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COMPARING QUANTUM DYNAMICAL ENTROPIES

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Abstract. Last years, the search for a good theory of quantum dynamical entropy has been very much intensified. This is not only due to its usefulness in quantum probability but mainly because it is a very promising tool for the theory of quantum chaos. Nowadays, there are several constructions which try to fulfill this need, some of which are more mathematically inspired such as CNT (Connes, Narnhofer, Thirring), and the one proposed by Voiculescu, others are more inspired by physics such as ALF (Alicki, Lindblad, Fannes). Therefore, a natural question arises whether there is a relation between all these different notions. In this paper we will indicate that the CNT entropy turns out to be smaller than the ALF dynamical entropy.

1. Introduction. The aim of this paper is to study the relation between the dynamical entropy as proposed by Alicki, Lindblad and Fannes (ALF) and the one proposed by Connes, Narnhofer, Thirring and Størmer (CNT). The main difference between both theories is that the one of CNT is more inspired by mathematics while the ALF theory got its inspiration in physics.

The CNT entropy couples the quantum system to a classical system and computes then the mutual information of the composed system. The classical system represents the measuring device. Although, it has all natural properties that one expects from a dynamical entropy, it is hard to compute in concrete models. ALF on the other hand, couples the quantum system to its coarse grained model which is again a quantum system. Hence, in this theory one tries to model a quantum mechanical system in a quantum manner. As coarse grained model one uses a quantum spin chain and the dynamics on the system translates into a shift on the quantum spin chain. This approach gives nice results in concrete models but it is much harder to prove general properties.

It turns out that there is a relation between the CNT and the ALF entropy in the sense that CNT can be expressed in terms of partitions of unity. This relation will then be

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used to prove the inequality between both after one time step. However, we still believe that this result can be extended to infinitely long times but we found a gap in our proof. Therefore, the result is formulated as a conjecture.

The paper is organized as follows. Section 2 reformulates the ALF entropy as a coupling with another quantum system. For the original construction we refer to [AF]. In order to be able to explain the relation between CNT and ALF we recall very briefly in section 3 the construction of CNT. In section 4 we present some examples to illustrate the inequality between both entropies. The most striking difference is found in the example of the free shift. The other two examples that we present are the quantum cat map and the shift on a quantum spin chain. In section 5 finally, the relation between both is shown. For proofs however we refer the interested reader to [TUY].

2. The ALF dynamical entropy. The goal of this section is to interprete the construction, as proposed in [AF], in terms of a coupling of the quantum dynamical system $(\mathcal{A}, \Theta, \omega)$ with the coarse grained model i.e. a quantum spin chain. By a coupling between a dynamical system and a coarse grained model, say $\mathcal{M}_k$, we will mean a completely positive map:

$$\Phi : \mathcal{M}_k \otimes \mathcal{A} \to \mathcal{A}.$$

Therefore, the starting point is a sequence of extreme completely positive, unity preserving mappings $\{\Phi_n\}_n$:

$$\Phi_n : \mathcal{M}_k^{\otimes[0,n-1]} \otimes \mathcal{A} \to \mathcal{A}.$$

To each mapping Φ_n, we associate a new mapping denoted by $\hat{\Phi}_n$:

$$\hat{\Phi}_n : \mathcal{M}_k^{\otimes[0,n-1]} \to \mathcal{A}$$

which is obtained as a restriction of the mapping Φ_n to $\mathcal{M}_k^{\otimes[0,n]}$, i.e. for all $A \in \mathcal{M}_k^{\otimes[0,n]}$

$$\hat{\Phi}_n(A) = \Phi_n(A \otimes \mathbb{1}_{\mathcal{A}}).$$

On the other hand, we denote the restriction of the map Φ_1 to the algebra $\mathcal{A}$ by Γ. The map $\Gamma : \mathcal{A} \to \mathcal{A}$ is hence defined as:

$$\Gamma(x) = \Phi_1(\mathbb{1}_{\mathcal{M}_k} \otimes x) \quad \text{for all} \quad x \in \mathcal{A},$$

and consequently completely positive. We require that this sequence of mappings satisfies the following conditions:

i) $\{\hat{\Phi}_n\}$ is a sequence of unity preserving completely positive mappings that are compatible: for $A \in \mathcal{M}_k^{\otimes[0,n]}$

