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INVARIANTS OF PIECEWISE-LINEAR KNOTS

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Abstract. We study numerical and polynomial invariants of piecewise-linear knots, with the goal of better understanding the space of all knots and links. For knots with small numbers of edges we are able to find limits on polynomial or Vassiliev invariants sufficient to determine an exact list of realizable knots. We thus obtain the minimal edge number for all knots with six or fewer crossings. For example, the only knot requiring exactly seven edges is the figure-8 knot.

1. Introduction. In the theory of knots and links, piecewise-linearity is usually regarded as a useful tool, a means to understanding knots which are most likely preferentially regarded as smooth. Here we shall take a different viewpoint, that piecewise-linear knots have interest in their own right and that piecewise-linear (or PL) invariants are of interest. The most elementary such invariant is simply the minimal number of edges required to form the given knot or link, and we will study this invariant (called the "edge number") in this paper.

Primary motivations for such a viewpoint are applications of knot theory in molecular biology and stereochemistry, in which the knot or link is used to model the physical structure of circular DNA or other large molecules [14], or of small molecules with carbon rings, such as cyclohexane [10]. It is often useful to regard the molecule as a graph made up of atoms as vertices and bonds as edges. Thus we regard the PL structure as a matter of primary interest, and one is concerned with such questions as how many edges are required to accomplish some knot type.

Because of this motivation, there is a variety of modifications one might make to a simple model of vertices and edges, such as labelling (or "coloring") edges or vertices, requiring fixed length or ratios of lengths along edges, or requiring adjacent edges to meet with certain fixed angles. In this paper we will generally ignore such interesting refinements in order to develop general tools.

After presenting some preliminaries and notation we consider the crossing number

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and (super)bridge number in Section 2. A very useful notion in this regard is the idea of a convenient projection or special position for a PL knot. In Section 3 we consider classical and neoclassical polynomial invariants of PL knots and show how to obtain limitations on these invariants given a fixed number of edges. In Section 4 we consider Vassiliev invariants (also called invariants of finite type). In the PL case we outline a recursive scheme for determining what the Vassiliev invariants of a given knot or link of n edges might be. As a consequence of the calculations and techniques of this section we show that the only knot with edge number equal to seven is the figure-8 knot (see also [8], [4]). We observe that these results (Theorem 5, Corollaries 8 and 9) complete the determination of the edge numbers $e(K)$ of all prime knots with crossing number six or smaller.

THEOREM 1. *Edge numbers of prime knots of six or fewer crossings are given as follows*:
$e(\textit{unknot}) = 3$,
$e(\textit{trefoil}) = 6$,
$e(\textit{figure-}8) = 7$,
$e(K) = 8$, *for* K *any prime five- or six-crossing knot*, *and*
$e(K) \geq 8$ *for any other knot.*

M. Meissen has constructed all prime seven-crossing knots with 9 edges, so that their edge numbers must be eight or nine, and she has constructed the knots 8_{19} and 8_{20} with eight edges, showing that their edge number is eight. See [6] for explicit coordinates and illustrations of these examples. Also, in a recent paper studying torus knots and links, G. T. Jin [3] has constructed the square and granny knots with eight edges.

In Section 5 we consider the space M_n of all knots with n edges. If we do not take into account sometimes appropriate equivalence relations, M_n is simply an open subset of $\mathbb{R}^{3n}$. It is a natural object to study [9]; path components are simply PL knot types and in some sense $M_n \to \mathcal{K}$, where $\mathcal{K}$ is the space of all knots, as $n \to \infty$. Many of the techniques of the preceding sections can be reinterpreted in this context. We will not pursue this in this article; in a forthcoming paper we will consider in detail the computation of the cohomology of M_n. We conclude with a bit of conjecture and speculation.

2. Projections; crossing and bridge numbers. We consider a knot in $\mathbb{R}^3$ consisting of n straight line segments, called edges and denoted $e_1, e_2, \ldots, e_n$. The vertices will be numbered $v_1, v_2, \ldots, v_n$, so that the edge e_i is $v_i v_{i+1}$ in an oriented sense. The polyhedral knot K (or K_n) is simply the union of the edges. We consider the space of all such knots as an open subspace M_n of $(\mathbb{R}^3)^n$. We also consider the discriminant Σ_n of singular PL knots, so that

$$M_n = (\mathbb{R}^3)^n \setminus \Sigma_n$$

We follow the convention of deleting the number of segments n whenever possible.

