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COMPLETELY POSITIVE MAPS ON COXETER GROUPS AND THE ULTRACONTRACTIVITY OF THE q-ORNSTEIN–UHLENBECK SEMIGROUP

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1. Coxeter groups. In this note we give an application of the following result on the symmetric group S_n:

THEOREM 1. *For fixed $n \in \mathbf{N}$ let us consider the permutation group S_n and denote by $\pi_i \in S_n$ $(i = 1, ..., n-1)$ the transposition between i and $i+1$. Furthermore, let operators $T_i \in B(\mathcal{H})$ $(i = 1, ...n-1)$ on some Hilbert space $\mathcal{H}$ be given, with the properties*:

(*i*) $T_i^* = T_i$ *for all* $i = 1, ..., n-1$;
(*ii*) $\|T_i\| \leq 1$ *for all* $i = 1, ..., n-1$;
(*iii*) *The T_i satisfy the braid relations*:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for all } i = 1, ..., n-2,$$
$$T_i T_j = T_j T_i \text{ for all } i, j = 1, ..., n-1 \text{ with } |i-j| \geq 2.$$

Now let us define a function

$$\varphi : S_n \longrightarrow B(\mathcal{H})$$

by quasi-multiplicative extension of

$$\varphi(e) = 1, \qquad \varphi(\pi_i) = T_i,$$

i.e. for a reduced word $S_n \ni \sigma = \pi_{i(1)} \dots \pi_{i(k)}$ we put $\varphi(\sigma) = T_{i(1)} \dots T_{i(k)}$. Then φ **is a completely positive map**, *i.e. for all $l \in \mathbf{N}, f_i \in \mathbf{C}S_n, x_i \in \mathcal{H}$ $(i = 1, \dots, l)$ we have*

$$\Big\langle \sum_{i,j=1}^{l} \varphi(f_j^* f_i) x_i, x_j \Big\rangle \geq 0.$$

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By our previous result from [BSp1], Theorem 1 is equivalent to the following:

THEOREM 2. *Under the assumptions of Theorem 1 the operator*

$$P^{(n)} = P_T^{(n)} = \sum_{\sigma\in S_n} \varphi(\sigma) =$$
$$= (1+T_1)(1+T_2+T_2T_1)\dots(1+T_{n-1}+T_{n-1}T_{n-2}+\dots+T_{n-1}\dots T_1)$$

satisfies

$$P^{(n)} \geq \prod_{k=2}^{n} c_k(q) > 0,$$

where

$$c_k(q) = (1-q^2)^{-1}\prod_{l=1}^{k}(1-q^l)(1+q^l)^{-1}.$$

Moreover, *by Gauss formula*

$$c_k(q) \geq c(q) = (1-q)^{-1}\prod_{l=1}^{\infty}(1-q^l)(1+q^l)^{-1} = (1-q)^{-1}\sum_{l=-\infty}^{+\infty}(-1)^l q^{l^2}.$$

In the proof we need the following lemma:

LEMMA 3. *If $T_i \in B(\mathcal{H})$ satisfy the braid relations of Theorem 1*, *then for $1 \leq r < k < n-1$*, *we have*

$$(T_{n-1}T_{n-2}\dots T_k)(T_{n-1}T_{n-2}\dots T_r) = T_{n-1}(T_{n-1}T_{n-2}\dots T_r)(T_{n-1}\dots T_{k+1}).$$

Proof. The proof of the Lemma follows by induction on k:

Let $k = n-2$. Then by the braid relations we get

$$\begin{aligned}
&(T_{n-1}\underbrace{T_{n-2})(T_{n-1}T_{n-2}}T_{n-3}\dots T_r) =\\
&= T_{n-1}\underbrace{T_{n-1}T_{n-2}T_{n-1}}T_{n-3}\dots T_r =\\
&= T_{n-1}(T_{n-1}T_{n-2}T_{n-3}\dots T_r)T_{n-1}.
\end{aligned}$$

The next step looks as follows:

$$\begin{aligned}
&(T_{n-1}T_{n-2}T_{n-3})(T_{n-1}T_{n-2}T_{n-3}\dots T_r) =\\
&= (T_{n-1}T_{n-2})(T_{n-1}\underbrace{T_{n-3}T_{n-2}T_{n-3}}\dots T_r) =\\
&= (T_{n-1}\underbrace{T_{n-2})(T_{n-1}T_{n-2}}T_{n-3}T_{n-2}T_{n-4}\dots T_r) =\\
&= T_{n-1}T_{n-1}T_{n-2}T_{n-1}T_{n-3}T_{n-2}(T_{n-4}\dots T_r) =\\
&= T_{n-1}T_{n-1}T_{n-2}T_{n-3}(T_{n-1}T_{n-2})(T_{n-4}\dots T_r) =\\
&= T_{n-1}(T_{n-1}T_{n-2}T_{n-3}\dots T_r)(T_{n-1}T_{n-2}). \ \blacksquare
\end{aligned}$$

