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# HOMFLY POLYNOMIALS AS VASSILIEV LINK INVARIANTS

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**Abstract.** We prove that the number of linearly independent Vassiliev invariants for an $r$-component link of order $n$, which derived from the HOMFLY polynomial, is greater than or equal to $\min\{n, [(n+r-1)/2]\}$.

**Introduction.** Let $V_n$ denote the vector space consisting of all Vassiliev knot invariants of order less than or equal to $n$. There is a filtration

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$$

in the entire space of Vassiliev knot invariants. Each $V_n$ is finite-dimensional. Vassiliev [V] studied for the special cases when $n$ is small: $V_0 = V_1$, which consists of a constant map (Propositions 3 and 5), and $V_2/V_1$ is a one-dimensional vector space, whose basis is the second coefficient of the Conway polynomial. The dimensions for small $n$ are found by using the computer by Bar-Natan and Stanford (cf. [BN; B1, p. 282 ]): For $n = 1, 2, 3, 4, 5, 6, 7$, $\dim V_n/V_{n-1} = 0, 1, 1, 3, 4, 9, 14$, respectively.

On the other hand, Bar-Natan (cf. [BN]) showed that the $n$th coefficient of the Conway polynomial is of order less than or equal to $n$. Birman and Lin [BL] and Gusarov [G] proved that the Jones, HOMFLY, and Kauffman polynomials of a knot can be interpreted as an infinite sequence of Vassiliev knot invariants, and as a corollary they proved that $\dim V_n/V_{n-1} \geq 1$ for every $n \geq 2$ using the HOMFLY polynomial [BL, Corollary 4.2 (i)].

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The paper is in final form and no version of it will be published elsewhere.

Stanford [S1] generalized this for a link. In the special case of the Jones polynomial, the statement is as follows: Let $V_K(t)$ be the Jones polynomial of a knot $K$. Set $t = e^x$ and expand $e^x$ via its Taylor series to obtain a power series expansion of $V_K(t)$:

$$V_K(e^x) = \sum_{n=0}^{\infty} u_n(K) x^n$$

Then the coefficient $u_n(K)$ of $x^n$ is a Vassiliev invariant of order less than or equal to $n$. Melvin and Morton [MM] have shown that the order is just $n$. From this, we see that the $n$th derivative of $V_K(t)$ evaluated at 1, $V_K^{(n)}(1)$, is a Vassiliev invariant of order $n$. See Theorem 1.

In this paper, we study Vassiliev link invariants derived from the HOMFLY polynomial in a similar form. Let $P_k^{(\ell)}(L;1)$ be the $\ell$th derivative of the $k$th coefficient polynomial of the HOMFLY polynomial of a link $L$ evaluated at 1. In particular, $P_k(L;1) = a_k(L)$, the $k$th coefficient of the Conway polynomial. We show that $P_k^{(\ell)}(L;1)$ is a Vassiliev link invariant of order $\max\{k+\ell, 0\}$; in the following, $P_k^{(\ell)}$ indicates this Vassiliev link invariant. Furthermore, we have:

Main Theorem. *Let $s = \min\{n, [(n+r-1)/2]\}$. Then the dimension of the subspace of the Vassiliev invariants for an $r$-component link of order $n$ spanned by the following Vassiliev invariants is $s$:*

$$P_{2i-r+1}^{(n+r-2i-1)}, \quad i = 0, 1, \ldots, s.$$

Here [ ] denotes the greatest integer function.

Let us restrict attention to knots. This theorem gives a lower bound of the dimension of the HOMFLY subspace of $V_n/V_{n-1}$ defined by Birman and Lin [BL, p. 264], where they give the bound for $n \leq 4$. Meng [Me] shows that the dimension of the HOMFLY subspace of $V_n/V_{n-1}$ is $[n/2]$ applying the bracket weight system. Also, Chmutov and Duzhin [CD] show $\dim V_n/V_{n-1} \leq (n-1)!$, and more recently, Ng [N] shows $\dim V_n/V_{n-1} \leq (n-2)!/2$ if $n \geq 6$.

This paper consists of seven sections. In Sect. 1, we define a Vassiliev link invariant and give some properties following Birman and Lin [BL], Birman [B1, B2] and Stanford [S1]. In Sect. 2, we show that $P_k^{(\ell)}$ is a Vassiliev link invariant of order $\max\{k+\ell, 0\}$ (Lemma 1). From the proof of this, we get a useful recursion formula (2.7) for calculating the $P_k^{(\ell)}$-value of the $(k+\ell)$-configuration. Using this formula, we calculate a family of configurations (Lemma 2), which is a key step for proving our main result. In Sect. 3, we give some results analogous to those in Sect. 2 for the Jones polynomial. It is known that Vassiliev knot invariants form an algebra, which means that the product of a Vassiliev invariant of order $\leq p$ and one of $\leq q$ is a Vassiliev invariant of order $\leq p+q$, which is shown by Lin (unpublished) and Bar-Natan [BN]. In Sect. 4, we prove this for a link (Theorem 2), and also give a formula for calculating the value of the product of Vassiliev invariants for a $(p+q)$-configuration (Proposition 9). In Sect. 5, we give a basis for the space $V_4$ in terms of the invariants derived from the HOMFLY polynomial by making use of the result of Birman and Lin [BL]. Using this we get various relations among polynomial invariants regarding them as Vassiliev invariants of small order. In Sect. 6, we give a relation among $P_{2i}^{(n-2i)}$'s (Theorem 3), which is obtained by generalizing some formulas given in Sect. 5. This theorem, together with Lemma 2 in Sect. 2, implies Main Theorem for a knot (Theorem 4). In Sect. 7, we generalize Theorem 4 to a link (Theorem 5), thereby completing the proof of Main Theorem.

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**1. Vassiliev link invariants.** An $r$-component link is the image of oriented $r$ circles under an embedding into an oriented 3-sphere $S^3$. A knot is a 1-component link. An $r$-component link is trivial if it is planar, which we denote by $U^r$; $U^1 = U$, which is a trivial knot, and $U^0 = \emptyset$.

An $r$-component *singular link* is the image of oriented $r$ circles under an immersion into $S^3$ whose only singularities are transverse double points. We assume that a double point on a singular link is a rigid (or flat) vertex, which means that there is a neighborhood around each double point in which the singular link is contained in a plane. Two $r$-component singular links with $n$ double points are equivalent if there is an isotopy of $S^3$ which takes one to the other and which preserves the orientation of each component and the rigidity of each double point. This equivalence relation is called rigid vertex isotopy.

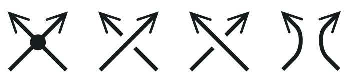
Let $v$ be an isotopy invariant of an $r$-component link, which takes values in the rational numbers $\mathbb{Q}$. Then $v$ can be uniquely extended to an $r$-component singular link invariant by the *Vassiliev skein relation*:

$$v(L_\times) = v(L_+) - v(L_-), \tag{1.1}$$

where $L_\times$ is a singular link with $x$ a double point and $L_+$, $L_-$ are ones obtained from $L_\times$ by replacing $x$ by a positive crossing and a negative crossing, respectively; see Fig. 1. Let $L^n = L_{x_1, x_2, \dots, x_n}$ be a singular link with $n$ double points $x_1, x_2, \dots, x_n$, and $L_{x_1(\epsilon_1), x_2(\epsilon_2), \dots, x_n(\epsilon_n)}$ be a non-singular link obtained from $L^n$ by replacing each double point $x_i$ by a positive crossing $x_i(+)$ or a negative crossing $x_i(-)$. We see that $v(L^n)$ is a linear combination of the $v$-values of $2^n$ links:

$$v(L^n) = \sum_{\epsilon_i = \pm} (-1)^{\mu(\epsilon)} v(L_{x_1(\epsilon_1), x_2(\epsilon_2), \dots, x_n(\epsilon_n)}), \tag{1.2}$$

where $\mu(\epsilon)$ is the number of minus signs in $\epsilon_1, \epsilon_2, \dots, \epsilon_n$; cf. [B2, (2)].