We are interested in studying the space M and in particular its path components, the set of (n-segment) knot types. This is somewhat similar to the approach of Vassiliev in [15]. While Vassiliev's interest is in the "stable" case we are interested in both the unstable and stable case. In particular, we want to exhibit an explicit description of M and Σ for small values of n, such as $4, 5, 6, 7$. This would include precise information about which knots can occur. It is hoped that this particular approximation of the space of all knots will prove useful in settling a number of the problems mentioned by Vassiliev and attacked by others.

In this paper we will use several slightly different approaches to the cases with n small. That is, we first examine how simple a projection one can find for a PL knot with few edges. It turns out the problem is manageable for these values of n, and we get detailed information about the relevant space. The second method applies some standard tools of knot theory to the problem. In particular, here we use Jones, and Conway polynomials as well as low order Vassiliev invariants, together with the classification of knots (and links) of low crossing number. This method implicitly begins to study the boundaries between knots, or in other words, various "pieces" of Σ.

2.1. *Projections.* There are advantages to the PL case when one considers knot projections. Naturally enough, one is interested in projections of the knot which enable one to simplify calculations. As a first step in understanding projections, we examine Σ in a somewhat cursory way.

In particular, how can n numbered vertices be arranged in space so that connecting the dots in cyclic order does *not* yield a simple closed curve? There are the following possibilities:

(i) two vertices coincide,
(ii) a vertex lies on an edge,
(iii) two edges partially coincide,
(iv) two or more edges intersect transversely at a point.

Notice that these conditions all define closed semialgebraic subsets of $\mathbb{R}^{3n}$, of various codimensions. The *discriminant* Σ is simply the union of those n-tuples of vertices satisfying one or more of the above four conditions. We will use the term *knot* to refer to a point $K \in M = \mathbb{R}^{3n} \setminus \Sigma$, while referring to a point $S \in \Sigma$ as a *singular knot*.

Definition 2. A *convenient projection* of a knot K is obtained by projecting perpendicular to the first edge e_1, then tilting slightly so that the edge e_1 (in projection) is not involved in any crossings, and in addition, so that e_2 and e_n do not cross.

It is an easy exercise to show that convenient projections exist. Note that in order to obtain such a projection one may have to move K slightly in the open set M in order to insure that the *diagram* of the knot has n edges with only over- and under-crossings. We will henceforth use such general position considerations without further comment.

The notion of convenient projection is a good example of a useful property of PL knotting. Using a convenient projection of the knot, one can essentially make an edge disappear or ensure that that edge is not involved in any crossings.

We first make some straightforward observations about crossing and (super)bridge numbers of PL knots. Recall that the crossing number $c(K)$ is the minimum number of crossings, taken over all projections and all knots within the given isotopy class. The bridge number $b(K)$ is the minimum number of relative maxima (in the y-direction of the projection plane), taken over all projections and all knots within the given isotopy class. To define the superbridge number $sb(K)$ [5] we first take the maximum number of relative maxima over all directions for a particular embedding, then minimize over the isotopy class. In [5] it was shown that for true knots $2 \leq b(k) < sb(K)$.

PROPOSITION 3. *Suppose K is a PL knot with n edges. Then*

(i) *if n is even,*

$$c(K) \leq (n-1)(n-4)/2$$

and if n is odd ($n > 3$),

$$c(K) \leq (n-1)(n-4)/2 - (n-3)/3$$

(ii) *For $n > 3$,*

$$b(K) \leq (n-2)/2 \quad \textit{and} \quad sb(K) \leq n/2.$$

COROLLARY 4. *A PL knot with five edges is unknotted.*

Proof (of proposition). Observe that in a projection of K with only transverse crossings, no edge can cross an adjacent edge. Thus there can be at most $n(n-3)/2$ crossings. Now consider a convenient projection of K: The edge e_1 does not intersect any other edges, and the edges e_2 and e_n do not cross. Thus there are at most $n(n-3)/2 - (n-3) - 1 = (n-1)(n-4)/2$ crossings as claimed.