$$\hat{\Phi}_{n+1}(A \otimes \mathbb{1}) = \hat{\Phi}_n(A).$$

ii) In order to make a coarse grained picture of the dynamical system $(\mathcal{A}, \Theta, \omega)$ on the spin chain $\mathcal{M}_k^{\otimes \mathbf{N}}$, in such a way that one time step of the dynamics is translated into a shift to the right on the spin model, we impose the following condition:

$$\hat{\Phi}_\infty \circ \mathbf{S}(A) = \Gamma \circ \Theta \circ \hat{\Phi}_\infty(A)$$

for A an arbitrary element in the quasi local algebra $\mathcal{M}_k^{\otimes \mathbf{N}}$. By $\mathbf{S}$ we denote the shift to the right on $\mathcal{M}_k^{\otimes \mathbf{N}}$ and by $\hat{\Phi}_\infty$ the extension of the maps $\hat{\Phi}_n$ to $\mathcal{M}_k^{\otimes \mathbf{N}}$. Because of compatibility, the relation $\omega^\infty \equiv \omega \circ \hat{\Phi}_\infty$ defines a state on the spin chain $\mathcal{M}_k^{\otimes \mathbf{N}}$. The quantum dynamical entropy of the system $(\mathcal{A}, \Theta, \omega)$ is then computed as the supremum of the entropy densities of the states ω^∞:

$$\mathsf{h}^{\mathrm{ALF}} = \sup_{\mathrm{CP}} \sigma(\omega^\infty) = \sup_{\mathrm{CP}} \lim_{n\to\infty} \frac{1}{n} \mathsf{S}(\omega^n).$$

CP stands for the set of all compatible completely positive mappings that are constructed by means of the mappings Φ_∞ and satisfy the conditions i) and ii). It is readily seen that partitions of unity are within this new setting. For a partition of unity of size k: $\mathcal{X} = (x_0, \dots, x_{k-1})$ the mappings above have the following explicit form:

$$\begin{aligned} \Phi_1 \quad &: \quad \mathcal{M}_k \otimes \mathcal{A} \to \mathcal{A} : [a_{i,j}]_{i,j} \mapsto \sum_{i,j} x_i^* a_{i,j} x_j \\ \Gamma \quad &: \quad \mathcal{A} \to \mathcal{A} : x \mapsto \sum_{i=0}^{k-1} x_i^* x x_i \\ \hat{\Phi}_1 \quad &: \quad \mathcal{M}_k \to \mathcal{A} : A \mapsto \sum_{i,j} A_{ij} x_i^* x_j. \end{aligned}$$

It follows immediately that the state $\omega^1 := \omega \circ \hat{\Phi}_1$ on $\mathcal{M}_k$ has density matrix $\rho[\mathcal{X}]$. At time t one recovers in this way the density matrices $\rho^{[0,t]}[\mathcal{X}]$. Hence, these couplings yield the original structure of ALF in [AF].

3. CNT entropy. We will follow here the approach as presented by J.L. Sauvageot and J.P. Thouvenot in [Sau] since it is more closely related to the notion of partition of unity. We start from the quantum probability space $(\mathcal{R}, \omega)$.

By a coupling between $(\mathcal{R}, \omega)$ and an Abelian algebra $\mathbf{B}$ is meant a state λ on the tensor product algebra $\mathcal{R} \otimes \mathbf{B}$ such that its restriction to $\mathcal{R}$ is ω.

For a finite dimensional algebra $\mathbf{B}$ and a coupling λ with $\mathcal{R}$, one can decompose any state ω on $\mathcal{R}$ according to the minimal projections δ_x in $\mathbf{B}$. Define $\omega_x(a) = \lambda(a \otimes \delta_x)$. Then, one obtains a decomposition of the state ω:

$$\omega = \sum_{x \in \mathbf{X}} \omega_x = \sum_{x \in \mathbf{X}} \mu(x) \hat{\omega}_x.$$

where $\mathbf{X}$ is the spectrum of $\mathbf{B}$ and $\mu(x) = \lambda(\mathbb{1} \otimes \delta_x)$. With the notations of above, the mutual information $\epsilon_\lambda(\mathcal{R}, \mathbf{P})$ associated to the coupling λ of $\mathcal{R}$ with a finite dimensional Abelian algebra $\mathbf{P}$ is given by

$$\epsilon_\lambda(\mathcal{R}, \mathbf{P}) = \mathsf{S}(\mu) + \sum_{x \in \mathbf{X}} \mathsf{S}(\omega_x, \omega).$$