$L_\times$ $L_+$ $L_-$ $L_0$

Fig. 1

We call $v$ a *Vassiliev* (*finite-type*) *link invariant* if it satisfies the following axiom:

(1.3) There exists an integer $n$ such that $v(L) = 0$ for any singular link $L$ with more than $n$ double points.

The smallest such an integer $n$ is the *order* of $v$. In the special case of a knot, this reduces to Vassiliev's knot invariant. Stanford [S1] introduces one more axiom in order to relate the values of $v$ on links with different number of components, which we do not adopt in this paper.

The following is an immediate consequence of (1.1).

PROPOSITION 1. *The value of a Vassiliev invariant of a singular link shown in Fig. 2 is zero.*

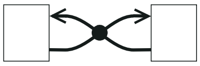


Fig. 2

The *$n$-configuration* which an $r$-component singular link with $n$ double points respects is the $n$ pairs of points on oriented $r$ circles; cf. [BL, p. 240; B1, p. 273; B2, p. 4]. We use a *chord-diagram of order* $n$ to represent it, that is, oriented $r$ circles with $n$ chords joining the paired points as in Figs. 4-6. We shall not distinguish strictly a chord-diagram from a configuration.

The following is due to Stanford [S1, Proposition 1.1]; cf. [B1, Lemma 1; B2, Proposition 1].

PROPOSITION 2. *Two $r$-component singular links with $n$ double points become equivalent after an appropriate series of crossing changes if and only if they respect the same $n$-configuration.*

In particular, any $r$-component link becomes trivial after an appropriate series of crossing changes. Thus we have

PROPOSITION 3. *A Vassiliev link invariant of order* 0 *is a locally constant map* (*i.e. it depends only on the number of components*).

The singular link shown in Fig. 2 respects the configuration given in Fig. 3, which we call *inadmissible*. A configuration is called *admissible* if it is not inadmissible. Thus for any inadmissible configuration, there is a singular link respecting it whose value of any Vassiliev link invariant is zero. (For a singular knot, such an immersion is called a *good model* in [BL, p. 242].)

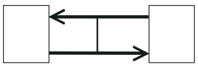


Fig. 3

Now we consider calculating a Vassiliev link invariant of a singular link with fixed number of components. Let us suppose that we have made a list of the distinct admissible $j$-configurations $\alpha_i^j$; $1 \le i \le s_j$, $j = 1, 2, \ldots$, and chosen, for each $\alpha_i^j$, a singular link $M_i^j$ respecting it. By Proposition 2, using a resolution tree, the value of a Vassiliev link invariant of a singular link is given as follows (cf. [LM, Proof of Theorem 2.4; B2, Proposition 2]):

PROPOSITION 4. *Let $v$ be a Vassiliev link invariant of order $\le m$, and $L^n$ a singular link respecting the admissible $n$-configuration $\alpha_p^n$, $n \le m$. Then*

$$v(L^n) \equiv v(M_p^n),$$

*where "≡" denotes equality up to a $\mathbb{Z}$-linear combination of* $v(M_i^j)$, $1 \le i \le r_j$, $n+1 \le j \le m$. *In particular*, *if* $m = n$, *then "≡" is "=", and so the* $v$*-value of a singular link with* $n$ *double points depends only on its configuration.*

Let $v$ be a Vassiliev invariant of order $\le n$, and $\alpha^n$ an $n$-configuration. Then by virtue of this proposition, we define $v(\alpha^n)$ by the $v$-value of any singular link respecting $\alpha^n$.

Since any 1-configuration is inadmissible, we have (cf. [CD, Examples 1.2.2 and 1.2.3]):

PROPOSITION 5. *A Vassiliev knot invariant of order* $\le 1$ *is a constant map*; $V_0 = V_1$.

There are linear relations among the $v$-values of singular links. It is known [V, S1, BN] that the finite set of 4-term relations suffice to determine a Vassiliev link invariant of order $m$. Thus we can find a consistent set of rational numbers $\{v(M_i^j) | 1 \le i \le s_j, j = 1, 2, \ldots, m\}$ such that we can determine an invariant; this assignment is called an *actuality table* for a Vassiliev link invariant. The method for making an actuality table for a knot is explained in [BL, B2].

**2. The HOMFLY polynomial.** The HOMFLY polynomial $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ [FYHLMO, PT] is an invariant of a link $L$, which is defined, as in [J], by the following formulas:

$$P(U; t, z) = 1; \tag{2.1a}$$

$$t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z), \tag{2.1b}$$

where $L_+$, $L_-$, $L_0$ are three links that are identical except near one point where they are as in Fig. 1; $L_+$ is obtained from $L_-$ by changing the crossing, and $L_0$ is obtained by smoothing the crossing.

By [LM, Proposition 22], the HOMFLY polynomial of an $r$-component link $L = K_1 \cup K_2 \cup \ldots \cup K_r$ is of the form

$$P(L; t, z) = \sum_{i=1}^{N} P_{2i-1-r}(L; t) z^{2i-1-r}, \tag{2.2}$$

where $P_{2i-1-r}(L; t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $(2i-1-r)$th *coefficient polynomial* of $P(L; t, z)$ and the powers of $t$ which appear in it are either all even or odd, depending on whether $r$ is odd or even. Let $P_k^{(\ell)}(L; t)$ be the $\ell$th derivative of $P_k(L; t)$. Note that $P_k^{(\ell)}(L; -t) = (-1)^{k+\ell} P_k^{(\ell)}(L; t)$. By [Kw, Lemma 1.7], if $1 \le i \le r-1$, then $P_{2i-1-r}(L; t)$ is divisible by $(t^{-1} - t)^{r-i}$. In particular, by [LM, Proposition 22],

$$P_{1-r}(L; t) = t^{2\lambda}(t^{-1} - t)^{r-1} \prod_{j=1}^{r} P_0(K_j; t), \tag{2.3}$$

where $\lambda$ is the total linking number of $L$ defined by $\lambda = \sum_{i<j} lk(K_i, K_j)$, and for a knot $K$, $P_0(K; 1) = 1$. Thus we have

PROPOSITION 6. *If* $L$ *is an* $r$*-component link*, $r \ge 2$, *then*

$$P_{2i-1-r}^{(m_i)}(L; 1) = \begin{cases} (r-1)!(-2)^{r-1} & \text{if } i = 1,\ m_1 = r-1; \\ 0 & \text{if } 1 \le i \le r-1,\ 0 \le m_i \le r-i-1. \end{cases}$$

Lemma 1. *$P_k^{(\ell)}(L;1)$ is a Vassiliev link invariant of order less than or equal to* $\max\{k+\ell, 0\}$.

Proof. First we prepare the formula (2.5) below. The equation (2.1b) implies

$$P_k(L_+;t) - P_k(L_-;t) = (t^2-1)P_k(L_-;t) + tP_{k-1}(L_0;t). \tag{2.4}$$

Differentiating the both sides $\ell$ times, we obtain

$$\begin{aligned} &P_k^{(\ell)}(L_+;t) - P_k^{(\ell)}(L_-;t) \\ &\qquad = (t^2-1)P_k^{(\ell)}(L_-;t) + 2\ell t P_k^{(\ell-1)}(L_-;t) + \ell(\ell-1)P_k^{(\ell-2)}(L_-;t) \\ &\qquad\quad + tP_{k-1}^{(\ell)}(L_0;t) + \ell P_{k-1}^{(\ell-1)}(L_0;t). \end{aligned}$$

Substituting $t=1$, this becomes

$$\begin{aligned} &P_k^{(\ell)}(L_+;1) - P_k^{(\ell)}(L_-;1) \\ &\quad = 2\ell P_k^{(\ell-1)}(L_-;1) + \ell(\ell-1)P_k^{(\ell-2)}(L_-;1) + P_{k-1}^{(\ell)}(L_0;1) + \ell P_{k-1}^{(\ell-1)}(L_0;1). \end{aligned} \tag{2.5}$$

We use induction on $k+\ell$. If $k+\ell \le 0$, then the lemma follows from Prosposition 5. Suppose that the lemma is true for $k+\ell < n$. Let $L_\times^{n+1}$ be a singular link with $n+1$ double points $x_1, x_2, \dots, x_n, x_{n+1}$, and $L_+^n$, $L_-^n$, $L_0^n$ be three singular links with $n$ double points obtained from $L_\times^{n+1}$; $L_+^n$ and $L_-^n$ by changing $x_{n+1}$ to a positive crossing $x_{n+1}(+)$ and a negative crossing $x_{n+1}(-)$, respectively, and $L_0^n$ by smoothing $x_{n+1}$.

