* if there exists an action of Z_n on S^3 which preserves the knot and the set of fixed points of the action is a circle disjoint from the knot.

Because $f(t + 1) = -f(t)$ we have

THEOREM 3.2. *An even Lissajous knot is Z_2-periodic and an odd Lissajous knot is strongly + amphicheiral.*

We will strengthen the above theorem by showing the following:

THEOREM 3.3. *In the even case the linking number of the axis of the Z_2-action with the knot is equal to ± 1.*

Sketch of proof ([5]). Consider f_1, f_2 defined as before (n_1, n_2 are co-prime odd numbers).

LEMMA 3.4. $(f_1, f_2)(t)$ *goes through* 0 *if and only if*

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k}{n_1 n_2} \quad \textit{for some integer } k.$$

Proof. (i) If $(f_1, f_2)(t)$ goes through 0, then for some integers h_1, h_2,

$$n_1 t + c_1 = \frac{1}{2} + h_1, \qquad n_2 t + c_2 = \frac{1}{2} + h_2,$$

thus

$$t = -\frac{c_1}{n_1} + \frac{1}{2n_1} + \frac{h_1}{n_1} = -\frac{c_2}{n_2} + \frac{1}{2n_2} + \frac{h_2}{n_2}.$$

Finally

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{h_2 n_1 - h_1 n_2}{n_1 n_2} + \frac{n_1 - n_2}{2n_1 n_2} = \frac{k}{n_1 n_2}$$

for some integer k.

(ii) If

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k}{n_1 n_2},$$

then we can find h_1 and h_2 such that $k = h_2 n_1 - h_1 n_2 + \frac{1}{2}(n_1 - n_2)$. Then, if we choose t such that $n_1 t + c_1 = \frac{1}{2} + h_1$, then $n_2 t + c_2 = \frac{1}{2} + h_2$. For this t, $f_1(t) = f_2(t) = 0$. ∎

LEMMA 3.5. *If (f_1, f_2) does not go through zero then there is a well defined linking number*, $lk(f_1, f_2)$, *of the curve with* 0, *and it is equal to* $+1$ *or* -1.

([5]) The detailed proof will be given elsewhere; see [Pr95].

P r o o f. The idea of the proof is as follows: We show that for a generic c_i the assumption of the theorem holds; and in the degenerate case, when the curve goes through 0, it also reaches antipodal corners of the square at which it bounces and "bends on itself". A small deformation of the parameters c_i naturally gives the linking number ± 1 and any deformation omitting 0 does not change this linking number.

More precisely: As proven in Section 1, the case of $\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{k}{n_1 n_2}$ is exactly the case when the curve reaches two antipodal corners of the square $[-1,1] \times [-1,1]$, bounces off at these corners and "bends on itself". In all other cases the curve $(f_1, f_2)(t)$ has $2n_1n_2 - n_1 - n_2$ double points and is embedded outside double points.

For simplicity (e.g. for better visualization of the curve but without loss of generality) let us consider the case of billiard curves. Then $(f_1, f_2)(t)$ is a piece-wise linear curve without horizontal segments. We can visualize the linking number by drawing an interval from $(0,0)$ to $(0,1)$ and counting signed intersections of the interval with segments of the billiard trajectory. (f_1, f_2) is a continuous function with respect to parameters c_i. Fix c_1 (we can assume $c_1 = 0$) and consider a curve (f_1, f_2') with parameter $c_2' = c_2 + \epsilon$ for small ϵ. When we move from $\epsilon = 0$ to ϵ close to (but different than) zero, then every segment splits into two segments symmetric with respect to $(0,0)$ (we also create new segments close to corners where the curve was "bent on itself"). With the exception of the segments which arise from the one going through $(0,0)$ all other split segments will cancel their contributions to the linking number. Thus the linking number will be ± 1. Any further deformation omitting $(0,0)$ will preserve the linking number. ∎

COROLLARY 3.6. *The Alexander polynomial of every Lissajous knot is a square modulo* 2.

P r o o f. For a strongly + amphicheiral knot the Alexander polynomial is a square by Hartley and Kawauchi [HK79]. For Z_2-periodic knots we can use the following theorem of Murasugi (assuming $k = \pm 1$). ∎

THEOREM 3.7 ([Mu71]). *Let L be an r-periodic oriented link with linking number k with the fixed point set axis. Then*

$$\Delta_L(t) \equiv \Delta^r_{L_*}(t)(1 + t + t^2 + \ldots + t^{|k|-1})^{r-1} \bmod r$$

where $L_* = L/Z_r$ and r is a prime number.