Now suppose n is odd. Consider any one of the edges except e_n, e_1, or e_2. Such an edge, say e_k, could possibly cross any of the $n-4$ edges $e_{k+2}, e_{k+3}, \ldots, e_n, \hat{e}_1, \ldots, e_{k-2}$. Note that $n-4$ is odd. If e_k crosses all of these edges, then v_{k+2} and v_{k-1} must be on opposite sides of the line in the projection plane which contains e_k. Hence e_{k-1} and e_{k+1} do not intersect. Thus any such edge e_k either does not cross one of the above $n-4$ edges, or e_{k-1} and e_{k+1} do not intersect. Notice that if we count the number of missing crossings, we have one for each of the $n-3$ edges, with each missing crossing counted at most three times. Thus we can subtract $(n-3)/3$ from the number of possible crossings.

The result for the (super)bridge number follows by noting that a knot with n vertices can have at most $n/2$ relative maxima, no matter what the direction of projection. Thus $sb(K) \leq n/2$, so by the result of Kuiper[5] $b(K) \leq n/2 - 1$.

Notice that for $n = 5$ this theorem yields a crossing number at most one, while for $n = 6$, $c(K) \leq 5$ and $b(K) \leq 2$, and for $n = 7$, $c(K) \leq 7$ and $b(K) \leq 2$. Since a knot with crossing number one is unknotted, we find that all knots with five edges are unknotted. We will use the fact that a 7-edge knot has a projection with seven or fewer crossings in a later section.

It is quite possible that in certain situations the above estimates can be improved. We include them here for their own interest, and because the form above provides the tools we need for later. Clearly, one may carry out similar computations for some other numerical invariants.

3. Polynomial invariants. In this and the next section we study finer invariants of the knots. Since we are considering the *space* of PL knots, it is natural to consider Vassiliev's invariants of finite type [15], which are based on the degree zero cohomology of the smooth knot space. Of the various polynomial invariants available, we choose to study the Conway and Jones polynomials. It is shown in [1] that the coefficients of the Conway polynomial are Vassiliev invariants, and in fact in Vassiliev's basis for his invariants, exactly the first two non-zero Conway coefficients appear by order four. In [2] (see also [1]) it is shown that the coefficients of the power series expansion of the Jones polynomial evaluated at e^x are also Vassiliev invariants.

In these sections we consider the problem of determining restrictions on these invariants, given restrictions on the number of edges of the knot. Such results, combined with the classification of knots of low crossing number and the Proposition above, allow us to determine the edge number for all knots of six or few crossings, and to determine all knots with seven or fewer edges.

3.1. *The Conway and Jones polynomials.* From our point of view the key property of these invariants is the recursive scheme for their computation. That is, if we follow the usual convention and consider the three links in Figure 1, the Conway polynomial $C(z)$

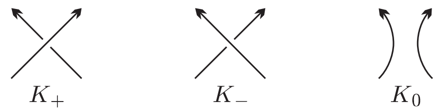

Fig. 1 (links are the same outside circles)

and the Jones polynomial $V(q)$ satisfy the relations

$$C(K_+) - C(K_-) = zC(K_0) \tag{1}$$

and

$$q^{-1}V(K_+) - qV(K_-) = \alpha(q)V(K_0) \tag{2}$$

where

$$\alpha(q) = q^{1/2} - q^{-1/2} \tag{3}$$

We will use these relations to study these polynomials for knots of six or seven edges. As a starting point, note that both polynomials take the value 1 for any knot with five edges, since such a knot is unknotted.

3.1.1. *Knots with six edges*

Theorem 5. *Any knot with six edges has the Conway and Jones polynomial of the unknot, the trefoil knot, or the mirror image of the trefoil knot. Hence, the only knots which can be realized with six edges are these three knots.*

It was observed in [10] that one can realize both trefoil knots with six edges.

Proof. Suppose we are given a knot K determined by the six vertices $v_1, v_2, \ldots, v_6$. We note that v_4, v_5, v_6 are the vertices of a triangle T. We attempt to isotope through PL knots by moving the vertex v_5 across T. If no other edges of K intersect T, then in fact we do obtain an isotopy of K to a five edge knot, so that K is unknotted.

We note that only the edges e_1 and e_2 can possibly intersect T, and that either edge can pass through in either the positive or negative direction. (We use the orientation on T

given above and the usual right-hand rule to determine the direction of intersection.) Our plan is to observe that we modify our given knot to the unknot with one or two crossing changes. By observing what happens at the crossing changes we obtain our result. We consider all cases in detail in view of the orientation subtleties.