By a finite partition of a unital Abelian C^*-algebra $\mathbf{B}$, we will mean a finite partition $\mathcal{C} = \{C_0, \dots, C_{k-1}\}$ of the spectrum $\mathbf{X}$ of $\mathbf{B}$ or in other words a finite dimensional unital subalgebra $\mathbf{P}$ of $\mathbf{B}$ or a finite family of projection operators $\{\chi_{C_i}\}_{i=0}^{k-1}$ whose sum equals $\mathbb{1}_{\mathbf{B}}$. The main contribution in [Sau] consists in proving the claim that one only has to look at stationary couplings in order to compute the CNT entropy.

A stationary coupling of the system $(\mathcal{R}, \Theta, \omega)$ with a classical dynamical system $(\mathbf{P}, \sigma)$ is a state λ on the C^*-algebra $\mathcal{R} \otimes \mathbf{P}$ that is invariant under the automorphism $\Theta \otimes \sigma$ and whose restriction to $\mathcal{R}$ is the state ω.

The entropy associated to a stationary coupling λ between the dynamical system $(\mathcal{R}, \Theta, \omega)$ and the classical couple $(\mathbf{P}, \sigma)$ is defined as:

$$\mathsf{h}(\mathbf{P}, \lambda) = \lim_{n\to\infty} \frac{1}{n} \epsilon_\lambda(\mathcal{R}, \bigvee_{k=1}^{n} \sigma^k \mathbf{P}). \tag{1}$$

The CNT dynamical entropy of the system $(\mathcal{R}, \Theta, \omega)$ is then given by the following formula:

$$\mathsf{h}_\omega^{\mathrm{CNT}}(\Theta) = \sup \mathsf{h}(\mathbf{P}, \lambda) \tag{2}$$

where the supremum has to be taken over all possible stationary couplings λ with all possible classical systems and for each λ over all possible finite partitions in the classical system.

4. Examples. Here, we present some examples to illustrate the theorem that we want to state later. We start with a model in which the difference between the ALF and the CNT entropy becomes extreme: the free shift. This model is not so important from the physical point of view but it is one of these systems for which the CNT entropy can be computed exactly.

Mathematically the model can be formulated as follows: let $\mathcal{A}$ be the universal C^*-algebra generated by the infinite number of generators $\{e_i | i \in \mathbf{Z}\}$ which satisfy the following rules:

$$e_i = e_i^* \tag{3}$$

$$e_i^2 = \mathbb{1}. \tag{4}$$

In $\mathcal{A}$ it is useful to consider the norm dense subalgebra $\mathcal{A}_0$ which consists of finite linear combinations of words. A word w is a monomial in the generators e_i which cannot be simplified by means of the relations 3, 4. In concreto, this means that a word of length n is of the following form: $e_{i_1} e_{i_2} \dots e_{i_n}$ with $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n$.

The free shift is defined on the generators as follows: $\Theta(e_i) = e_{i+1}$. Θ extends to an automorphism on $\mathcal{A}$. There is a unique tracial state on this algebra which is given by: $\tau(\mathbb{1}) = 1$, $\tau(w) = 0$. τ is invariant under the shift and we will take it as the invariant state of the dynamical system. The von Neumann algebra $\mathcal{R}$ associated to the C^*-algebra $\mathcal{A}$ is the II_1 factor $\mathrm{L}(\mathrm{F}_\infty)$ obtained from the left regular representation of the free group F_∞ in an infinite number of generators.

THEOREM 1. *Let $\mathcal{R}$ be the von Neumann algebra generated by the free generators with the free shift Θ and invariant trace τ. Then,*

$$\mathsf{h}^{\mathrm{ALF}}_{(\mathcal{R},\Theta,\tau)} = \infty.$$

Proof. As explained in section 2, we can start from a partition of unity. Choose the following partition

$$\chi = \left(\frac{e_1}{\sqrt{k}}, \frac{e_1 e_2 e_1}{\sqrt{k}}, \cdots, \frac{e_1 e_2 e_1 \cdots e_2 e_1}{\sqrt{k}} \right).$$

Then, by the shift it is transformed into

$$\Theta(\mathcal{X}) = \left(\frac{e_2}{\sqrt{k}}, \frac{e_2 e_3 e_2}{\sqrt{k}}, \dots, \frac{e_2 e_3 e_2 \cdots e_3 e_2}{\sqrt{k}} \right).$$