Next we need the following important lemma:

LEMMA 4. *Let $T_i \in B(\mathcal{H})$ and*

$$R_k(T_1,\dots,T_{k-1}) = R_k = 1+T_{k-1}+T_{k-1}T_{k-2}+\dots+T_{k-1}T_{k-2}\dots T_1,$$

where $k = 2,3,\dots,n$. Then

$$(a)\quad R_k(1-T_{k-1}T_{k-2}\dots T_1)=$$
$$=(1-T_{k-1}^2T_{k-2}\dots T_1)(1+T_{k-1}+T_{k-1}T_{k-2}+\dots+T_{k-1}T_{k-2}\dots T_2)=$$
$$=(1-T_{k-1}^2T_{k-2}\dots T_1)R_{k-1}(T_2,T_3,\dots,T_{k-1}),$$

$$(b)\quad R_n(1-T_{n-1}T_{n-2}\dots T_2T_1)(1-T_{n-1}T_{n-2}\dots T_2)\dots(1-T_{n-1})=$$
$$=(1-T_{n-1}^2T_{n-2}\dots T_2T_1)(1-T_{n-1}^2T_{n-2}\dots T_2)\dots(1-T_{n-1}^2T_{n-2})(1+T_{n-1}).$$

P r o o f. Let us start with the case $k=3$. Since $R_3=1+T_2+T_2T_1$, we have

$$\begin{aligned}R_3(1-T_2T_1)&=1+T_2-T_2^2T_1-T_2T_1T_2T_1=\\&=1+T_2-T_2^2T_1-T_2^2T_1T_2=\\&=(1-T_2^2T_1)(1+T_2).\end{aligned}$$

Now we consider the case $k=4$. By natural calculations using Lemma 3 we get

$$\begin{aligned}R_4(1-T_3T_2T_1)&=(1+T_3+T_3T_2)-(T_3^2T_2T_1)(1+T_3+T_3T_2)=\\&=(1-T_3^2T_2T_1)(1+T_3+T_3T_2).\end{aligned}$$

Therefore, using the case $k=3$, we have

$$R_4(1-T_3T_2T_1)(1-T_3T_2)=(1-T_3^2T_2T_1)(1-T_3^2T_2)(1+T_2).$$

Repeating this process we get the proof of the Lemma. ■

This implies the next lemma.

LEMMA 5. *If*

$$\begin{aligned}P^{(n)}&=\sum_{\sigma\in S_n}\varphi_T(\sigma)=P^{(n-1)}(1+T_{n-1}+\dots+T_{n-1}\dots T_1)=\\&=P^{(n-1)}R_n=R_2R_3\dots R_n,\end{aligned}$$

and $\|T_i\|\le q<1$, *then*

$$\|R_n^{-1}\|\le(1-q)^{-1}\prod_{k=1}^{n-1}(1+q^k)\prod_{k=3}^{n}(1-q^k)^{-1}.\tag{$**$}$$

P r o o f. By Lemma 4 we have

$$R_n=\prod_{k=1}^{n-2}(1-T_{n-1}^2T_{n-2}\dots T_k)(1+T_{n-1})\prod_{l=n-1}^{1}(1-T_{n-1}\dots T_l)^{-1}.$$

But, since $\|T_i\|<q<1$, therefore

$$\|(1-T_{n-1}\dots T_{(n-1)-k})^{-1}\|\le(1-q^k)^{-1}$$

and we infer the estimation of Lemma 5. ■

Now we can state Theorem 2 in a stronger version.

THEOREM 6. *If* $\|T_i\|\le q<1$ *and the assumptions of Theorem 1 are satisfied, then*

$$(i)\quad P^{(n)}\ge\omega(q)(P^{(n-1)}\otimes 1),$$

where

$$\omega(q)^2 = (1-q^2)^{-1}\prod_{k=1}^{\infty}(1-q^k)(1+q^k)^{-1}.$$

$$(ii) \qquad P^{(n)} \le \frac{1}{1-q}(P^{(n-1)}\otimes 1).$$

P r o o f. The proof follows from the following considerations:

(a) We know from the results of [BSp1] that

$$P^{(n)} \ge 0.$$

Since, by Lemma 5, $\|R_n^{-1}\| \le \frac{1}{c}$ for some $c > 0$, therefore

$$\|(R_n^{-1})^* R_n^{-1}\| \le \frac{1}{c^2},$$

and this implies

$$R_n R_n^* \ge c^2.$$

But, because

$$P^{(n)} = (P^{(n-1)}\otimes 1)R_n,$$

and $P^{(n)} = P^{(n)*}$, we obtain

$$[P^{(n)}]^2 = P^{(n-1)}R_n R_n^* P^{(n-1)} \ge c^2[P^{(n-1)}]^2$$

and hence

$$P^{(n)} \ge c(P^{(n-1)}\otimes 1), \text{ where } c = \omega(q).$$

(b) The statement (ii) of Theorem 2 follows from the two facts:

$$P^{(n)} = P^{(n-1)}R_n$$

and

$$R_n = 1 + T_{n-1} + T_{n-1}T_{n-2} + \ldots + T_{n-1}\ldots T_1.$$

Therefore $\|R_n\| < \frac{1}{1-q}$ and again as before we have

$$P^{(n)} \ge \frac{1}{1-q}(P^{(n-1)}\otimes 1).$$

So, the proof of Theorem 6 is complete. ■

This theorem is also valid for all finite and affine Coxeter groups (for more details see [BSp4]). Theorem 1 comes from investigations in harmonic analysis on groups (see [B1], [BSz]) and on perturbed cannonical commutation relations. In the paper with R. Speicher ([BSp1]) we considered the following relations

$$c_i c_j^* - q c_j^* c_i = \delta_{ij}\mathbf{1}$$

for a real q with $|q| \le 1$, and we needed essentially the fact that the function

$$\varphi : S_n \longrightarrow \mathbf{C}, \qquad \pi \longmapsto q^{|\pi|}$$

is a positive definite function for all n, where $|\pi|$ denotes the number of inversions of π. For other proofs of that result see [BKS, BSp1, BSp2, BSp4, BSz, Spe, Z].

R. Speicher in [Spe] considered more general commutations relations

$$d_i d_j^* - q_{ij} d_j^* d_i = \delta_{ij}\mathbf{1}$$

for

$$-1 \leq q_{ij} = q_{ji} \leq 1,$$

and he founded the existence of a Fock representation by central limit arguments. Our construction of the q_{ij} relations depends on some operator T which is a self-adjoint contraction on a Hilbert space $\mathcal{H}$ and satisfies the braid or Yang-Baxter relations of the following form:

$$T_1 T_2 T_1 = T_2 T_1 T_2,$$

where $T_1 = T \otimes \mathbf{1}$ and $T_2 = \mathbf{1} \otimes T$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ are the natural amplifications of T to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.

From Theorem 1 we get more general construction of deformed commutation relations of the Wick form:

$$d_i d_j^* - \sum_{r,s} t_{js}^{ir} d_r^* d_s = \delta_{ij}\mathbf{1}$$

(see also Jorgensen et al. [JSW] and [BSp4] for similar considerations).

2. Applications. Next we examine the deformed commutation relations from an operator spaces' point of view. If we assume that $\|T\| = q \leq 1$ and if we take $G_i = d_i + d_i^*$, then we prove that the operator space generated by the G_i is completely isomorphic to the canonical operator Hilbert space $\mathcal{R} \cap \mathcal{C}$, which means

$$\Big\| \sum_{i=1}^N a_i \otimes G_i \Big\| \approx \max \Big(\Big\| \sum_{i=1}^N a_i a_i^* \Big\|^{1/2}, \Big\| \sum_{i=1}^N a_i^* a_i \Big\|^{1/2} \Big)$$

for all bounded operators $a_1, ..., a_N$ on some Hilbert space. This generalizes the Theorem of Haagerup and Pisier [HP], who obtained that result for free creation and annihilation operators, (see also [VDN] and [Buch]). As another application of our construction we have obtained a large class of *non-injective* von Neumann algebras, when considering the von Neumann algebra $VN(G_1, ..., G_N)$ generated by $G_1, ..., G_N$. For more details see [BSp4, BKS].