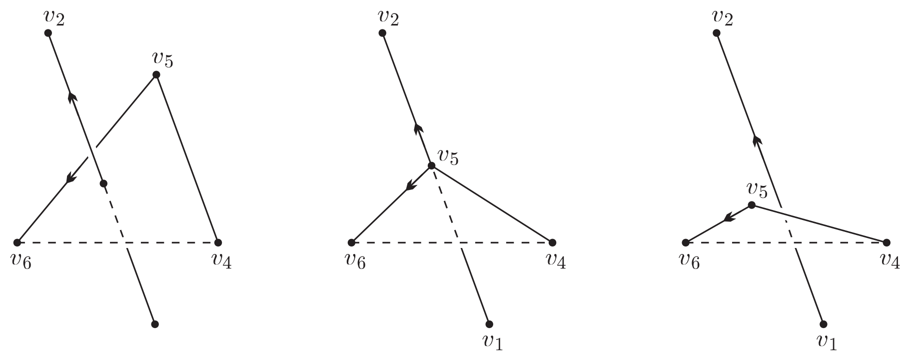

Fig. 2

First, suppose e_1 passes through T in the positive sense. In Figure 2 we show the situation just before, as, and after the vertex v_5 passes through the edge e_1. Note that we change from K_+ to K_-. Now suppose that the other edge e_2 does not pass through T. Then K_- is the unknot (it has five edges), and from (1) we see that

$$C(K_+) - 1 = zC(K_0)$$

so that we need to understand $C(K_0)$. From Figure 2 it is easily seen that $C(K_0)$ is a link with two components, consisting of the PL knots v_1, v_5', v_6 and v_4, v_5'', v_2, v_3. The first of these components is an unknotted planar triangle and the second is unknotted and has four edges. We prove that the associated link is either the unlink of two unknotted circles, or is a (+)-Hopf link (see Figure 3). In fact, note that the edge v_2, v_3 misses the

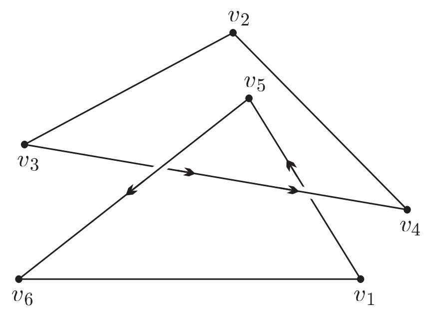

Fig. 3. A (+)-Hopf link

triangle bounded by v_1, v_5', v_6. Thus, if the two components of the link are in fact to truly link, the edge v_3, v_4 must intersect this triangle. With respect to the induced orientation of this triangle, v_4 is *above* its plane. Thus v_3, v_4 passes from below to above, and this means that the linking is in the (+) sense as claimed.

Thus $C(K_0)$ is either 0 or z respectively, so that $C(K_+) = 1$ or $1 + z^2$.

Next we refer again to Figure 2, with e_1 passing through negatively. Then we change from K_- to K_+. Again supposing that e_2 does not intersect T, we know that K_+ is the unknot (since it has five edges). Arguing as before, we see that K_0 is a link with two components, either the trivial unlinked pair, or a Hopf link. In the linked case, the analogous argument shows that it is a $(-)$-Hopf link, so that $C(K_0) = -z$, so that the original knot K has $C(K) = 1 + z^2$ as claimed.

Finally, we observe that if both e_1 and e_2 intersect T, then the triangle T_1 with vertices v_1, v_2, v_3 intersects T in a line segment in the interior of T, and the triangle T_1 intersects no edges of K. For, note that only the edges e_4 and e_5 can possibly intersect $\mathrm{int}(T)$, and since T and T_1 intersect as they do, this is impossible. Thus since T_1 misses all edges of K one can isotope across T_1 to show that K is unknotted.

We turn next to the Jones polynomial, but first note that already we have shown that only the unknot and trefoil can be realized in M_6, since any knot with six edges has a projection with five or fewer crossings, and among the knots of five or fewer crossings, the Conway polynomial characterizes the unknot and trefoil knots.

