QUANTUM PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 43
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

SINGLETON INDEPENDENCE

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Abstract. Motivated by the central limit problem for algebraic probability spaces arising from the Haagerup states on the free group with countably infinite generators, we introduce a new notion of statistical independence in terms of inequalities rather than of usual algebraic identities. In the case of the Haagerup states the role of the Gaussian law is played by the Ullman distribution. The limit process is realized explicitly on the finite temperature Boltzmannian Fock space. Furthermore, a functional central limit theorem associated with the Haagerup states is proved and the limit white noise is investigated.

1. Introduction. Let F_∞ be the free group on countably infinite generators $\{g_j\,;\, j \in \mathbf{N}\}$ and $\mathcal{A}$ the group $*$-algebra. For simplicity we adopt the following notation: for $j \in \mathbf{N}$

1991 *Mathematics Subject Classification*: Primary 60A05; Secondary 60F17,81Q99.

Research supported by Grant-in-Aid for Scientific Research No. 09640178 and No. 09440057, Ministry of Education, Japan

The paper is in final form and no version of it will be published elsewhere.

and $\varepsilon = \pm 1$ we put

$$\alpha = (j, \varepsilon), \qquad \alpha^* = (j, -\varepsilon), \qquad g_\alpha = g_j^\varepsilon.$$

A product $x = g_{\alpha_1} \cdots g_{\alpha_k}$, $k \geq 1$, is called a *reduced word* if $\alpha_i \neq \alpha_{i+1}^*$ for $1 \leq i < k$. In that case k is called the *length* of x and is denoted by $|x|$. The identity has length zero by definition: $|e| = 0$. Following Figà–Talamanca and Picardello [14] (see also Chiswell [12] and Lyndon [23]) a state φ on $\mathcal{A}$ is called a *Haagerup state* if

(i) $\varphi(e) = 1$ and $|\varphi(g_j)| \leq 1$ for all g_j;
(ii) $\varphi(g_j^{-1}) = \overline{\varphi(g_j)}$;
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$ for any $x, y \in F_\infty$ with $|xy| = |x| + |y|$.

Examples are given by the one-parameter family of states φ_γ, $0 \leq \gamma \leq 1$, defined by

$$\varphi_\gamma(x) = \gamma^{|x|}, \qquad x \in F_\infty. \tag{1}$$

This is due to Haagerup [17]. With the notation $0^0 = 1$ the case $\gamma = 0$ corresponds to the tracial state φ_0 on $\mathcal{A}$ characterized by:

$$\varphi_0(x) = \begin{cases} 1, & \text{if } x = e, \\ \\ 0, & \text{if } x \in F_\infty,\ x \neq e. \end{cases} \tag{2}$$

We consider the two sums

$$a_N^+ = \frac{1}{\sqrt{N}} \sum_{j=1}^N g_j, \qquad a_N^- = \frac{1}{\sqrt{N}} \sum_{j=1}^N g_j^{-1}, \tag{3}$$

and the limit of their *mixed momenta*:

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\varepsilon_1} \cdots \tilde{a}_N^{\varepsilon_k}), \qquad k \geq 1, \quad \varepsilon_1, \cdots, \varepsilon_k \in \{\pm\}, \tag{4}$$

where $\tilde{a}_N^\varepsilon = a_N^\varepsilon - \varphi_{\lambda/\sqrt{N}}(a_N^\varepsilon)$. In the previous paper [4] we proved the existence of the limit and obtained an explicit realization of the GNS space of the limit by means of a *finite temperature* analogue of the usual Boltzmannian Fock space. This finite temperature analogue, which was first introduced by Fagnola [13], appears also in the stochastic limit of quantum electrodynamics at finite temperature [1, 3] and, hence, possesses a similar characteristic as the finite temperature (or universally invariant) Brownian motion. As for the symmetrized random variable $Q_N = a_N^+ + a_N^-$, Hashimoto [18] investigated the limit $\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{Q}_N^k)$ for any $k \geq 1$ and $\lambda > 0$, and proved that it coincides with the k-th moment of

$$u_\lambda(s)ds = \frac{1}{2\pi}\, \chi_{[-2-\lambda, 2-\lambda]}(s)\, \frac{\sqrt{(2+\lambda+s)(2-\lambda-s)}}{1-\lambda s}\, ds$$

which belongs to the Ullman family of probability measures introduced in connection with potential theory. Beyond potential theory the Ullman distributions also have emerged naturally in quantum probability and in physics, see e.g., [1, 10, 19].