A general element of the partition $\Theta^{n-1}\mathcal{X} \circ \cdots \Theta\mathcal{X} \circ \mathcal{X}$ can be written symbolically as follows:

$$[e_n \cdots e_n] \times [e_{n-1} \cdots e_{n-1}] \times \ldots \times [e_1 \cdots e_1].$$

This yields the following matrix elements for the m-step evolved density matrix $\rho^{[0,m-1]}[\mathcal{X}]$: put $\mathbf{i} = (i_0, \dots, i_{m-1})$ and $\mathbf{j} = (j_0, \dots, j_{m-1})$ then:

$$\rho^{[0,m-1]}[\mathcal{X}]_{\mathbf{i},\mathbf{j}} = \frac{1}{k^m} \tau([e_1 \ldots e_1]_{j_o} \ldots [e_m \ldots e_m]_{j_{m-1}} [e_m \ldots e_m]_{i_{m-1}} \ldots [e_1 \ldots e_1]_{i_0})$$

A small computation shows that with the previous definitions this density matrix is the trace on $\mathcal{M}_k^{[0,m-1]}$. The entropy density of this state is therefore $\log k$. Since the size k of the starting partition $\mathcal{X}$ was arbitrary, this yields:

$$\mathsf{h}^{\mathrm{ALF}}_{(\mathcal{R},\Theta,\tau)} = \infty. \ \blacksquare$$

It was computed by E. Størmer [Sto] that for this model the CNT entropy equals zero in spite of the fact that the algebra increases enormously under the shift. Moreover, it was proven by R. Alicki and H. Narnhofer in [AN] that for the same model also $\mathsf{h}^{\mathrm{ALF}}_{(\mathcal{R},\Theta^2,\tau)} = \infty$ whereas for the CNT entropy this entropy is also zero by additivity.

As a second example, we present the quantum Arnold cat map. For a detailed discussion we refer the interested reader to [AAFT], [AFTA],[FT], [TUY]. The algebras describing these systems are the rotation algebras which are generated by two unitaries u and v satisfying the following twisted commutation relation:

$$uv = e^{2i\pi q} vu.$$

It is useful to consider the Weyl like operators

$$W(\mathcal{X}) = e^{-iqmn/2} u^m v^n \quad \mathcal{X} = \begin{pmatrix} m \\ n \end{pmatrix}.$$

In this setting the cat dynamics looks like a quasifree dynamics on the CCR:

$$\Theta_T W(\chi) = W(T\chi) \quad \text{for} \quad \mathrm{T} \in \mathrm{SL}(2, \mathbf{Z}).$$

Again, it is easy to verify that Θ_T extends to an automorphism on the rotation algebras generated by the unitaries u and v. The matrix T has two eigenvalues λ_+ and λ_- satisfying $\lambda_+ \lambda_- = 1$. The gauge automorphism:

$$\gamma_\alpha(W(\mathcal{X})) = e^{i\langle \alpha, \mathcal{X}\rangle} W(\mathcal{X})$$

leads to the following horocycle rule

$$\Theta_T \circ \gamma_\alpha = \gamma_{T\alpha} \circ \Theta_T.$$

Therefore, the numbers $\log \lambda_+$ and $\log \lambda_-$ may be interpreted as Lyapunov exponents. Finally, we mention that there is a tracial state τ on the von Neumann algebras generated by the unitaries u, v, which we will denote by $\mathcal{R}^q$. We formulate the main theorem on the entropy of this system without proof. For a proof, we refer the reader to [TUY], [FT].

THEOREM 2. *The dynamical entropy of the system* $(\mathcal{R}^q, \Theta_T, \varphi)$ *equals its positive Lyapunov exponent* $\log \lambda_+$, *for any normal state* φ *on* $\mathcal{R}^q$ *and partitions chosen in the algebra of Schwartz functions on the non-commutative torus.*

It was proven in [BNS] that the CNT entropy of this system is zero if the deformation parameter q is an irrational number. For rational deformation parameters it equals the positive Lyapunov exponent.