The argument for the Jones polynomial is precisely the same, but using (2). Thus we have

$$q^{-1}V(K_+) - q = \alpha(q)V(K_0)$$

where $V(K_+)$ is to be determined and K_0 is the trivial link of two components, or a $(+)$-Hopf link. These two links have Jones polynomials $-q^{1/2} - q^{-1/2}$ and $-q^{1/2} - q^{5/2}$ respectively, so that

$$V(K_+) = 1 \quad \text{or} \quad V(K_+) = q^1 + q^3 - q^4$$

The latter is of course the Jones polynomial of the $(+)$-trefoil knot.

The other possible direction of intersection can give the Jones polynomial of the $(-)$-trefoil.

We could use similar techniques as the number of edges increases. In the next section we will do so, but using Vassiliev invariants in place of polynomial invariants.

4. Vassiliev invariants. In this section we will use techniques similar to the above to determine Vassiliev invariants through order four for all knots with seven or fewer crossings. We will use Vassiliev's own basis (or "actuality table") as is found in [15]. This particular basis is handy for several reasons, not the least of which is the table of invariants for all prime knots through seven crossings prepared by Vassiliev. In addition, with this basis the order two invariant is just the coefficient of z^2 in the Conway polynomial and the third invariant of order four is the negative of the coefficient of z^4. This provides a check to our computations, and allows one to see the relationship of the invariants.

4.1. *Brief description of Vassiliev's invariants.* The invariants described by Vassiliev in [15] are numerical invariants of knots which extend to invariants of immersions of the circle with transverse double points, or nodes. As pointed out in the introduction of [1], given a numerical invariant for knots, one can define numerical invariants V for immersions $K_\bullet$ with nodes by the formula

$$V(K_\bullet) = V(K_+) - V(K_-)$$

Here K_+ and K_- are formed from $K_\bullet$ by changing one transversal crossing to a positive or negative crossing in the projection.

Then an invariant is defined to be of "finite type" (order m) if V vanishes for all immersions with more than m nodes. Vassiliev invariants turn out to be exactly those of finite type with the above definition. The question, of course, is how does one find and compute such invariants. Vassiliev solves this problem completely. The invariants themselves are certain zero-th cohomology classes of the complement of the space of all knots. By appropriate duality, such classes correspond to higher degree cohomology classes of the discriminant in the knot space (the discriminant is the set of all smooth maps of the circle into space which are not embeddings). Such a cohomology class can be thought of as a linear combination of various "pieces" of the discriminant. These pieces are made up of nodal immersions.

The actual computation of the invariants proceeds via an "actuality table" which involves a number of choices. In this paper we will use Vassiliev's original actuality table, for the reasons mentioned above. The idea of the computation is to consider a generic path in the space of knots, running from the given knot to the unknot. Such a path will several times transversely cross the discriminant at one of its generic or nonsingular points, that is, at an immersion with a single node. At each crossing the value of the invariant changes (depending on orientation) by the value of the invariant on the particular piece of the discriminant. In turn, the values on these nonsingular parts of the discriminant change as one crosses through binodal immersions, and so forth.

Thus the actuality table is a list of values of the invariants for relevant nodal curves. An invariant of order m vanishes on curves with more than m nodes, so that one need only consider those curves with m or fewer nodes when computing such an invariant.

We will present determination of the five invariants of order four or less for PL knots with seven or fewer edges. As a consequence we obtain the edge number for all knots with six or fewer crossings. We will also present a recursive scheme for computation of Vassiliev invariants of any PL knot.

We follow Vassiliev in writing these invariants as $[a; b; c, d, e]$, where the values inside the brackets are rational (actually, half-integer) numbers. The semicolons separate invariants of different orders. Thus, a is the order two invariant, while c, d, e are the three invariants of order four. We note the following: a is the coefficient of the quadratic term in the Conway polynomial, while $-e$ is the coefficient of the degree four term. Vassiliev has normalized the invariants so that the even order invariants are the same for mirror image knots, while the odd order invariants change by a factor of -1. Thus if the knot K is achiral (equivalent to its mirror image), all odd order invariants vanish. Tables such as those of Vassiliev and T. Stanford [13] suggest that these invariants are very powerful in detecting chirality.