COROLLARY 3.8. (i) ([BHJS94]) *The Arf invariant of the Lissajous knot is* 0.

(ii) *A nontrivial torus knot is not a Lissajous knot.*

(iii) *For $\eta_z = 2$ a Lissajous knot is a two bridge-knot and its Alexander polynomial is congruent to* 1 *modulo* 2.

P r o o f. (i) It follows from Corollary 3.6 and the fact that the Arf invariant is, modulo 2, the first nontrivial coefficient of the Conway–Alexander polynomial.

(ii) The Alexander polynomial of a nontrivial torus knot is not a square modulo 2.

(iii) A Lissajous knot with $\eta_z = 2$ has two maxima so it is a two-bridge knot. If we divide it by the Z_2-action we get a one-bridge knot as an orbit (thus trivial knot). Finally, we use the Murasugi theorem ($\Delta_{L_*} = 1$). ∎

Motivated by the case of Z_2-periodic knots we propose

CONJECTURE 3.9. *Turks head knots* (*e.g. the closure of the 3-string braids* $(s_1\bar{s}_2)^{2k+1}$) *are not Lissajous.* (Observe that they are strongly + amphicheiral.)

We do not think, as the above conjecture shows, that the converse to Theorem 3.2 holds. However for Z_2-periodic knots it may hold (the method sketched in Section 0.4 of [BHJS94] may work).

PROBLEM 3.10. *Let K be a Z_2-periodic knot*, *such that the linking number of the axis of the Z_2-action with K is equal to ± 1. Is K an even Lissajous knot*?

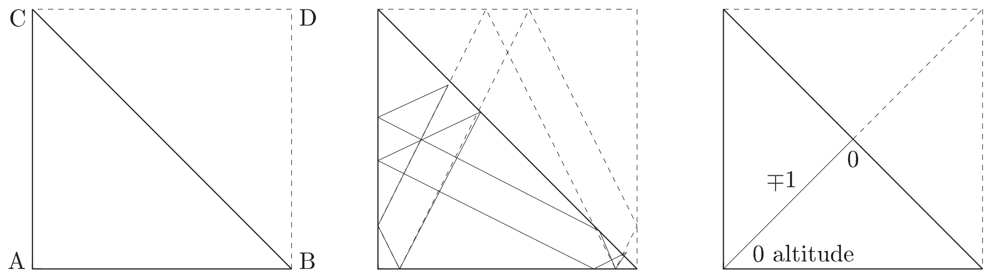
The first prime knots (in the knot tables [Ro76]) which may or may not be Lissajous are 7_5, 8_3, 8_6.

We can divide billiard knots in the cylinder into two classes, even and odd, depending on the number of segments in the projection into the base. Let $f_{1,2}(t)$ parametrize the projection of the knot into the base, and $f_3(t)$ describe the vertical direction, where $t \in [-1,1]$. We have $f_{1,2}(t+\frac{2}{n}) = e^{2\pi\frac{k}{n}} f_{1,2}(t)$, where $n > 2k$; and $f_3(t+\frac{1}{m}) = -f_3(t)$. n is the number of segments in the projection of the knot into the base and m is the number of maxima of the trajectory of K.

THEOREM 3.11. *An even billiard knot in a cylinder is either Z_2-periodic or strongly+ amphicheiral*, *depending on whether it has an even or odd number of maxima. In the Z_2-periodic case the linking number of an axis of the Z_2 action with the knot is equal to $\pm k$.*

Proof. For even n one has $f_{1,2}(t+1) = -f_{1,2}(t)$ and $f_3(t+1) = f_3(t)$ or $-f_3(t)$ depending on whether the knot has an even or odd number of maxima; t is considered modulo 2. The value of the linking number is immediately visible from the diagram (compare Fig. 2.4(b) for the case $n=8,\ k=3$). Thus Theorem 3.11 follows. ∎

A similar result also holds for an isosceles right triangular prism.



(a) (b) (c)

Fig. 3.1

Let Δ_{ABC} denote a floor (isosceles right triangle) of the prism. We choose coordinates $A=\{-1,-1\}, B=\{1,-1\}$, and $C=\{-1,1\}$ for the vertices of the triangle; Fig. 3.1(a). Let $f_{1,2}(t)$ $(t \in [-1,1])$ be a parametrization of a closed trajectory of a ball in Δ_{ABC} disjoint from corners of the triangle. This trajectory can be "unfolded" into the trajectory in the square $\square_{ABCD} = ([-1,1]\times[-1,1])$; Fig. 3.1(b), which can be parametrized by

$(f_1(t), f_2(t))$ where $f_i(t) = f(n_i t + c_i)$ and $f(t)$ is the "sawtooth" function of Fig. 1.1. If $(f_{1,2}(t), f_3(t))$ describes a knot in the prism then the associated "unfolded" curve $(f_1(t), f_2(t), f_3(t))$ describes a knot in a cube. In particular n_1, n_2, and n_3 are pairwise relatively prime integers.