Finally, we comment very briefly on the dynamical entropy for the shift on a quantum spin chain. The algebra describing the system is a UHF algebra which we will denote by $\bigotimes_{i\in\mathbf{Z}} M_d$. The shift Θ maps an element that lives on a finite volume Λ of $\mathbf{Z}$ onto one that lives on the volume $\Lambda+1$. As invariant state, one can take any translation invariant state ω on $\bigotimes_{i\in\mathbf{Z}} M_d$. In [FT], [TUY], we constructed a subalgebra $\mathcal{A}^{\infty}_{q,\mathrm{fin}}$ of elements whose tails converge to the identity as $e^{-q|j|}$, $j \in \mathbf{Z}$. If one restricts the choice of partitions to the subalgebra $\mathcal{A}^{\infty}_{q,\mathrm{fin}}$ then one has the following result:

THEOREM 3. *The dynamical entropy of the sytem* $(\bigotimes_{i\in\mathbf{Z}} M_d, \Theta, \omega)$ *equals*

$$\sigma(\omega) + \log d$$

where $\sigma(\omega)$ *is the entropy density of the state* ω.

It can be proven that under certain clustering conditions of the state ω the CNT entropy of this system is equal to the entropy density of the state ω. For a completely random system, i.e. the invariant state is the trace, the ALF entropy is twice as large as the CNT entropy: $\log d$.

5. Relation between $\mathsf{h}^{\mathrm{CNT}}$ and $\mathsf{h}^{\mathrm{ALF}}$. The following lemma indicates that there is a relation between the CNT construction and the notion of partitions of unity.

LEMMA 1. *Let* ω *be a faithful, normal state on* $\mathcal{R}$ *that is decomposed into positive functionals* ω_i:

$$\omega = \sum_{i=1}^{n} \omega_i.$$

Then, there exists a partition of unity $\mathcal{X}$ *in* $\mathcal{R}$ *which realizes this decomposition. Moreover,* $\omega \circ \Gamma_{\mathcal{X}} = \omega$ *where* $\Gamma_{\mathcal{X}}$ *denotes the completely positive map associated to* $\mathcal{X}$.

The maps Φ_1 associated to a partition $\mathcal{X}$ induce states $\omega^{\mathcal{X}}$ on $\mathcal{R} \otimes \mathcal{M}_k$ as follows. Take $[a_{i,j}]_{i,j}$ in $\mathcal{R} \otimes \mathcal{M}_k$ then:

$$\omega^{\mathcal{X}}([a_{i,j}]_{i,j}) = (\omega \circ \Phi_1)([a_{i,j}]_{i,j}). \tag{5}$$

It can be easily seen that the restriction of $\omega^{\mathcal{X}}$ to $\mathcal{R}$ is the state ω and its marginal on $\mathcal{M}_k$ gives the density matrix is $\rho[\mathcal{X}]$. We introduce the diagonalization mapping.

$$\Gamma^d : \mathcal{M}_n \otimes \mathcal{R} \to \mathbf{C}^n \otimes \mathcal{R} : [a_{i,j}]_{i,j} \mapsto [a_{i,i}\delta_{i,j}]_{i,j}. \tag{6}$$

The map Γ^d defined in equation 6 is a unity preserving completely positive map. Given a state ω on a von Neumann algebra $\mathcal{R}$ and a partition $\mathcal{X}$ in $\mathcal{R}$ that leaves ω invariant, we denote by ω_i the positive functional that is defined by the relation: $\omega_i(z) \equiv \omega(x_i^* z x_i)$.

From the definition of the state $\omega^{\mathcal{X}}$ in (5) and of the map Γ^d in (6) we can derive:

$$\omega^{\mathcal{X}} \circ \Gamma^d([a_{i,j}]_{i,j}) = \sum_{i=1}^{n} \omega_i(a_{i,i}),$$

or stated in another way:

$$\omega^{\mathcal{X}} \circ \Gamma^d = \bigoplus_{i=1}^{n} \omega_i.$$

On the other hand:

$$(\rho[\mathcal{X}] \otimes \omega) \circ \Gamma^d = \rho_d[\mathcal{X}] \otimes \omega$$

where $\rho_d[\mathcal{X}]$ is the diagonal matrix $\rho[\mathcal{X}] = [\omega(x_i^* x_j)\delta_{i,j}]_{i,j}$.

In order to compare $\mathsf{h}^{\mathrm{ALF}}$ with $\mathsf{h}^{\mathrm{CNT}}$ we will use the notion of relative entropy together with the Uhlmann monotonicity theorem [OP]. In particular, we will have to compute $\mathsf{S}(\omega^{\mathcal{X}}, \omega \otimes \rho[\mathcal{X}])$ and $\mathsf{S}(\omega^{\mathcal{X}} \circ \Gamma, \omega \otimes \rho[\mathcal{X}]_d)$. On the technical level this means that we need the modular operators $\Delta_{\omega^{\mathcal{X}}, \omega\otimes\rho[\mathcal{X}]}$ and $\Delta_{\omega^{\mathcal{X}}\circ\Gamma, \omega\otimes\rho[\mathcal{X}]_d}$. Therefore, we construct the GNS representations of these states.