From (1.2), we have

$$\begin{aligned} P_k^{(\ell)}(L_\times^{n+1};1) &= \sum_{\epsilon=(\epsilon_1,\dots,\epsilon_n,\epsilon_{n+1})} (-1)^{\mu(\epsilon)} P_k^{(\ell)}(L_{x,x_{n+1}(\epsilon_{n+1})};1) \\ &= \sum_{\epsilon'=(\epsilon_1,\dots,\epsilon_n)} (-1)^{\mu(\epsilon')} \left( P_k^{(\ell)}(L_{x,x_{n+1}(+)};1) - P_k^{(\ell)}(L_{x,x_{n+1}(-)};1) \right), \end{aligned}$$

where $x = (x_1(\epsilon_1), x_2(\epsilon_2), \dots, x_n(\epsilon_n))$. Using (2.5), this becomes

$$\begin{aligned} &P_k^{(\ell)}(L_\times^{n+1};1) \\ &\quad = \sum_{\epsilon'=(\epsilon_1,\dots,\epsilon_n)} (-1)^{\mu(\epsilon')} \Big( 2\ell P_k^{(\ell-1)}(L_{x,x_{n+1}(-)};1) + \ell(\ell-1)P_k^{(\ell-2)}(L_{x,x_{n+1}(-)};1) \\ &\qquad\qquad + P_{k-1}^{(\ell)}(L_{x,x_{n+1}(0)};1) + \ell P_{k-1}^{(\ell-1)}(L_{x,x_{n+1}(0)};1) \Big). \end{aligned}$$

Again using (1.2), we have

$$\begin{aligned} &P_k^{(\ell)}(L_\times^{n+1};1) \\ &\quad = 2\ell P_k^{(\ell-1)}(L_-^n;1) + \ell(\ell-1)P_k^{(\ell-2)}(L_-^n;1) + P_{k-1}^{(\ell)}(L_0^n;1) + \ell P_{k-1}^{(\ell-1)}(L_0^n;1). \end{aligned} \tag{2.6}$$

If $k+\ell = n$, then by the inductive hypothesis, the right-hand side is zero, thereby completing the proof. ∎

If $k+\ell = n+1$, then (2.6) implies the recursion formula:

$$P_k^{(\ell)}(\alpha^{n+1}) = 2\ell P_k^{(\ell-1)}(\alpha_-^n) + P_{k-1}^{(\ell)}(\alpha_0^n), \tag{2.7}$$

where $\alpha^{n+1}$, $\alpha_-^n$, $\alpha_0^n$ are the configurations respecting $L_\times^{n+1}$, $L_-^n$, $L_0^n$, respectively. Regard $\alpha^{n+1}$, $\alpha_-^n$, $\alpha_0^n$ as chord-diagrams. Then $\alpha_-^n$ is obtained from $\alpha^{n+1}$ by deleting the chord $c$ corresponding to the double point $x_{n+1}$, and $\alpha_0^n$ is obtained from $\alpha^{n+1}$ by chainging the chord $c$ as in Fig. 4. Thus the $P_k^{(\ell)}$-value of any configuration of order $k+\ell$ is given as a $\mathbb{Z}$-linear combination of $P_{-r}^{(r)}(U^{r+1};1)$, which is equal to $r!(-2)^r$ by Proposition 6.

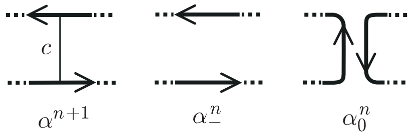



Fig. 4

EXAMPLE 1. Let $\sigma^2$ and $\tau^1$ be the chord-diagrams shown in Fig. 5. Deleting a chord from $\sigma^2$, it becomes inadmissible. So using (2.7), we have

$$P_0^{(2)}(\sigma^2) = P_{-1}^{(2)}(\tau^1) = 4P_{-1}^{(1)}(U^2) = -8.$$

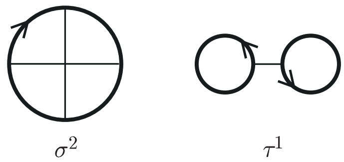



Fig. 5

The Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$ [C] of an oriented $r$-component link $L$ is given by

$$\nabla_L(z) = P(L;1,z),$$

and is of the form

$$\nabla_L(z) = \sum_{i=0}^{N} a_{r+2i-1}(L) z^{r+2i-1},$$

where $a_{r+2i-1}(L) \in \mathbb{Z}$.

From Lemma 1, $a_n(L)(= P_n^{(0)}(L;1))$ is a Vassiliev link invariant of order $\leq n$. The recursion formula, which follows from (2.7), is easy:

$$a_n(\alpha^n) = a_{n-1}(\alpha_0^{n-1}). \tag{2.8}$$

Since

$$a_0(U^r) = \begin{cases} 1 & \text{if } r = 1; \\ 0 & \text{if } r > 1, \end{cases}$$

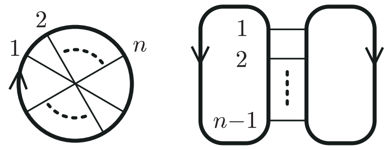
the $a_n$-value of any $n$-configuration is either 1 or 0.

EXAMPLE 2. Let $\sigma^n$ and $\tau^{n-1}$ be the chord-diagrams shown in Fig. 6. Applying (2.8), we have

$$a_n(\sigma^n) = a_{n-1}(\tau^{n-1}) = a_{n-2}(\sigma^{n-2}).$$

Since $a_1(\sigma^1) = 0$ and $a_0(\sigma^0) = a_0(U) = 1$, we obtain

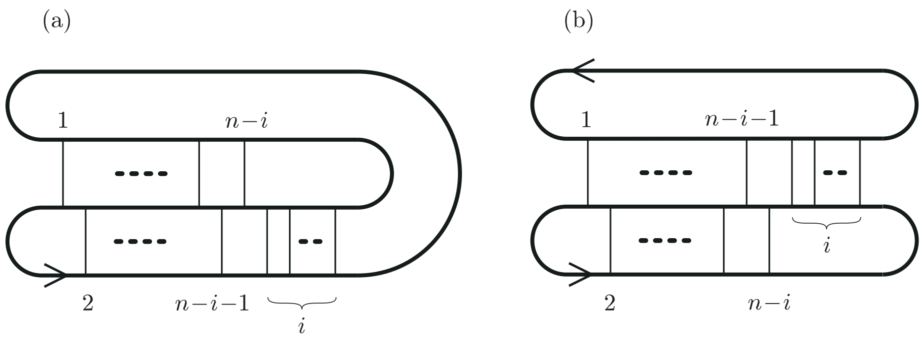
$$a_n(\sigma^n) = a_{n-1}(\tau^{n-1}) = \begin{cases} 1 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$




$\sigma^n$ $\tau^{n-1}$

Fig. 6

Let $A_i^n$, $n \geq 2$, $1 \leq i \leq n-1$, be an $n$-configuration for a circle and $B_i^n$, $n \geq 1$, $1 \leq i \leq n$, be one for two circles, which are represented by the chord-diagrams shown in Figs. 7(a) and 7(b), respectively.




$A_i^n$ $B_i^n$

Fig. 7

Lemma 2. *Suppose that* $n = k + \ell$.