4.2. *The recursive scheme.* We have seen this pattern in Section 3. Suppose we are given a PL knot with n edges. We choose three consecutive vertices and attempt to isotope across the corresponding triangle to a PL knot with $n-1$ edges. The ending knot, since it has fewer edges, has invariants already computed. At the point of each self-intersection we have an immersion $K_\bullet$ with one node, and we look up its invariants in the table.

In fact, of course, the values for $K_\bullet$ will not be in the table, but must be computed. The crucial point is that $K_\bullet$ will be fairly simple. For example, when $n = 6$ $K_\bullet$ is a graph with seven edges and a single four-valent vertex. One can think of $K_\bullet$ as a PL *embedding* of a figure-8 graph in which one of the loops is PL with three edges and the other PL with four edges. Thus $K_\bullet$ is quite restricted. Thus the key step in proceeding from n to $n+1$ is the study of such restricted $K_\bullet$.

4.3. *Calculations for $n = 6$.* We have actually done the work for this calculation in Section 3. Here we add one further piece of information and write down the answer.

THEOREM 6. *If K is a PL knot with six edges, its Vassiliev invariants are one of the following*:

(i) $[0; 0; 0, 0, 0]$,
(ii) $[1; -1; 0, 0, 0]$,
(iii) $[1; 1; 0, 0, 0]$.

Proof. Suppose we are given a knot K determined by the six vertices $v_1, v_2, \ldots, v_6$. We note that v_4, v_5, v_6 are the vertices of a triangle T. As before, we attempt to isotope through PL knots by moving the vertex v_5 across T. If no other edges of K intersect T, then in fact we do obtain an isotopy of K to a five-edge knot, so that K is unknotted, and (i) holds.

If e_1 passes through T in the positive sense then in Figure 3 we show the situation just before, as, and after the vertex v_5 passes through the edge e_1. Note that we change from K_+ to K_-. Now suppose that the other edge e_2 does not pass through T. Then K_- is a five-edge unknot, and

$$V(K_+) - V(K_-) = V(K_\bullet)$$

implies that

$$V(K) = V(K_+) = V(K_\bullet)$$

so that we need to understand $V(K_\bullet)$. From Figure 2 it is easily seen that $K_\bullet$ is a figure-8 graph with two loops, the PL knots v_1, v_5', v_6 and v_4, v_5', v_2, v_3. The first of these loops is an unknotted planar triangle and the second is unknotted, since it has four edges. An analysis as in Section 3 shows that $K_\bullet$ must be one of the two embedded graphs shown in Figure 4. Both crossings are positive in the second graph because in order for the edge v_3v_4 to intersect the triangle which is the convex hull of v_1, v_5', v_6, the vertex v_3 must lie below the plane of this triangle. This holds, because we already know that v_4 is above the plane.

If the first of the alternatives holds, then $V(K_\bullet) = [0; 0; 0, 0, 0]$. If the second holds, then $V(K_\bullet) = [1; -1; 0, 0, 0]$.

Next we refer again to Figure 2, with e_1 passing through negatively. Then we change from K_- to K_+. Again supposing that e_2 does not intersect T, we know that K_+ is the unknot (since it has five edges). Arguing as before, we see that $V(K_\bullet) = [1; 1; 0, 0, 0]$.

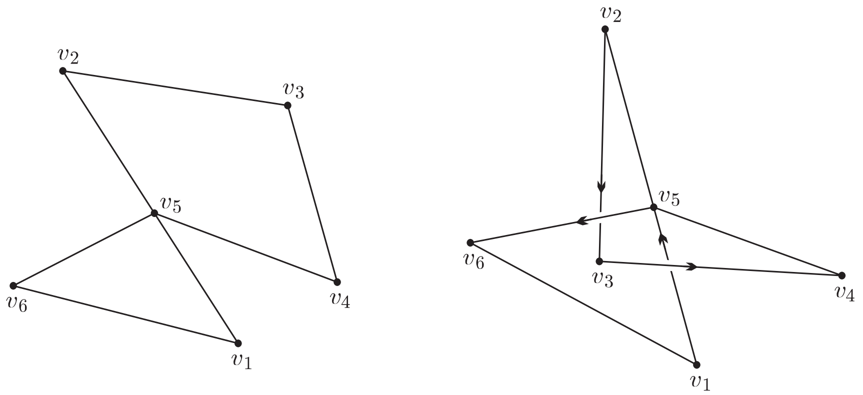

Fig. 4

Finally, we observe that if both e_1 and e_2 intersect T, then the triangle T_1 with vertices v_1, v_2, v_3 intersects T in a line segment in the interior of T, and the triangle T_1 intersects no edges of K. For, note that only the edges e_4 and e_5 can possibly intersect $\operatorname{int}(T)$, and since T and T_1 intersect as they do, this is impossible. Thus since T_1 misses all edges of K one can isotope across T_1 to show that K is unknotted.