First of all it is well known that if the triple $(\mathcal{H}_{\omega_1}, \pi_1, \omega_1)$ is the GNS representation of the algebra $\mathcal{A}_1$ with respect to the state ω_1 and $(\mathcal{H}_{\omega_2}, \pi_2, \omega_2)$ is that of $\mathcal{A}_2$ with respect to ω_2 then the GNS representation of the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ with respect to the product state $\omega_1 \otimes \omega_2$ is given by $(\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}, \pi_1 \otimes \pi_2, \Omega_1 \otimes \Omega_2)$.

The following theorem tells us explicitly how the GNS representations of the states $\omega \otimes \rho[\mathcal{X}]$ and $\omega \otimes \rho[\mathcal{X}]_d$ can be constructed.

Corollary 1. *Let ρ be a density matrix on $\mathcal{M}_k$ and let $\{\lambda_0, \ldots, \lambda_{d-1}\}$ be the set of non-zero eigenvalues with corresponding eigenvectors $\{e_0, \ldots, e_{d-1}\}$. Consider also a state ω on a C^*-algebra $\mathcal{A}$ with GNS triplet $(\mathcal{H}_\omega, \pi, \Omega)$. Then the GNS triplet of the algebra $\mathcal{M}_k \otimes \mathcal{A}$ with respect to the product state $\rho \otimes \Omega$ is given by*:

$$\Big(\mathbf{C}^k \otimes \mathbf{C}^d \otimes \mathcal{H}_\omega, \quad \sum_{i=0}^{d-1} \lambda_i^{1/2} e_i \otimes e_i \otimes \Omega, \quad \pi_1 \otimes \pi\Big)$$

where d is the number of nonzero eigenvalues of the matrix ρ and

$$\pi_1 \otimes \pi : \mathcal{M}_k \otimes \mathcal{A} \to \mathcal{M}_k \otimes \mathcal{M}_d \otimes \mathcal{B}(\mathcal{H}_\omega) : A \otimes x \mapsto A \otimes \mathbb{1}_d \otimes \pi(x).$$

Another representation that is missing is that of the state $\omega^{\mathcal{X}}$. It is again not hard to verify that the algebra $\mathcal{M}_k \otimes \mathcal{A}$ can be represented on the Hilbert space $\mathbf{C}^k \otimes \mathbf{C} \otimes \mathcal{H}_\omega$. The representation map π is then given by:

$$\pi : \mathcal{M}_k \otimes \mathcal{A} \to \mathcal{M}_k \otimes \mathbf{C} \otimes \mathcal{B}(\mathcal{H}_\omega) : A \otimes x \mapsto A \otimes \mathbb{1} \otimes x$$

and the state $\omega^{\mathcal{X}}$ by the vector:

$$\sum_{i=0}^{k-1} e_i \otimes \xi \otimes \Omega.$$

Denote by Ω_ρ the cyclic vector $\sum_{i=0}^{k-1} \lambda_i^{1/2} e_i \otimes e_i \otimes x_i \Omega$ representing the state $A \otimes x \mapsto (\operatorname{Tr} \rho[\mathcal{X}] A)\omega(x)$. Here the λ_i are the eigenvalues of $\rho[\mathcal{X}]$ and the e_i are the corresponding eigenvectors. In order to compute $\mathsf{S}(\Omega_{\mathcal{X}}, \Omega_\rho)$, we have to construct the relative modular operator $\Delta_{\Omega_{\mathcal{X}},\Omega_\rho}$. This is the content of the next lemma.