A different generalization of the notion of independence called (φ, ψ)-independence, also with the motivation from the Haagerup functions on F_∞, was proposed by Bożejko and the corresponding central limit theorem was later proved by Bożejko and Speicher [11], see also [10]. In particular, in the notations (1) and (2), the generators of F_∞ are

$(\varphi_\gamma, \varphi_0)$-independent for any $0 \le \gamma \le 1$. Using this fact, Bożejko [8] was able to give a different proof of the combinatorial part of our result [4, Theorem 5.2] by a direct verification of the conditions of [11, Theorem 2]. It is noticeable that our notion of independence discussed in this paper bears an analytical nature on the basis of inequalities rather than an algebraic identities as in the case of (φ, ψ)-independence, see Section 7. Nevertheless, (φ, ψ)-independence is more general since it covers, in principle, limit states whose mixed momenta are defined by partitions with subsets of cardinality higher than 1 or 2. However, the conditions for the validity of central limit theorems in [11] is given by a set of countably many limits which do not give much insight on the mechanism underlying the validity of this theorem.

A direct comparison of the two notions of independence is not obvious: our notion and corresponding limit theorem are stated in the context of general algebras, while Theorem 2 in [11] is stated on free products a priori; our conditions allow only singeltons and pair partions to survive while in principle (we do not know any concrete published example) the conditions in [11] allow partitions with subsets of arbitrary cardinarity; on the other hand the partitions in [11] have to satisfy a generalized non-crossing condition, while in our case non-negligible pair partitions can be crossing.

In view of the results obtained in the present paper, together with several related works [5, 6, 7, 15, 20, 24] and others, we may conjecture that, underlying any central limit theorem arising in the harmonic analysis on discrete groups or, more generally, on discrete graphs, there should be an appropriate notion of *independence* or of *weak dependence* and the corresponding *fully quantum* central limit theorem, in this connection see also [9, 16, 21, 25]. This conjecture is supported by several examples and a more detailed study shall appear elsewhere.

Acknowledgements. The authors are grateful to Professor M. Bożejko for interesting discussion on the connection between his notion of ψ-independence and the one introduced in this paper.

2. Singleton condition. In order to prove a central limit theorem with the method of moments it is necessary to observe that *only a few* singletons give a non-zero contribution to the limit. The role of the singleton condition was first pointed out by von Waldenfels [28], [29]. The content of this section is rather standard and is included for completeness.

Definition 1. Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{C}$ a C^*-algebra with norm $|\cdot|$, and $E : \mathcal{A} \to \mathcal{C}$ a real linear map. A finite or countably infinite set of sequences

$$(b_n^{(1)})_{n=1}^\infty, \quad (b_n^{(2)})_{n=1}^\infty, \cdots, (b_n^{(j)})_{n=1}^\infty, \cdots$$

of elements in $\mathcal{A}$ with mean $E(b_n^{(j)}) = 0$ is said to satisfy the *singleton condition* with respect to E if for any choice of $k \ge 1$, $j_1, \cdots, j_k \in \mathbf{N}$, and $n_1, \cdots, n_k \in \mathbf{N}$

$$E\big(b_{n_1}^{(j_1)} \cdots b_{n_k}^{(j_k)}\big) = 0 \tag{5}$$

holds whenever there exists an index n_s which is different from all other ones, i.e., such that $n_s \ne n_t$ for $s \ne t$.

In the above definition the condition $E(b_n^{(j)}) = 0$ is, in fact, a consequence of (5). The singleton condition is equivalent to the usual independence in the classical case and follows from free independence [27]. We may generalize the (E, ψ)-independence [10] by replacing the condition $E(b_n^{(j)}) = 0$ with $\psi(b_n^{(j)}) = 0$.