4.4. *Calculation for* $n = 7$. The idea here is precisely the same as above. The two key elements are the analysis of how the triangle intersects the rest of the knot, and of the imbedded graphs resulting at these intersections. The graphs resulting are again figure-8 graphs. In one case, each loop has four edges, while in the other one loop has three and the other has five edges. Notice that in both cases the individual loops are unknotted.

THEOREM 7. *Suppose K is a PL knot with seven edges. Then the Vassiliev invariants of K through order four are* $[a; b; 0, d, 0]$.

COROLLARY 8. *The only knots which can be formed with seven edges are the unknot, the two trefoil knots, and the figure-8 knot.*

This result also appears in [8] and [4].

Proof (of corollary). A seven-edge knot has a projection with seven or fewer crossings by Proposition 3. Thus, the result follows for *prime* knots by examining Vassiliev's table [15]. It follows for all knots with seven or fewer crossings by noting that the theorem shows that the Conway polynomial (through degree four) is simply $1 + az^2 + 0z^4 + \ldots$ But the Conway polynomial multiplies for composition of knots, and one can check that no possible true composite knots with seven or fewer crossings can have such a polynomial.

COROLLARY 9. *The edge number for all five and six-crossing prime knots is eight. The edge number for any knot requiring more than six crossings is at least eight.*

Proof. The previous corollary shows that the edge number is at least eight. K. Millett [7] has shown that all five and six-crossing prime knots can be realized with eight edges.

Proof (of theorem). There are two key points in the proof. The first is the consideration of exactly how various edges can intersect the triangles formed by successive vertices. The second is the question of how the relevant graphs are embedded.

In order to consider the first question, we need some notation. Let T_i denote the triangular (two-dimensional) region which is the convex hull of the vertices v_i, v_{i+1}, v_{i+2}, with this orientation.

LEMMA 10. *After possibly cyclically renumbering the vertices, one of the following three cases holds*:

(i) $\mathrm{Int}(T_5)$ *intersects none of the edges of* K.

(ii) $\mathrm{Int}(T_5)$ *intersects exactly one of the edges of* K, *namely* e_1, e_2, *or* e_3; *or*

(iii) $\mathrm{Int}(T_5)$ *intersects* e_1 *and* e_3 *but does not intersect any other edges.*

Proof. Consider T_5. We note that $\mathrm{Int}(T_5)$ cannot intersect any of the edges e_4, e_5, e_6, e_7. Suppose both e_1 and e_2 intersect $\mathrm{Int}(T_5)$ non-trivially. Then consider the triangle T_1. Since $T_1 \cap T_5$ must be a line segment in $\mathrm{Int}(T_5)$, the only edge which can possibly intersect $\mathrm{Int}(T_1)$ is e_4. Thus, after renumbering we have either case (i) or case (ii).

A similar argument applies in the case that the two consecutive edges e_2 and e_3 intersect T_5 non-trivially. Thus the only possibilities are those stated. ∎

Let us now complete the proof of Theorem 7. The proof is by explicit computation; we describe below a way to organize the cases involved. First, by Lemma 10 there are three main cases.

CASE (i): T_5 intersects no edges of the knot. Then the knot is isotopic to a knot with six edges, and Theorem 6 applies.

CASE (ii): T_5 intersects exactly one edge e_1, e_2, or e_3 of the knot. First suppose that edge is e_1 or e_3. Then, letting the intersection point be labelled $\star$, we have the associated graph $K_\bullet$. We may think of the graph $K_\bullet$ as consisting of two loops in space, sharing the point $\star$ in common. One loop has three edges, with vertices we will call $\star, a_2, a_3$. The other loop has five edges labelled $\star, b_2, \ldots, b_5$. Orientation is given as inherited from K, and following increasing subscripts. We consider the projection of $K_\bullet$ into a plane perpendicular to the line segment a_2a_3. One then easily sees that the algebraic intersection number of the loop $\star, b_2, \ldots, b_5$ with the triangle spanned by $\star, a_2, a_3$ is 0 or ± 1. Explicit computation of the possible cases show that in this case the order four Vassiliev invariants of $K_\bullet$ are all zero. Therefore we can isotope K to a PL knot with six edges through $K_\bullet$, and thus the order four Vassiliev invariants of K are all zero.