LEMMA 2. *The relative modular operator $\Delta_{\Omega_{\mathcal{X}},\Omega_\rho}$ equals*:

$$\Delta_{\Omega_{\mathcal{X}},\Omega_\rho} = \sum_{i,j=0}^{k-1} E_{i,j} \otimes \rho[\mathcal{X}]^{-1} \otimes x_i \Delta_\Omega x_j^*$$

where Δ_Ω is the modular operator of the vector Ω representing the state ω. Moreover,

$$\sum_{i,j} E_{i,j} \otimes \mathbb{1} \otimes x_i \Delta_\Omega x_j^* \Omega_{\mathcal{X}} = \Omega_{\mathcal{X}}.$$

Analogously, it follows that the cyclic vector representing the state $\omega^{\mathcal{X}} \circ \Gamma$ is given by $\Omega_{\mathcal{X}}^\Gamma := \sum_{i=0}^{k-1} e_i \otimes e_i \otimes x_i \Omega$ in the representation $\pi : A \otimes x \mapsto A \otimes \mathbb{1} \otimes x$ and the vector representing the state $\rho_d[\mathcal{X}] \otimes \omega$ is $\Omega_\rho^d := \sum_{i=0}^{k-1} \mu(i) e_i \otimes e_i \otimes \Omega$ where $\mu(i) = \omega(x_i^* x_i)$. Another small computation shows then that

$$\Delta_{\Omega_{\mathcal{X}}^\Gamma,\Omega_\rho^d} = \sum_{j=0}^{k-1} E_{jj} \otimes \rho_d[\mathcal{X}]^{-1} \otimes \Delta_{\omega_j,\omega}. \tag{7}$$

At this point, we are in a position where we are able to prove the following proposition which provides us with the basic inequality to prove the relation between $\mathsf{h}^{\mathrm{ALF}}$ and $\mathsf{h}^{\mathrm{CNT}}$ after one time step.

PROPOSITION 4. *Let $\mathcal{R}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and let ω be a state on $\mathcal{R}$ and Γ be the completely positive map from equation 6. Consider also a partition of unity $\mathcal{X}$ such that $\omega \circ \Gamma_{\mathcal{X}} = \omega$. Then*

$$\mathsf{S}(\rho[\mathcal{X}]) \geq \mathsf{S}(\{\omega_i(\mathbb{1})\}_{i=0}^{k-1}) + \sum_{i=0}^{k-1} \mathsf{S}(\omega, \omega_i).$$

Proof. We exploit the Uhlmann monotonicity theorem [OP] and the previous lemmas.

$$\mathsf{S}(\omega^{\mathcal{X}}, \rho[\mathcal{X}] \otimes \omega) \geq \mathsf{S}(\omega^{\mathcal{X}} \circ \Gamma, \rho_d[\mathcal{X}] \otimes \omega).$$

The left hand side of the inequality yields:

$$\langle \Omega_{\mathcal{X}}, \log\Big(\sum_{i,j=0}^{k-1} E_{i,j} \otimes \mathbb{1} \otimes x_i \Delta_\Omega x_j^* \Big) \Omega_{\mathcal{X}} \rangle - \langle \Omega_{\mathcal{X}}, \mathbb{1} \otimes \log \rho[\mathcal{X}] \otimes \mathbb{1}\, \Omega_{\mathcal{X}} \rangle.$$

By the previous lemma we know that the vector $\Omega_{\mathcal{X}}$ is invariant under the operator $\sum_{i,j} E_{i,j} \otimes \mathbb{1} \otimes x_i \Delta_\Omega x_j^*$. Therefore the first term of the previous equation is zero. Since the restriction of the state $\Omega_{\mathcal{X}}$ to the algebra $\mathcal{M}_k$ is the density matrix $\rho[\mathcal{X}]$, the second term and hence the whole equation equals $\mathsf{S}(\rho[\mathcal{X}])$. Because

$$\log \Delta_{\Omega_{\mathcal{X}}^d,\Omega_\rho^d} = -\mathbb{1} \otimes \log \rho_d[\mathcal{X}] \otimes \mathbb{1} + \sum_{j=0}^{k-1} E_{jj} \otimes \mathbb{1} \otimes \log \Delta_{\omega_j,\omega},$$

the right hand side becomes:

$$\mathsf{S}(\mu) + \sum_{i=0}^{k-1} \mathsf{S}(\omega_i, \omega).$$

where μ is the measure $(\mu(0), \ldots, \mu(k-1))$ and $\mu(i) = \omega(x_i^* x_i)$. So that finally we have:

$$\mathsf{S}(\rho[\mathcal{X}]) \geq \mathsf{S}(\mu) + \sum_{i=0}^{k-1} \mathsf{S}(\omega_i, \omega). \quad \blacksquare \tag{8}$$