DEFINITION 2. We say that sequences $(b_n^{(1)}), (b_n^{(2)}), \cdots$ of elements of $\mathcal{A}$ satisfy the condition of *boundedness of the mixed momenta* if for each $k \in \mathbf{N}$ there exists a positive constant $\nu_k \geq 0$ such that

$$\left| E\big(b_{n_1}^{(j_1)} \cdots b_{n_k}^{(j_k)}\big) \right| \leq \nu_k \tag{6}$$

for any choice of $n_1, \cdots, n_k$ and $j_1, \cdots, j_k$.

Given a sequence $b = (b_n)_{n=0}^{\infty} \subset \mathcal{A}$, we put

$$S_N(b) = \sum_{n=1}^{N} b_n. \tag{7}$$

LEMMA 1. *Let $(b_n^{(1)}), (b_n^{(2)}), \cdots$ be sequences of elements of $\mathcal{A}$ satisfying the condition of boundedness of the mixed momenta. Then, for any $\alpha > 0$ it holds that*

$$\begin{aligned} &\lim_{N\to\infty} E\left(\frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha}\right) \\ &= \lim_{N\to\infty} N^{-\alpha k} \sum_{\alpha k \leq p \leq k} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,p\} \\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b_{\sigma\circ\pi(1)}^{(1)} \cdots b_{\sigma\circ\pi(k)}^{(k)}\right), \end{aligned} \tag{8}$$

in the sense that one limit exists if and only if the other does and the limits coincide. (The limit is understood in the sense of norm convergence in $\mathcal{C}$.)

P r o o f. Expanding the product explicitly by means of (7), we obtain

$$E\left(\frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha}\right) = N^{-\alpha k} \sum_{j_1,\cdots,j_k=1}^{N} E\left(b_{j_1}^{(1)} \cdots b_{j_k}^{(k)}\right). \tag{9}$$

Note that the sum may be taken over all mappings $j : \{1, \cdots, k\} \to \{1, \cdots, N\}$. We shall split the sum according to the cardinality of the range of j. Suppose that j has a range of p elements, $1 \leq p \leq k$. Then there exist a unique surjective map $\pi : \{1, \cdots, k\} \to \{1, \cdots, p\}$ and a unique order-preserving map $\sigma : \{1, \cdots, p\} \to \{1, \cdots, N\}$ such that $j = \sigma \circ \pi$. Then (9) becomes

$$N^{-\alpha k} \sum_{1 \leq p \leq k} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,p\} \\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b_{\sigma\circ\pi(1)}^{(1)} \cdots b_{\sigma\circ\pi(k)}^{(k)}\right). \tag{10}$$

For the assertion (8) it is sufficient to show that, whenever $p < \alpha k$, one has

$$\lim_{N\to\infty} N^{-\alpha k} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,p\} \\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b_{\sigma\circ\pi(1)}^{(1)} \cdots b_{\sigma\circ\pi(k)}^{(k)}\right) = 0. \tag{11}$$

It follows immediately from (6) that

$$\left| \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,p\} \\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b^{(1)}_{\sigma\circ\pi(1)} \cdots b^{(k)}_{\sigma\circ\pi(k)}\right) \right|$$

$$\le \nu_k \left| \left\{ \begin{matrix} \pi : \{1,\cdots,k\} \to \{1,\cdots,p\} \\ \text{surjective} \end{matrix} \right\} \right| \left| \left\{ \begin{matrix} \sigma : \{1,\cdots,p\} \to \{1,\cdots,N\} \\ \text{order-preserving} \end{matrix} \right\} \right|. \qquad (12)$$

Note that

$$\lim_{N\to\infty} N^{-p} \left| \left\{ \begin{matrix} \sigma : \{1,\cdots,p\} \to \{1,\cdots,N\} \\ \text{order-preserving} \end{matrix} \right\} \right| = \lim_{N\to\infty} N^{-p} \binom{N}{p} = \frac{1}{p!}. \qquad (13)$$