Now suppose that T_5 hits exactly the edge e_2 and no other edges. Then $K_\bullet$ consists of two loops of four edges each, meeting at $\star$. We number the vertices $\star, a_2, a_3, a_4$ and $\star, b_2, b_3, b_4$ in a manner similar to case (i). Now project to a plane perpendicular to the line segment a_2a_4. Again, the algebraic intersection number of the two loops is 0 or ± 1. Explicit computation of possible cases shows that the order four invariants are of the form $[0, d, 0]$. Thus in this case K has Vassiliev invariants through order four of form $[a; b; 0, d, 0]$.

CASE (iii): In this case the edges e_1 and e_3 are hit in succession. Thus as in the first part of the proof for case (ii) all order four Vassiliev invariants are zero.

EXAMPLE 11. K. Smith [12] gave the following realization of the figure-8 knot with seven edges:

$$\begin{array}{ll} v_1 = (0,-1,0) & v_5 = (1,1,9) \\ v_2 = (0,2,0) & v_6 = (-1/2,1,-4) \\ v_3 = (-2,0,1) & v_7 = (1,-1,7) \\ v_4 = (1,0,1) & \end{array}$$

We leave it as an exercise to analyze this particular knot in view of the proof of Theorem 7 above.

5. The space of all PL knots; a bit of speculation. We briefly consider the space M_n of all knots with n edges. By identifying a knot K with its ordered n-tuple of vertices in three space we see that actual (embedded) PL knots with n edges thus form this open dense subset of $\mathbb{R}^{3n}$. A path component of M_n is thus simply a PL knot type and is contained in a path component of the space of all knots. In some sense $M_n \to \mathcal{K}$, where $\mathcal{K}$ is the space of all knots, as $n \to \infty$. Many of the results of the preceding sections can be reinterpreted in this context, and our techniques can be used to gather information about the number of path components of M_n. There is one path component for $n \leq 5$, and there are at least three (the unknot and the two trefoils) for $n = 6$. For the next case $n = 7$ we know that there are at least four path components, but there is no *a priori* reason that there could not be, say, two path components of figure-8 knots. In a similar way, there is no particular reason to expect that any two unknots with six edges are connected by a path in M_6. In a forthcoming paper we will consider in detail the determination of the path components of M_n and its variants, using cohomological techniques.

The most obvious realm of speculation given these initial results is certainly the relationship of the edge number of a knot to classical numerical invariants, such as the crossing number or the bridge number. It is known [2] that the unknotting number is not a Vassiliev invariant. We will observe below that the edge number also is not a Vassiliev invariant. The crossing number gives the traditional partial ordering of knots, as in knot tables. One could also use the edge number. It is natural to ask to what extent these two orderings are compatible.

With respect to this question, let us observe that Proposition 3 may be interpreted as a suggestion that asymptotically the crossing number is proportional to the square of the edge number. In [8], S. Negami showed that $e \leq 2c$. Thus by rearranging the inequality of Proposition 3 one obtains the bounds

$$\frac{5 + (25 + 8(c-2))^{1/2}}{2} \leq e \leq 2c$$

Asymptotically this inequality becomes $\sqrt{2c} \leq e \leq 2c$

The following example arose from discussions with J. Simon and J. Przytycki:

EXAMPLE 12. Consider the torus knots $L(2,q)$. It is known [5] that $sb(L(2,q)) = \dot{4}$, $q \geq 5$. It is a classical fact [11] that $b(L(2,q)) = 2$ and $c(L(2,q)) = q$. Thus in this family, as $q \to \infty$ we have constant bridge and superbridge number, while the crossing number, and hence edge number, grows without bound. On the other hand, by the inequality above the edge number grows no faster than $2q$. This fact can be used, arguing as in [2], to show that the edge number is not a Vassiliev invariant.

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