The term $\mathsf{S}(\{\omega_i(\mathbb{1})\}_{i=0}^{k-1})$ can be seen as the entropy of a classical measure on the finite state space $\mathbf{Z}_k$. In order to give a full proof of the fact that $\mathsf{h}^{\mathrm{ALF}} \geq \mathsf{h}^{\mathrm{CNT}}$ we should also implement the time evolution. This is not so easy as we thought at first sight. One has to prove that the fine decompositions of the state ω after n time steps can be made by means of completely positive mappings Φ_n and this turns out to be highly non-trivial problem. However, because of all the examples we still believe that $\mathsf{h}^{\mathrm{ALF}} \geq \mathsf{h}^{\mathrm{CNT}}$ and therefore it has to be considered as a conjecture.

6. Information theory. We will show here that the quantum dynamical entropy can be used to estimate the capacity of a quantum channel through which one sends messages consisting of a classical alphabet.

We consider the following situation: a message has to be encoded by quantum means (photons and optical fibers) and to be decoded in the final station. We will assume that the messages consist of letters belonging to a finite alphabet $\mathbf{A} = \{1, 2, \ldots, m\}$. The encoding transmission device is a quantum dynamical system for which we will take the shift on a quantum spin chain. The encoding procedure will be modeled by a perturbation of the reference state ω. Such a perturbation will be realized by a completely positive mapping generated by a partition of unity $\mathcal{X}$. Hence, to each letter $\alpha \in \mathbf{A}$ there is associated a partition of unity $\mathcal{X}^\alpha = (x_0^\alpha, \ldots, x_{k(\alpha)-1}^\alpha)$. Denoting by $(\pi, \mathcal{H}_\omega, \Omega)$ the GNS representation of $(\mathcal{A}, \omega)$, we consider the following perturbation of the state ω in order to encode a letter α:

$$\omega \circ \Gamma_{\mathcal{X}^\alpha}(z) = \mathrm{Tr} \sum_{i=0}^{k^{(\alpha)}} |\pi(x_i^\alpha)\Omega\rangle\langle\pi(x_i^\alpha)\Omega|\pi(z).$$

We model the receiver by a measuring device, i.e. a set of observables $\mathcal{Z} = \{z_i | \, i = 0, \ldots, m-1\}$ such that $\sum_{i=0}^{k-1} z_i = \mathbb{1}$. The standard notion that measures the amount of information that can be sent through a communication channel is

$$\mathbf{I}(\alpha|\beta) = \mathsf{S}(\mathsf{P}^i) + \mathsf{S}(\mathsf{P}^o) - \mathsf{S}(\mathsf{P}^{io}).$$

Here, we denoted by P^i the input probability measure for letters belonging to $\mathbf{A}$, and by P^o the output probability measure. The measure $\mathsf{P}^{io}(\alpha|\beta)$ gives the so called input-output probability. It can be easily verified that the measures P^i and P^o are the marginals of the measure P^{io} and hence one has by subadditivity and monotonicity of the entropy the following upper bound on the information capacity:

$$0 \leq \mathbf{I}(\alpha|\beta) \leq \min\{\mathsf{S}(\mathsf{P}^i), \mathsf{S}(\mathsf{P}^o)\}$$

However, a more useful inequality is the Holevo-Levitin inequality for the information capacity. It provides the following upper bound:

$$\mathbf{I}(\alpha|\beta) \leq \mathsf{S}(\sum_{\alpha\in\mathbf{A}} \mathsf{P}^i(\alpha)\rho^\alpha) - \sum_{\alpha\in\mathbf{A}} \mathsf{P}^i(\alpha)\mathsf{S}(\rho^\alpha)$$

where $\rho^\alpha = \sum_{i=0}^{k-1} |\pi(x_i^\alpha)\Omega\rangle\langle\pi(x_i^\alpha)\Omega|$. The quantity that determines the efficiency of the communication channel is the averaged amount of information per time unit $\mathbf{J}(\alpha|\beta)$. The input probability measure is a measure on the classical spin chain $\mathbf{A}^{\mathbf{N}}$. As dynamical system that models the transmission device we take the shift on a quantum spin chain. By means of the Holevo-Levitin inequality the following theorem then easily follows:

THEOREM 5. *The speed of information transmission* $\mathbf{J}(.|.)$ *is bounded above by the dynamical entropy* $\mathsf{h}^{\mathrm{ALF}}_{(\mathcal{M}_d^{\otimes\mathbf{Z}},\Theta,\omega)}$.

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