(i)
$$P_k^{(\ell)}(A_i^n) = \begin{cases} (i-1)!2^{i-1} & \textit{if } \ell = i-1; \\ -(i+1)!2^{i+1} & \textit{if } \ell = i+1; \\ 0 & \textit{otherwise}, \end{cases}$$

*where* $k = 0, 2, \ldots, 2[n/2]$.

(ii)
$$P_k^{(\ell)}(B_i^n) = \begin{cases} (i-1)!2^{i-1} & \textit{if } \ell = i-1; \\ -(i+1)!2^{i+1} & \textit{if } \ell = i+1; \\ 0 & \textit{otherwise}, \end{cases}$$

*where* $k = -1, 1, \ldots, 2[(n+1)/2] - 1$.

Proof. First, we consider the case $i = 1$. In the same way as Example 1, we have

$$P_k^{(\ell)}(A_1^n) = P_{k-1}^{(\ell)}(B_1^{n-1}) = P_{k-2}^{(\ell)}(A_1^{n-2}).$$

So if $i = 1$, the lemma is true by Examples 1 and 2.

Suppose that $i > 1$. Applying (2.7), we have

$$P_k^{(\ell)}(A_i^n) = 2\ell P_k^{(\ell-1)}(A_{i-1}^{n-1}), \quad P_k^{(\ell)}(B_i^n) = 2\ell P_k^{(\ell-1)}(B_{i-1}^{n-1}),$$

which are equal to

$$2^{i-1}\frac{\ell!}{(\ell-i+1)!}P_k^{(\ell-i+1)}(A_1^{n-i+1});$$
$$2^{i-1}\frac{\ell!}{(\ell-i+1)!}P_k^{(\ell-i+1)}(B_1^{n-i+1}),$$

respectively. From the $i=1$ case, we obtain the results. ■

Let $\gamma^{n-1}$ be an $(n-1)$-configuration, and $c_1$ and $c_2$ be its two chords. Let $\gamma_i^n$, $i=1,2$, be an $n$-configuration obtained from $\gamma^{n-1}$ by adding a new chord parallel to $c_i$ as shown in Fig. 8. Applying (2.7) and (2.8), we have immediately

Proposition 7. *If* $k+\ell=n$, *then*

$$P_k^{(\ell)}(\gamma_1^n)=P_k^{(\ell)}(\gamma_2^n).$$

*In particular*,

$$a_n(\gamma_1^n)=a_n(\gamma_2^n)=0.$$

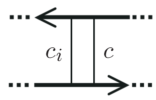
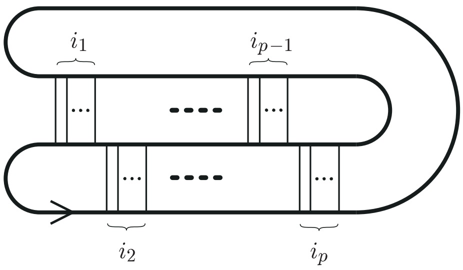



Fig. 8

Let $A^n(i_1,i_2,\dots,i_p)$ be an $n$-configuration represented by the chord-diagram shown in Fig. 9, where $p$ is even, $i_1+i_2+\cdots+i_p=n$, and $i_1,\, i_2,\dots,i_p\geq 1$. When $p=n-j+1$, $i_1=i_2=\ldots=i_{n-j}=1$ and $i_{n-j+1}=j$, it coincides with $A_j^n$. Therefore, Proposition 7 implies that if $k+\ell=n$, then

$$P_k^{(\ell)}(A^n(i_1,i_2,\dots,i_{n-j+1}))=P_k^{(\ell)}(A_j^n). \tag{2.9}$$




$A^n(i_1,i_2,\dots,i_p)$

Fig. 9

**3. The Jones polynomial.** The Jones polynomial $V(L;t)\in\mathbb{Z}[t^{\pm1/2}]$ [J] of an oriented link $L$ is given by

$$V(L;t)=P(L;t,t^{1/2}-t^{-1/2}). \tag{3.1}$$

The aim of this section is to prove the following:

Theorem 1. *$V^{(n)}(L;1)$ is a Vassiliev link invariant of order $n$.*

Noting that

$$V(L;1) = (-2)^{r-1} \tag{3.2}$$

for an $r$-component link $L$ [J, (12.1)], we can prove the following in the same way as Lemma 1.

Lemma 3. *$V^{(n)}(L;1)$ is a Vassiliev link invariant of order less than or equal to $n$.*

From the proof of Lemma 3, we get the recursion formula which is similar to (2.7) and (2.8):

$$V^{(n+1)}(\alpha^{n+1}) = 2(n+1)V^{(n)}(\alpha^n_-) + (n+1)V^{(n)}(\alpha^n_0), \tag{3.3}$$

where $\alpha^{n+1}$, $\alpha^n_-$, $\alpha^n_0$ are the same as in (2.7). Using (3.2) and (3.3), we may calculate the $V^{(n)}$-values of the configurations given in Fig. 7 (cf. Lemma 2):

Lemma 4.

$$V^{(n)}(A^n_i) = V^{(n)}(B^n_i) = -3 \cdot 2^{i-1}(n!).$$

Using this, we obtain an analogous result to Proposition 7:

Proposition 8.

$$V^{(n)}(\gamma^n_1) = V^{(n)}(\gamma^n_2) = 2nV^{(n-1)}(\gamma^{n-1}).$$

This yields an analogous formula to (2.9):

$$V^{(n)}(A^n(i_1, i_2, \dots, i_{n-j+1})) = V^{(n)}(A^n_j). \tag{3.4}$$

Let $\alpha^n$ be an $n$-configuration, and $\alpha^n \sqcup U$ denote the $n$-configuration represented by the disjoint union of $\alpha^n$ and a circle. Then we have:

Lemma 5.

$$V^{(n)}(\alpha^n \sqcup U) = -2V^{(n)}(\alpha^n).$$

Proof. If $L$ is a link, then

$$V(L \sqcup U;t) = (-t^{1/2} - t^{-1/2})V(L;t),$$

and so

$$V^{(n)}(L \sqcup U;t) = \sum_{i=0}^{n} (-t^{1/2} - t^{-1/2})^{(i)} V^{(n-i)}(L;t).$$

By Lemma 3, if $k < n$, then

$$V^{(k)}(\alpha^n) = 0,$$

and thus we obtain the result. ∎

Proof of Theorem 1. By Lemma 3, it suffices to show that there exists an $n$-configuration $\alpha^n$ for $r$ circles such that $V^{(n)}(\alpha^n) \neq 0$ for each $n$ and $r$. Lemma 4 shows this for $r = 1, 2$. Note that $V_1 = V_0$. For $r > 2$, we have

$$V^{(n)}(B^n_i \sqcup U^{r-2}) = \left(-3 \cdot 2^{i-1}(n!)\right)(-2)^{r-2} \neq 0$$

by Lemmas 4 and 5, and thus the proof is complete. ∎

R e m a r k. Melvin and Morton [MM] prove Theorem 1 for a knot using the configuration $A^n_{n-1}$.

**4. The product of Vassiliev link invariants.** Let $v$ and $w$ be Vassiliev link invariants. Then the product $v\cdot w$ is defined by $(v\cdot w)(L)=v(L)w(L)$, for a non-singular link $L$.