THEOREM 3.12. *Let* $(f_{1,2}(t), f_3(t))$ *describe a knot*, K, *in an isosceles right triangular prism. If* n_1 *and* n_2 *are odd then*:

(i) n_3 *has to be odd*, *except for the degenerate case desribed in Fig.* 3.1(*c*). *Here* $n_3 = 2$ *and* $n_1 = n_2 = 1$; *furthermore*, *the knot in the triangualar prism is trivial*, *but the "unfolded knot" in the cube has one singularity* (*at the point* $(0, 0, s)$ *for some* s).

(ii) K *is* Z_2*-periodic and the linking number of an axis of the* Z_2*-action with the knot is equal to* ± 1.

Proof. (i) Since $(f_1, f_2)(t+1) = -(f_1, f_2)(t)$, it follows that if $(f_1, f_2)(t)$ is in Δ_{ABC} then $(f_1, f_2)(t+1)$ is in Δ_{ADC}. Thus $f_{1,2}(t+1)$ and $f_{1,2}(t)$ are symmetric with respect to the diagonal $(-1, -1), (1, 1)$. Furthermore the above symmetry preserves the orientation of the trajectory (in Δ_{ABC}). If n_3 is even then $f_3(t+1) = f_3(t)$ and therefore the knot K is symmetric with respect to the plane of the rectangle $[(-1, -1), (0, 0)] \times [-1, 1]$ and the symmetry preserves the orientation of K. The only way for n_3 to be even is for K to be disjoint from the symmetry plane or fully to lie on the plane. The first case is impossible and the second leads to the degenerate situation described in Theorem 3.12(i).

(ii) If n_3 is odd then $f_3(t+1) = -f_3(t)$ and thus the line going through $(-1, -1, 0)$ and $(0, 0, 0)$ is the axis of symmetry for K, disjoint from K. Therefore K is Z_2-periodic. To find the linking number of the axis with K we follow the method of the proof of Theorem 3.3 and, in particular, the proof of Lemma 3.5. Our knot is a deformation of the singular closed curve whose projection goes through $(0, 0)$, reaches to the corners of the square where it bounces and "bends on itself". A small deformation of the parameters naturally gives the linking number ± 1 and any deformation omitting $(0, 0)$ does not change this linking number. ∎

4. Further speculations. One possibility of extending the family of billiard knots in a cube is to relax the condition on $\mathcal{F}$ from the first section. We no longer require that our trajectories are time-reversible. We will give two examples below, the second motivated by movement in ponds of a unicellular organism *Paramecium Caudatum* also called Slipper Animalcule.

DEFINITION 4.1. Consider the family $\mathcal{F}'$ of continuous fuctions $f : R \to R$ which satisfy the following properties:

(i) $f(t + 1) = -f(t)$ (skew-period 1),
(ii) $f(0) = 1$ (initial condition),
(iii) f is strictly decreasing on the interval $[0, 1]$.

As before, we define $f_i(t) = f(n_i(t) + c_i)$. We say that a knot is an $\mathcal{F}'$ *knot* if it can be parametrized by $(f_1(t), f_2(t), f_3(t))$.

As a consequence of our conditions we immediately have the following:

(iv) f has period 2; $f(k) = (-1)^k$ for any integer k.

(v) For any $f \in \mathcal{F}'$, there is a unique number $b \in (0,1)$ such that $f(b) = f(b+1) = 0$.

(vi) $\mathcal{F}'$ is a convex space; that is, if $f, g \in \mathcal{F}'$, then $(1-s)f + sg$ is in $\mathcal{F}'$ for any $s \in [0,1]$.

We do not know whether every $\mathcal{F}'$ knot is a billiard knot but we can extend Theorem 3.2 to this family of knots.

THEOREM 4.2. *If* $(f_1(t), f_2(t), f_3(t))$ *parametrize a knot* K *in* R^3 *then*:

(i) *If all* n_i*'s are odd then* K *is strongly* $+$ *amphicheiral.*

(ii) *If the* n_i*'s are pairwise relatively prime numbers and* n_3 *is even then* K *is a* Z_2*-periodic knot and the linking number between the axis of the* Z_2*-action and* K *is equal to* ± 1.