3. The ultracontractivity of the q-second quantization functor Γ_q. Let $T : \mathcal{H} \longrightarrow \mathcal{K}$ be a contraction beetween real Hilbert spaces. Then the linear map defined on elementary tensors by

$$F_q(T)(f_1 \otimes \ldots \otimes f_n) = Tf_1 \otimes \ldots \otimes Tf_n$$

extends to a contraction from q-Fock spaces $F_q(\mathcal{H})$ to $F_q(\mathcal{K})$. Here $F_q(\mathcal{H})$ is the completion of the full Fock space $\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$ with respect to the new scalar product

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_n \rangle_q = \delta_{n,m} \sum_{\sigma \in S_n} q^{inv(\sigma)} \langle f_{\sigma(1)}, g_1 \rangle \ldots \langle f_{\sigma(n)}, g_n \rangle.$$

The creation operators are defined as:

$$c^*(f_0)(f_1 \otimes \ldots \otimes f_n) = f_0 \otimes f_1 \otimes \ldots \otimes f_n, \quad f_j \in \mathcal{H}$$

and $c(f) = [c^*(f)]^*$.

Let $G(f) = c(f) + c^*(f)$ for $f \in \mathcal{H}$. Let $\Gamma_q(\mathcal{H})$ be the von Neumann algebra generated by $G(f)$, $f \in \mathcal{H}$, and

$$\tau_q(S) = \langle S\Omega, \Omega\rangle_q, \quad S \in \Gamma_q(\mathcal{H}).$$

One can show that τ_q is a trace on $\Gamma_q(\mathcal{H})$.

If $\dim \mathcal{H} = \infty$, then we showed that $\Gamma_q(\mathcal{H})$ is a factor.

If $e_1, e_2, \ldots, e_N$ is an orthonormal basis of $\mathcal{H}$, then we put $G_i = G(e_i)$, $(i = 1, \ldots, N$, $N = \infty, 1, 2, \ldots)$. In this setting the following theorem holds:

Theorem 7 ([BKS], Theorem 2.1.1). *Let T be as above, then there exists a unique map $\Gamma_q(T) : \Gamma_q(\mathcal{H}) \longrightarrow \Gamma_q(\mathcal{K})$ such that $\Gamma_q(T)(X)\Omega = F_q(T)(X\Omega)$ for every $X \in \Gamma_q(\mathcal{H})$. The map $\Gamma_q(T)$ is bounded, normal, unital, completely positive and trace preserving.*

We note that Γ_q is a functor, namely if $S : \mathcal{H} \longrightarrow \mathcal{K}$ and $T : \mathcal{K} \longrightarrow \mathcal{J}$ are contractions, then $\Gamma_q(ST) = \Gamma_q(S)\Gamma_q(T)$.

If $\mathcal{H}$ is a real Hilbert space and $T_t = e^{-t}I$ for $t \geq 0$, then the completely positive maps $P_t^q = \Gamma_q(T_t)$, $t \geq 0$, on $\Gamma_q(\mathcal{H})$, form a semigroup, called the *q-Ornstein-Uhlenbeck* semigroup. The q-Ornstein-Uhlenbeck semigroup extends to a semigroup of contractions of the non-commutative L^p spaces, which are symmetric on L^2 . Its infinitesimal generator on L^2 is the number operator N^q, i.e. $P_t = exp(-tN^q)$, where N^q is the unbounded self-adjoint operator defined as $N^q\Omega = 0$ and

$$N^q f_1 \otimes \ldots \otimes f_n = n f_1 \otimes \ldots \otimes f_n, \quad f_1, \ldots, f_n \in \mathcal{H}.$$

Ph. Biane [Bia] proved Nelson's hypercontractivity of the q-Ornstein-Uhlenbeck semigroup P_t, extending the results of Nelson and Gross. In that paper Ph. Biane also showed ultracontractivity for $q = 0$ using some results of the author (see [B2]). Now we prove the ultracontractivity of that semigroup for all $q \in [-1, 1]$.

Theorem 8. *Let X be in the eigenspace of N^q, with eigenvalue n. Then*

(*i*) $\|X\|_{L^\infty} \leq C(q)(n+1)\|X\|_L^2$;

(*ii*) *For $t \geq 0$, P_t maps L^2 into $L^\infty = VN_q(G_1 \ldots G_N)$ and for $t \leq 1$*

$$\|P_t^q\|_{L^2 \to L^\infty} \leq c_q t^{-3/2}.$$

(*iii*) (**Poincaré-Sobolev inequality**). *If $Q_q(X) = \langle XN^qX\Omega, \Omega\rangle$ is a non-commutative complete Dirichlet form (on an appropriate domain) on $L^2(\Gamma_q(\mathcal{H}), \tau_q)$, in the sense of [DL], then there exists a constant $c_q \geq 0$ such that for all X in the domain of Q_q we have*

$$\|X\|_{L^3}^2 \leq c_q(|\tau_q(X)|^2 + Q_q(X)).$$

For the details of the proof of this theorem see [B3] and [Bia].

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