LEMMA 6. *Let $L^n = L_{x_1(\times),x_2(\times),\ldots,x_n(\times)}$ be a singular link with $n$ double points $x_1(\times)$, $x_2(\times)$, …, $x_n(\times)$. Then*

$$(v\cdot w)(L^n)=\sum_{(\epsilon_i,\epsilon'_i)} v(L_{x_1(\epsilon_1),x_2(\epsilon_2),\ldots,x_n(\epsilon_n)})w(L_{x_1(\epsilon'_1),x_2(\epsilon'_2),\ldots,x_n(\epsilon'_n)}),$$

*where each pair $(\epsilon_i,\epsilon'_i)$ is either $(+,\times)$ or $(\times,-)$, and the sum runs over the $2^n$ possible choices.*

P r o o f. We prove by induction on $n$. When $n=0$, the lemma is just the definition. Suppose that the lemma is true for $n$. By the Vassiliev skein relation (1.1), we have

$$\begin{aligned}&(v\cdot w)(L_{x_1(\times),x_2(\times),\ldots,x_n(\times),x_{n+1}(\times)})\\&\qquad=(v\cdot w)(L_{x_1(\times),x_2(\times),\ldots,x_n(\times),x_{n+1}(+)})-(v\cdot w)(L_{x_1(\times),x_2(\times),\ldots,x_n(\times),x_{n+1}(-)}).\end{aligned}$$

By the inductive hypothesis, this becomes:

$$\sum_{(\epsilon_i,\epsilon'_i)} v(L_{x,x_{n+1}(+)})w(L_{x',x_{n+1}(+)})-\sum_{(\epsilon_i,\epsilon'_i)} v(L_{x,x_{n+1}(-)})w(L_{x',x_{n+1}(-)}),$$

where $x=(x_1(\epsilon_1),x_2(\epsilon_2),\ldots,x_n(\epsilon_n))$ and $x'=(x_1(\epsilon'_1),x_2(\epsilon'_2),\ldots,x_n(\epsilon'_n))$. This is equal to

$$\begin{aligned}&\sum_{(\epsilon_i,\epsilon'_i)}\big(v(L_{x,x_{n+1}(+)})w(L_{x',x_{n+1}(+)})-v(L_{x,x_{n+1}(-)})w(L_{x',x_{n+1}(-)})\big)\\&\qquad=\sum_{(\epsilon_i,\epsilon'_i)}\big(v(L_{x,x_{n+1}(+)})w(L_{x',x_{n+1}(+)})-v(L_{x,x_{n+1}(+)})w(L_{x',x_{n+1}(-)})\\&\qquad\quad+v(L_{x,x_{n+1}(+)})w(L_{x',x_{n+1}(-)})-v(L_{x,x_{n+1}(-)})w(L_{x',x_{n+1}(-)})\big)\\&\qquad=\sum_{(\epsilon_i,\epsilon'_i)}\big(v(L_{x,x_{n+1}(+)})\big(w(L_{x',x_{n+1}(+)})-w(L_{x',x_{n+1}(-)})\big)\\&\qquad\quad+\big(v(L_{x,x_{n+1}(+)})-v(L_{x,x_{n+1}(-)})\big)w(L_{x',x_{n+1}(-)})\big).\end{aligned}$$

Again from the Vassiliev skein relation, this becomes

$$\begin{aligned}&\sum_{(\epsilon_i,\epsilon'_i)}\big(v(L_{x,x_{n+1}(+)})w(L_{x',x_{n+1}(\times)})+v(L_{x,x_{n+1}(\times)})w(L_{x',x_{n+1}(-)})\big)\\&\qquad=\sum_{(\epsilon_i,\epsilon'_i)} v(L_{x,x_{n+1}(\epsilon_{n+1})})w(L_{x',x_{n+1}(\epsilon'_{n+1})}).\end{aligned}$$

We have completed the proof of Lemma 6. ■

This lemma implies immediately the following, which is proved for a knot by Lin (unpublished) and Bar-Natan [BN].

Theorem 2. *If $v$ and $w$ are Vassiliev link invariants of orders less than or equal to $p$ and $q$, respectively, then the product $v \cdot w$ is a Vassiliev link invariant of order less than or equal to $p+q$.*

Let $\alpha^{p+q}$ be a chord-diagram of order $p+q$ for $r$ circles, and $C$ the set of its $p+q$ chords. For a subset $S$ of $C$ with $\#S = p$, let $\alpha^p_S$ denote a chord-diagram of order $p$ consisting of $r$ circles and the chords in $S$.

Proposition 9. *Let $v$ and $w$ be Vassiliev link invariants of orders $p$ and $q$, respectively. Then*

$$(v \cdot w)(\alpha^{p+q}) = \sum_{S \sqcup \bar{S} = C} v(\alpha^p_S) w(\alpha^q_{\bar{S}}),$$

*where $S \sqcup \bar{S}$ is the disjoint union of $S$ and $\bar{S}$.*

Proof. This follows from Lemma 6 when $n = p+q$. ∎

Example 3. We calculate $a_2^2 (= a_2 \cdot a_2)$, the square of $a_2$, the coefficient of $z^2$ in the Conway polynomial. Let $\alpha^4$ be a chord-diagram for a circle with $C = \{c_1, c_2, c_3, c_4\}$ a set of its chords. Applying Proposition 9, we have

$$\begin{aligned} &a_2^2(\alpha^4) \\ &\quad = 2\left(a_2(\alpha^2_{\{c_1,c_2\}}) a_2(\alpha^2_{\{c_3,c_4\}}) + a_2(\alpha^2_{\{c_1,c_3\}}) a_2(\alpha^2_{\{c_2,c_4\}}) + a_2(\alpha^2_{\{c_1,c_4\}}) a_2(\alpha^2_{\{c_2,c_3\}})\right). \end{aligned}$$

Using this, we obtain the following:

$$\begin{aligned} a_2^2(\sigma^4) &= 6a_2(\sigma^2)^2 = 6; \\ a_2^2(A_1^4) &= 2a_2(\sigma^2)^2 = 2; \\ a_2^2(A_3^4) &= 0; \\ a_2^2(A^4(2,2)) &= 4a_2(\sigma^2)^2 = 4, \end{aligned}$$

where $\sigma^2$ and $\sigma^4$ are given in Figs. 5 and 6 (Exapmles 1 and 2), $A_1^4$ and $A_3^4$ in Fig. 7, and $A^4(2,2)$ in Fig. 9.

Remark. Hoste [H] gives a formula for $a_{r-1}(L)$ with $L$ an $r$-component link in terms of the linking numbers; more precisely, $a_{r-1}(L)$ is a polynomial of degree $r-1$ in the linking numbers of the sublink of $L$. In particular, if $r = 2$, then $a_1(L)$ is the linking number of $L$ (cf. [Kf, p. 24]). By Theorem 2 and the result in [S2], we see that $a_{r-1}(L)$ is a Vassiliev invariant of order less than or equal to $r-1$.

**5. Vassiliev knot invariants of order $\leq$ 4.** In this section, we study a Vassiliev knot invariant of order $\leq 4$, making use of the result of Birman and Lin [BL, Example 3.9].

The only admissible 2-configuration is $\sigma^2$ shown in Fig. 5, which are denoted by the symbol **22** in [BL]. Let $M^2$ be the singular knot of order 2 shown in Fig. 10 respecting it.

There are two admissible 3-configurations: $A_2^3$ (Fig. 7) and $\sigma^3$ (Fig. 6). In [BL], they are denoted by the symbols **232** and **333**, respectively, and it is shown that if $v_3$ is a

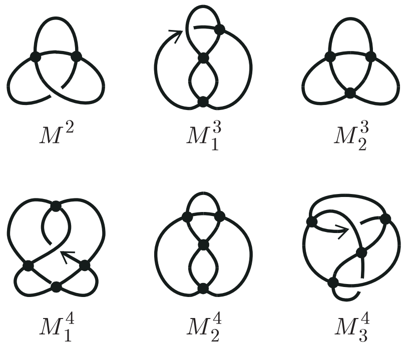



Fig. 10

Vassiliev invariant of order 3, then

$$v_3(\sigma^3) = 2v_3(A_2^3). \tag{5.1}$$

Let $M_1^3$ and $M_2^3$ be the singular knots of order 3 shown in Fig. 10 respecting $A_2^3$ and $\sigma^3$.