Proof. (i) We use the condition (i) to get $f_i(t+1) = -f_i(t)$; thus K is preserved by the central symmetry of R^3.

(ii) As in the case of the proof of Theorem 3.2, the proof of Theorem 4.2(ii) will be completed if we prove the following lemma. ∎

LEMMA 4.3. *If a closed planar curve* $(f_1(t), f_2(t))$, *for co-prime odd integers* n_1, n_2, *does not go through* $(0,0)$ *then the linking number*, $lk(f_1, f_2)$, *of the curve with* $(0,0)$ *is equal to* $+1$ *or* -1.

Proof. The closed planar curve $(f_1(t), f_2(t))$ goes through 0 if and only if $n_1 t + c_1 = b + h_1$ and $n_2 t + c_2 = b + h_2$ for some integers h_1, h_2. Thus

$$\frac{c_2}{n_2} - \frac{c_1}{n_1} = \frac{h_2}{n_2} - \frac{h_1}{n_1} + b\left(\frac{n_1 - n_2}{n_1 n_2}\right).$$

One should notice that h_1, h_2 are unique in the sense that if for different t (say t') one gets h_1', h_2' then $h_1' = h_1 + sn_1$ and $h_2' = h_2 + sn_2$ for some integer s. Thus t' is congruent to t or $t+1$ modulo 2. Without loss of generality we can assume that $c_1 = 0$. Now consider the open annulus parametrized by $c_2 \in [0,2]/(0 \sim 2)$, $b \in (0,1)$. We are considering all b corresponding to functions $f \in \mathcal{F}'$, $f(b) = 0$. The subspace of the annulus for which (f_1, f_2) goes through 0 is of codimension 1. This is a collection of intervals given by the condition $b(n_1 - n_2) = n_1 c_2 - n_1 h_2 + n_2 h_1$. Each of the components of the complement of the intervals contains a point with coordinate b equal to $\frac{1}{2}$. Thus the theorem reduces to the case $f(\frac{1}{2}) = 0$, as the linking number for every path-connected region of the annulus is constant. Now consider a function g which satisfies also $g(-t) = g(t)$ (for example $\cos(\pi t)$), and let $g^s = sg(t) + (1-s)f(t)$. Of course $g^s(\frac{1}{2}) = 0$, thus for given c_2, if $f(t)$ omits zero then $g^s(t)$ also omits zero. Therefore, by homotopy invariance of the linking number and by Lemma 3.5, $lk(f_1(t), f_2(t)) = lk(g_1(t), g_2(t)) = \pm 1$ as required. ∎

Paramecium Caudatum. Members of the genus *Paramecium* are unicellular organisms and hence placed in the phylum Protozoa. Paramecium Caudatum is the "slipper-shaped animalcule" of the early microscopists that is widely distributed and extensively studied. It commonly measures 170-290 microns. The species is world-wide in distribution and commonly found in ponds and bodies of stagnant and fresh water. The Paramecium Caudatum can swim in a straight line. When it strikes a solid object, it backs away and tries a new direction [Wi53]. According to J. Dembowski, the angle of reflection off the object is approximately constant.

We will propose two types of trajectories, motivated by movement of Paramecium Caudatum, and so relaxing the condition that the angle of incidence be equal to the angle of reflection.

Definition 4.4. (i) We consider a *Paramecium room* to be a Cartesian product $M\times N$ of a 2-dimensional Riemannian manifold M (with a piecewise smooth boundary) and a 1-dimensional Riemannian manifold N. A Paramecium moves along a geodesic line in $M\times N$ with constant speed until it hits the boundary. Then it reflects in such a way that in a horizontal direction the reflection angle is constant (say $\alpha \leq \pi/2$) when a vertical wall is hit, and the vertical component changes its sign when the floor or ceiling is hit.

(ii) *A room of constant reflection angle* is a Riemannian manifold M with a piecewise smooth boundary such that a particle moves along a geodesic line in M with constant speed until it hits the boundary. Then it reflects in such a way that it stays on the plane of the vector normal to the boundary and the trajectory before the reflection, and the value of the reflection angle is constant. If the particle hits a corner or hits a boundary along the vector normal to the boundary, its further motion is usually not defined.

The natural setting for Paramecium knots (simple closed trajectories in a Paramecium room) is a polygonal prism. The advantage of the Paramecium knot setting is that the segments of the projection of the trajectory have only a finite number of directions, so the case of irrational polygons is not much more difficult than the rational one. We will end this section by showing that the set of Paramecium knots (with $\alpha=\pi/4$) in rectangular prisms is the same as the set of billiard knots in a cube (or, according to Corollary 1.4, Lissajous knots). Fig. 4.1(b) represents the knot of type 5_2 [Ro76], as a billiard knot in a cube ($f_1(t)=f(2t-\frac{1}{6}), f_2(t)=f(3t), f_3(t)=f(7t+\frac{1}{4})$). Fig. 4.1(a) represents the knot 5_2 as a Paramecium knot ($\alpha=\pi/4$) in a rectangular prism.