Let $v$ be a Vassiliev knot invariant of order $\le 4$ and $K$ be a knot. There are seven admissible 4-configurations, and the $v$-value of any 4-configuration is determined by those of the three 4-configurations $A_3^4$, $A^4(2,2)$, $A_1^4$ shown in Figs. 7 and 9. They are denoted by the symbols **2442**, **3533**, and **2332**, respectively in [BL]. Let $M_1^4$, $M_2^4$, $M_3^4$ be the singular knots of order 4 shown in Fig. 10 respecting them.

Therefore, Proposition 4 implies

$$v(K) = [\,v(U) \quad v(M^2) \quad v(M_1^3) \quad v(M_1^4) \quad v(M_2^4) \quad v(M_3^4)\,] \begin{bmatrix} 1 \\ p \\ q \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{5.2}$$

where $p$, $q$, $r_1$, $r_2$, $r_3$ are integers.

Let $3_1$, $4_1$, $5_1$, $5_2$ be the knots in the table of [R]. We denote by $K!$ the mirror image of the knot $K$. So $3_1$ and $3_1!$ denote the left- and right-hand trefoil knots, respectively, and $4_1$ is the figure-eight knot. Using the Vassiliev skein relatin (1.1), we have

$$\begin{aligned} &[\,v(M^2) \quad v(M_1^3) \quad v(M_1^4) \quad v(M_2^4) \quad v(M_3^4)\,] \\ &= [\,v(U) \quad v(3_1!) \quad v(3_1) \quad v(4_1) \quad v(5_1!) \quad v(5_2!)\,] \begin{bmatrix} -1 & -2 & 3 & -4 & 0 \\ 1 & 1 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \end{aligned} \tag{5.3}$$

In Table 1, we give the values of the Vassiliev invariants of order less than or equal to 4 derived from the HOMFLY and the Jones polynomials. Many of them are already given in Examples 1–3.

| | $a_2$ | $P_0^{(2)}$ | $V^{(2)}$ | $P_2^{(1)}$ | $\frac{P_0^{(3)}}{24}$ | $\frac{V^{(3)}}{18}$ | $a_4$ | $P_2^{(2)}$ | $\frac{P_0^{(4)}}{24}$ | $\frac{V^{(4)}}{24}$ | $a_2^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $3_1!$ | 1 | −8 | −6 | 2 | −1 | −1 | 0 | 2 | −1 | −1 | 1 |
| $4_1$ | −1 | 8 | 6 | 0 | −1 | −1 | 0 | 0 | 5 | 4 | 1 |
| $M^2$ | 1 | −8 | −6 | 2 | −1 | −1 | 0 | 2 | −1 | −1 | 1 |
| $M_1^3$ | 0 | 0 | 0 | 2 | −2 | −2 | 0 | 2 | 4 | 3 | 2 |
| $M_1^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | −12 | 0 |
| $M_2^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | −16 | −12 | 4 |
| $M_3^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −8 | 0 | −3 | 2 |

**Table 1**

From (5.2) and Table 1, we have:

$$\begin{bmatrix} a_2(K) \\ P_0^{(3)}(K;1)/24 \\ a_2(K)^2 \\ a_4(K) \\ P_0^{(4)}(K;1)/24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 4 & -16 & -16 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

and thus we have

$$\begin{bmatrix} p \\ q \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 & 0 \\ -3/16 & -3/8 & -1/4 & 1/2 & -1/16 \\ 0 & 1/4 & 1/4 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_2(K) \\ P_0^{(3)}(K;1)/24 \\ a_2(K)^2 \\ a_4(K) \\ P_0^{(4)}(K;1)/24 \end{bmatrix}. \tag{5.4}$$

Also we have:

$$\begin{bmatrix} P_0^{(2)}(K;1) \\ V_K^{(2)}(1) \\ P_2^{(1)}(K;1) \\ V_K^{(3)}(1)/18 \\ P_2^{(2)}(K;1) \\ V_K^{(4)}(1)/24 \end{bmatrix} = \begin{bmatrix} -8 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 2 & 2 & 8 & 8 & -8 \\ -1 & 3 & -12 & -12 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{5.5}$$

Substituting (5.4) to (5.5), we obtain:

$$P_0^{(2)}(K;1) = -8a_2(K); \tag{5.6}$$

$$V_K^{(2)}(1) = -6a_2(K) \quad ([\text{Mu1}]); \tag{5.7}$$

$$P_2^{(1)}(K;1) = a_2(K) - \frac{1}{24}P_0^{(3)}(K;1) \quad ([\text{Mi}]); \tag{5.8}$$

$$V_K^{(3)}(1) = \frac{3}{4}P_0^{(3)}(K;1) \quad ([\text{Mi}]); \tag{5.9}$$

$$P_2^{(2)}(K;1) = -\frac{1}{2}a_2(K) - \frac{1}{12}P_0^{(3)}(K;1) - 8a_4(K) - \frac{1}{48}P_0^{(4)}(K;1); \tag{5.10}$$

$$V_K^{(4)}(1) = -6a_2(K) - 72a_4(K) + 18P_0^{(4)}(K;1). \tag{5.11}$$

Combining (5.2)–(5.4), we obtain:

$$v(K) = [\, v(U) \quad v(3_1!) \quad v(3_1) \quad v(4_1) \quad v(5_1!) \quad v(5_2!) \,]\, X \begin{bmatrix} 1 \\ a_2(K) \\ P_0^{(3)}(K;1)/24 \\ a_2(K)^2 \\ a_4(K) \\ P_0^{(4)}(K;1)/24 \end{bmatrix}, \tag{5.12}$$

where

$$X = \begin{bmatrix} 1 & -9/16 & -9/8 & -7/4 & 7/2 & -3/16 \\ 0 & 17/16 & 7/8 & 1 & -1 & 3/16 \\ 0 & 0 & 1/4 & 1/4 & -1/2 & 0 \\ 0 & -5/16 & 3/8 & 3/4 & -3/2 & 1/16 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -3/16 & -3/8 & -1/4 & -3/2 & -1/16 \end{bmatrix}.$$

We consider $v_2$, a Vassiliev knot invariant of order $\leq 2$. Then (5.2) becomes

$$v_2(K) = v_2(U) + pv_2(M^2). \tag{5.13}$$

Then using (5.3) and (5.4), we have

$$v_2(K) = [\, v_2(U) \quad v_2(3_1!) \,] \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_2(K) \end{bmatrix}. \tag{5.14}$$

This is given in [La, Proposition 4.2.9], where $V_2(K) = a_2(K)$ and $v_2(U)$ is determined to be zero.

Next, we consider $v_3$, a Vassiliev knot invariant of order $\leq 3$. Then (5.2) becomes

$$v_3(K) = v_3(U) + pv_3(M^2) + qv_3(M_1^3). \tag{5.15}$$

Then using (5.3) and (5.4), we have

$$v_3(K) = [\, v_3(U) \quad v_3(3_1!) \quad v_3(4_1) \,] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ a_2(K) \\ P_0^{(3)}(K;1)/24 \end{bmatrix} \tag{5.16}$$

Substituting (5.8) to (5.16), we obtain

$$v_3(K) = [\, v_3(U) \quad v_3(3_1!) \quad v_3(4_1) \,] \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1/2 \\ 0 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ a_2(K) \\ P_2^{(1)}(K;1) \end{bmatrix}, \tag{5.17}$$

which is the first formula in [La, Proposition 4.3.10] with $V_3(K) = P_2^{(1)}(K;1)/2$.

Substituting (5.1) to (5.2), we have

$$v_3(K) = v_3(U) + pv_3(M^2) + \frac{q}{2} v_3(M_2^3). \tag{5.18}$$

Using the Vassiliev skein relation (1.1), we have

$$v_3(M_2^3) = v_3(3_1!) - v_3(3_1), \tag{5.19}$$

and thus we obtain

$$v_3(K) = [\, v_3(U) \quad v_3(3_1!) \quad v_3(3_1) \,] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3/4 & -1/4 \\ 0 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 \\ a_2(K) \\ P_0^{(3)}(K;1)/24 \end{bmatrix}. \tag{5.20}$$

Substituting (5.8) to (5.20), we obtain the second formula in [La, Proposition 4.3.10]:

$$v_3(K) = [\, v_3(U) \quad v_3(3_1!) \quad v_3(3_1)\,] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 1/4 \\ 0 & 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 1 \\ a_2(K) \\ P_2^{(1)}(K;1) \end{bmatrix}. \tag{5.21}$$

**6. A relation among $P_k^{(\ell)}$.** From (5.6), (5.8) and (5.10), we have

$$\frac{1}{2!2^2} P_0^{(2)}(K;1) + a_2(K) = 0;$$
$$\frac{1}{3!2^3} P_0^{(3)}(K;1) + \frac{1}{2} P_2^{(1)}(K;1) = \frac{1}{2} a_2(K);$$
$$\frac{1}{4!2^4} P_0^{(4)}(K;1) + \frac{1}{2!2^2} P_2^{(2)}(K;1) + a_4(K) = -\frac{5}{16} a_2(K) + \frac{1}{4} P_2^{(1)}(K;1).$$

We can generalize these formulas. Let $\varphi_m$ be a Vassiliev invariant for an $r$-component link $L$ defined by

$$\varphi_{k-r+1}(L) = \sum_{i=0}^{[k/2]} \frac{1}{(k-2i)!2^{k-2i}} P_{2i-r+1}^{(k-2i)}(L;1).$$

By Lemma 1, $\varphi_{k-r+1}$ is a Vassiliev link invariant of order $\le \max\{k-r+1, 0\}$. However, we shall prove:

THEOREM 3. $\varphi_{k-r+1}$, $k = 0, 1, 2, \ldots$, *is a Vassiliev invariant for an $r$-component link of order less than or equal to* $\max\{k-r, 0\}$.

In order to prove Theorem 3, we study a Vassiliev link invariant of order less than or equal to one, which may be derived from [Mu2]. The only admissible 1-configuration for an $r$ circles, $r \ge 2$, is represented by the union of $\tau^1$ (Fig. 4) and $r-2$ circles, $\tau^1 \sqcup U^{r-2}$. Using this, we show the following:

PROPOSITION 10. *Let $v$ be a Vassiliev invariant of order less than or equal to one for an $r$-component link, $r \ge 2$. Then for an $r$-component link $L$, it holds that*

$$v(L) = v(U^r) + \lambda v(\tau^1 \sqcup U^{r-2}),$$

*where $\lambda$ is the total linking number of $L$.*

Proof. From Proposition 4, we have

$$v(L) = v(U^r) + m v(\tau^1 \sqcup U^{r-2}),$$

where $m$ is an integer. Since $V^{(1)}(L;1)$ is a Vassiliev invariant of order $\le 1$, we have

$$V^{(1)}(L;1) = V^{(1)}(U^r;1) + m V^{(1)}(\tau^1 \sqcup U^{r-2}).$$

By applying (3.3), this becomes

$$V^{(1)}(L;1) = V^{(1)}(U^r;1) + 2m V(U^r;1) + m V(U^{r-1};1).$$

Using (3.2) and $V^{(1)}(L;1) = -3(-2)^{r-2}\lambda$ [J, (12.2); Mu1], we obtain

$$-3(-2)^{r-2}\lambda = 2m(-2)^{r-1} + m(-2)^{r-2},$$

from which we get $m = \lambda$, and the proof is complete. ∎

Proof of Theorem 3. We prove by induction on $k$. If $k \le r-1$, then this follows from Lemma 1. We show that $\varphi_1$ is of order zero. Let $L$ be an $r$-component link. If $r=1$, then

$$\varphi_1(L) = \frac{1}{2}P_0^{(1)}(L;1) = 0.$$

Suppose that $r \ge 2$. Since the order of $\varphi_1$ is $\le 1$, from Proposition 10, we have

$$\varphi_1(L) = \varphi_1(U^r) + \lambda\varphi_1(\tau^1 \sqcup U^{r-2}).$$

Using (2.7), we have

$$P_{2i-r+1}^{(r-2i)}(\tau^1 \sqcup U^{r-2}) = 2(r-2i)P_{2i-r+1}^{(r-2i-1)}(U^r;1) + P_{2i-r}^{(r-2i)}(U^{r-1};1),$$

and so

$$\varphi_1(\tau^1 \sqcup U^{r-2}) = \sum_{i=0}^{[r/2]}\left(\frac{P_{2i-r+1}^{(r-2i-1)}(U^r;1)}{(r-2i-1)!2^{r-2i-1}} + \frac{P_{2i-r}^{(r-2i)}(U^{r-1};1)}{(r-2i)!2^{r-2i}}\right),$$

which is zero by Proposition 6. Therefore, $\varphi_1$ is a constant map.

From (2.2), we have

$$P(L;y+1,y) = \sum_{i=0}^{n} P_{2i-r+1}(L;y+1)y^{2i-r+1}. \tag{6.1}$$

We expand $P_{2i-r+1}(L;y+1)$ via its Taylor series:

$$P_{2i-r+1}(L;y+1) = \sum_{j=0}^{\infty}\frac{P_{2i-r+1}^{(j)}(L;1)}{j!}y^j. \tag{6.2}$$

Then we obtain a power series expansion of $P(L;y+1,y)$:

$$P(L;y+1,y) = \sum_{k=0}^{\infty}\Phi_{k-r+1}(L)y^{k-r+1}, \tag{6.3}$$

where

$$\Phi_{k-r+1}(L) = 2^{k-r+1}\varphi_{k-r+1}(L). \tag{6.4}$$

The equation (2.1b) implies

$$\begin{aligned} P(L_+;y+1,2y) - P(L_-;y+1,2y) \\ = (y^2+2y)P(L_-;y+1,2y) + 2(y^2+y)P(L_0;y+1,2y). \end{aligned} \tag{6.5}$$

Then from (6.3), we have

$$\Phi_\ell(L_+) - \Phi_\ell(L_-) = \Phi_{\ell-2}(L_-) + 2\Phi_{\ell-1}(L_-) + 2\Phi_{\ell-2}(L_0) + 2\Phi_{\ell-1}(L_0). \tag{6.6}$$

Assume that $\Phi_k$ is a Vassiliev invariant of order $\le \max\{k-1,0\}$ for each $k(<\ell)$. Then using (6.6), we can prove that $\Phi_\ell(L)$ is of order $\le \ell-1$ in a similar way to the proof of Lemma 1. This completes the proof of Theorem 3. ∎

From this theorem, for a knot $K$ we have

$$\sum_{i=0}^{[n/2]}\frac{1}{(n-2i)!2^{n-2i}}P_{2i}^{(n-2i)}(K;1) \equiv 0$$

in $V_n/V_{n-1}$. Therefore, by Lemma 2 (i), we obtain:

THEOREM 4. *The dimension of the subspace of $V_n/V_{n-1}$ spanned by the following Vassiliev invariants of order $n$ is $[n/2]$:*

$$P_{2i}^{(n-2i)}(K;1), \qquad i = 0, 1, \dots, [n/2].$$

Now we reconsider the Jones polynomial of a knot $K$. From (2.2) and (3.1), we have

$$V(K;t) = \sum_{k=0}^{N} \psi_k(t) P_{2k}(K;t),$$

where $\psi_k(t) = (t^{1/2} - t^{-1/2})^{2k}$. Then we obtain

$$V^{(n)}(K;t) = \sum_{k=0}^{N} \left( \sum_{i=0}^{n} \binom{n}{i} \psi_k^{(i)}(t) P_{2k}^{(n-i)}(K;t) \right).$$