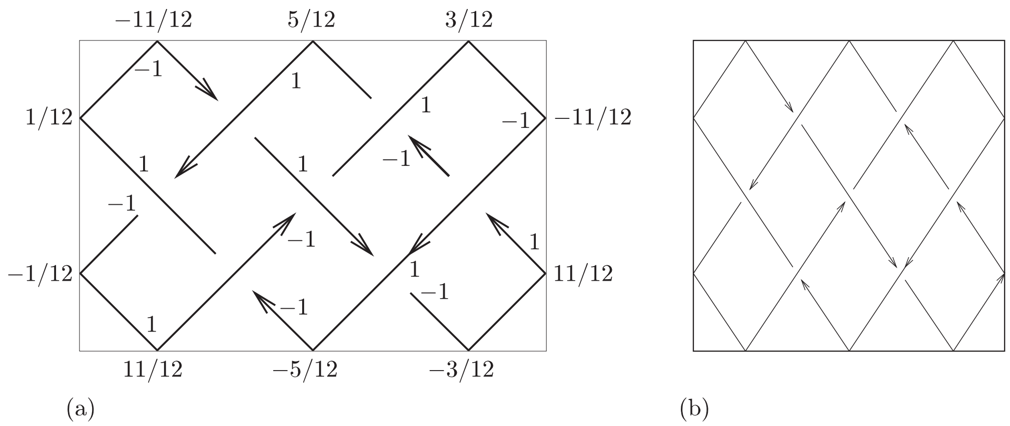


Fig. 4.1. The knot 5_2.

Theorem 4.5. *There is a bijection between billiard knots* (*in a cube*) *of type* n_1, n_2, n_3 *and Paramecium knots in a rectangular prism* $([-n_2,n_2]\times[-n_1,n_1]\times[-1,1])$, *for* $\alpha=\pi/4$.

Proof. Consider the automorphism (homothety) of R^3 given by $(x,y,z) \to (n_2x, n_1y, z)$. Then a billiard trajectory in a cube is sent to a Paramecium trajectory in a rectangular prism (with $\alpha = \pi/4$). The map is a bijection between the trajectories so the proof is complete. ∎

A similar fact also holds for constant reflection angle knots.

THEOREM 4.6. *There is a bijection between billiard knots* (*in a cube*) *of type* n_1, n_2, n_3 *and constant reflection angle knots in the rectangular prism* $([-n_2n_3, n_2n_3] \times [-n_1n_3, n_1n_3] \times [-n_1n_2, n_1n_2])$, *for* $\alpha = \pi/4$.

In Figure 4.2, we show an interesting example of a billiard link (also a Paramecium link with $\alpha = \pi/3$) in a regular triangular prism. This 3-dimensional figure of the link 9^2_{24} [Ro76] was made by M. Veve. This link is interesting because A. Hatcher and A. Reid [Re91] showed that its complement has a hyperbolic structure and the group of the knot is an arithmetic group (subgroup of $PSL_2(O_2)$, where O_2 is the ring of integers in the imaginary quadratic number field $Q(\sqrt{-2})$).

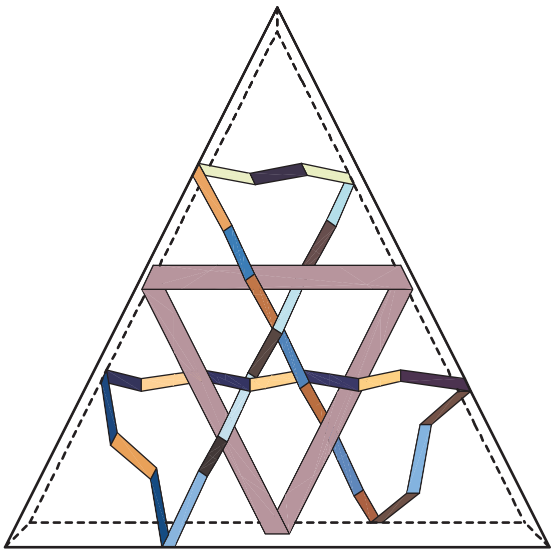


Fig. 4.2. The link 9^2_{24} as a billiard (also Paramecium) link.

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