Since

$$\psi_k^{(i)}(1) = \begin{cases} 0 & \text{if } i < 2k; \\ (-1)^i \frac{i!(i-k-1)!}{(i-2k)!(k-1)!} & \text{if } i \geq 2k, \end{cases}$$

we obtain

$$\begin{aligned} (6.7) \quad V^{(n)}(K;1) &= P_0^{(n)}(K;1) + \sum_{k=1}^{[n/2]} \left( \sum_{i=2k}^{n} \binom{n}{i} \psi_k^{(i)}(1) P_{2k}^{(n-i)}(K;1) \right) \\ &= P_0^{(n)}(K;1) + \sum_{k=1}^{[n/2]} \left( \sum_{i=2k}^{n} (-1)^i \frac{n!(i-k-1)!}{(n-i)!(i-2k)!(k-1)!} P_{2k}^{(n-i)}(K;1) \right). \end{aligned}$$

In particular, we obtain

$$\begin{aligned} V_K^{(2)}(1) &= P_0^{(2)}(K;1) + 2a_2(K); \\ V_K^{(3)}(1) &= P_0^{(3)}(K;1) + 6P_2^{(1)}(K;1) - 6a_2(K); \\ V_K^{(4)}(1) &= P_0^{(4)}(K;1) + 12P_2^{(2)}(K;1) - 24P_2^{(1)}(K;1) + 24a_2(K) + 24a_4(K), \end{aligned}$$

cf. (5.6)–(5.11). Furthermore, combining with Theorem 3, we have

$$V^{(n)}(K;1) \equiv \sum_{k=1}^{[n/2]} \frac{n!(1-2^{2k})}{(n-2k)!} P_{2k}^{(n-2k)}(K;1) \tag{6.8}$$

in $V_n/V_{n-1}$.

**7. The link case.** Let $V_n^r$ denote the vector space consisting of all Vassiliev invariants for an $r$-component link of order less than or equal to $n$. We consider the subspace of $V_n^r$ that is spanned by

$$P_{2i-r+1}^{(n+r-2i-1)}, \quad i = 0, 1, \dots, \left[ \frac{n+r-1}{2} \right].$$

By Theorem 3, these are linearly dependent in $V_n^r/V_{n-1}^r$. If $r - n \geq 3$, then by Proposition 6, $P_{2i-r+1}^{(n+r-2i-1)}$ is a zero-map for $i = n+1, \dots, [(n+r-1)/2]$.

THEOREM 5. *Let $s = s(n,r) = \min\{n, [(n+r-1)/2]\}$ and $r \geq 2$. The dimension of the subspace of $V_n^r/V_{n-1}^r$ spanned by the following Vassiliev invariants of order $n$ is $s$:*

$$P_{2i-r+1}^{(n+r-2i-1)}, \qquad i = 0, 1, \dots, s.$$

We prove this theorem by making use of the space of singular links, which is dual to the space of Vassiliev link invariants. This space, which we denote by $(V_n^r)^*$, is the vector space over $\mathbb{Q}$ generated by equivalent classes of $r$-component singular links, subject to the following relations:

$$L_\times = L_+ - L_-; \tag{7.1}$$

$$L = 0 \text{ if } L \text{ has more than } n \text{ vertices.} \tag{7.2}$$

First we consider the knot case. Put

$$e_j = \frac{1}{(n-2j)!2^{n-2j}}(A^n_{n-1} + A^n_{n-3} + \cdots + A^n_{n-2j+1}).$$

Then from Lemma 2 (i), we have

$$P_{2i}^{(n-2i)}(e_j) = \delta_{ij},$$

where $i, j = 1, 2, \ldots, [n/2]$ and $\delta_{ij}$ denotes the Kronecker delta. Namely, $e_1$, $e_2, \ldots, e_s$ is the dual basis of $P_2^{(n-2)}$, $P_4^{(n-4)}, \ldots$, $P_{2s}^{(n-2s)}$, where $s = [n/2]$. Note that $P_0^{(1)}$ is a zero-map.

The following is analogous to Lemma 5.

LEMMA 7. *Suppose that $k + \ell = n$. Then*

$$P_k^{(\ell)}(\alpha^n \sqcup U) = -2\ell P_{k+1}^{(\ell-1)}(\alpha^n).$$

We denote by $\alpha^n \vdash U$ an $(n+1)$-configuration obtained by joining $\alpha^n$ and a circle $U$ with a new chord.

LEMMA 8. *Suppose that $k + \ell = n + 1$. Then*

$$P_k^{(\ell)}(\alpha^n \vdash U) = -4\ell(\ell-1)P_{k+1}^{(\ell-2)}(\alpha^n) + P_{k-1}^{(\ell)}(\alpha^n).$$

Proof. Applying (2.7), we have

$$P_k^{(\ell)}(\alpha^n \vdash U) = 2\ell P_k^{(\ell-1)}(\alpha^n \sqcup U) + P_{k-1}^{(\ell)}(\alpha^n).$$

Using Lemma 7, we obtain the result. ∎

Proof of Theorem 5. It is sufficient to prove: There exist vectors $e_j$ in $(V_n^r)^*$ such that

$$P_{2i-r+1}^{(n+r-2i-1)}(e_j) = \delta_{ij}, \qquad P_{1-r}^{(n+r-1)}(e_1) \neq 0,$$

where $i, j = 1, 2, \ldots, s$.

We shall use induction on $r$ $(\geq 2)$. Put

$$e_j = \frac{1}{(n-2j+1)!2^{n-2j+1}}(B^n_n + B^n_{n-2} + \cdots + B^n_{n-2j+2}).$$

Then from Lemma 2 (ii), we have

$$P_{2i-1}^{(n-2i+1)}(e_j) = \delta_{ij},$$

where $i, j = 1, 2, \ldots, [(n+1)/2]$. Also

$$P_{-1}^{(n+1)}(e_1) = \frac{-(n+1)!2^{n+1}}{(n-1)!2^{n-1}} = -4n(n+1).$$

Thus the statement holds for $r = 2$.

Suppose that the statement holds for an $r$-component singular link. Put $s = s(n, r)$. We have two cases:

(i) $s(n, r+1) = s$.
(ii) $s(n, r+1) = s+1$.

Case (i). First note that $n+r$ is odd. Using Lemma 7, we have

$$\begin{aligned} P^{(n+(r+1)-2i-1)}_{2i-(r+1)+1}(e_j \sqcup U) &= P^{(n+r-2i)}_{2i-r}(e_j \sqcup U) \\ &= -2(n+r-2i)P^{(n+r-2i-1)}_{2i-r+1}(e_j) \\ &= -2(n+r-2i)\delta_{ij}; \\ P^{(n+(r+1)-1)}_{1-(r+1)}(e_1 \sqcup U) &= P^{(n+r)}_{-r}(e_1 \sqcup U) \\ &= -2(n+r)P^{(n+r-1)}_{1-r}(e_1) \\ &\neq 0, \end{aligned}$$

where $i, j = 1, 2, \ldots, s$. Thus

$$\frac{-1}{2(n+r-2j)}(e_j \sqcup U), \qquad j = 1, 2, \ldots, s$$

is the desired vectors in $(V^{r+1}_n)^*$.

Case (ii). The codition yields that $n+r$ is even and $n \geq (n+r)/2 = s+1$. We have

$$P^{(n+(r+1)-2i-1)}_{2i-(r+1)+1}(e_j \sqcup U) = \begin{cases} -2(n+r-2i)\delta_{ij} & \text{if } i, j = 1, 2, \ldots, s; \\ -2(n+r)P^{(n+r-1)}_{1-r}(e_1)(\neq 0) & \text{if } i = 0,\ j = 1; \\ P^{(0)}_n(e_j \sqcup U) = 0 & \text{if } i = s+1,\ j = 1, 2, \ldots, s. \end{cases}$$

Since $s(n-1, r) = s$, by the inductive hypothesis, there exists $f \in (V^r_{n-1})^*$ such that $P^{(0)}_{n-1}(f) = 1$. Then using Lemma 8, we have $P^{(0)}_n(f \vdash U) = P^{(0)}_{n-1}(f) = 1$. Thus from $e_j \sqcup U$ $(j = 1, 2, \ldots, s)$ and $f \vdash U$, we can construct desired set of vectors in $(V^{r+1}_n)^*$. ■

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