Then (11) follows immediately from (12) and (13). ∎

LEMMA 2. *Notations and assumptions being the same as in Lemma 1, assume that the sequences* $(b^{(j)}_n)$ *satisfies the singleton condition with respect to* E. *Then*

$$\lim_{N\to\infty} E\left(\frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha} \right) = 0 \qquad (14)$$

takes place if $\alpha > 1/2$ *or if* $\alpha = 1/2$ *and* k *is odd. If* $\alpha = 1/2$ *and* k *is even, say* $k = 2n$, *the left hand side of* (14) *is equal to the limit*

$$\lim_{N\to\infty} N^{-n} \sum_{\substack{\pi:\{1,\cdots,2n\}\to\{1,\cdots,n\} \\ 2-1 \text{ map}}} \sum_{\substack{\sigma:\{1,\cdots,n\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b^{(1)}_{\sigma\circ\pi(1)} \cdots b^{(2n)}_{\sigma\circ\pi(2n)}\right). \qquad (15)$$

Moreover, the following Gaussian bound takes place:

$$\limsup_{N\to\infty} \left| E\left(\frac{S_N(b^{(1)})}{N^{1/2}} \cdot \frac{S_N(b^{(2)})}{N^{1/2}} \cdots \frac{S_N(b^{(2n)})}{N^{1/2}} \right) \right| \le \frac{(2n)!}{2^n n!} \nu_{2n}. \qquad (16)$$

Proof. We use the same notation as in the proof of Lemma 1. For each surjective map $\pi : \{1,\cdots,k\} \to \{1,\cdots,p\}$ put $S_j = \pi^{-1}(j)$, $1 \le j \le p$. If $|S_j| = 1$ for some j,

$$\sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} E\left(b^{(1)}_{\sigma\circ\pi(1)} \cdots b^{(k)}_{\sigma\circ\pi(k)}\right) = 0$$

by the singleton condition. Suppose that $|S_j| \ge 2$ for all j. Then

$$k = \sum_{j=1}^{p} |S_j| \ge 2p.$$

This condition is incompatible with $p \ge \alpha k$ if $\alpha > 1/2$ or if $\alpha = 1/2$ and k is odd. Thus (14) follows from (8).

Suppose that $\alpha = 1/2$ and $k = 2n$. Then the limit of the left hand side of (8) exists if and only if the limit of the right hand side of (8) exists and, in that case, it is reduced to (15). Finally, (15) is dominated in norm by

$$\lim_{N\to\infty} N^{-n} \nu_{2n} \left| \left\{ \begin{matrix} \pi : \{1,\cdots,n\} \to \{1,\cdots,2n\} \\ \text{2-1 map} \end{matrix} \right\} \right| \binom{N}{n} = \frac{(2n)!}{2^n n!} \nu_{2n},$$

as desired. ∎

3. Entangled ergodic theorems. Following [4] we illustrate here the natural role of the non-crossing partitions in the proof of central limit theorems under some singleton conditions and we show how this naturally leads to the idea of *entangled ergodic theorems.*

Definition 3. Let $(S_1, \cdots, S_p)$ be a partition of $\{1, \cdots, k\}$ and put

$$\underline{s}_j = \min\{s \in S_j\}, \qquad \overline{s}_j = \max\{s \in S_j\}.$$

Then S_j is called *non-crossing* if for any $h = 1, \cdots, p$,

$$(\underline{s}_j, \overline{s}_j) \cap (\underline{s}_h, \overline{s}_h) \neq \phi \Leftrightarrow (\underline{s}_j, \overline{s}_j) \subseteq (\underline{s}_h, \overline{s}_h) \quad \text{or} \quad (\underline{s}_h, \overline{s}_h) \subseteq (\underline{s}_j, \overline{s}_j).$$

The set S_j is said to belong to the *non-crossing component* of a partition if, whenever $(\underline{s}_h, \overline{s}_h) \subseteq (\underline{s}_j, \overline{s}_j)$ it follows that S_h is non crossing. The partition $(S_1, \cdots, S_p)$ is called *totally crossing* if no two consecutive indices belong to the same set S_j.

Definition 4. Let $\mathcal{A}$ and E be as in Definition 1. For each $j \in \mathbf{N}$ let $(b_n^{(j)})$ be a sequence of elements of $\mathcal{A}$. These sequences are said to satisfy the *entangled ergodic theorem* with respect to E if for any $n \in \mathbf{N}$ and any totally crossing pair partition

$$\{1, \cdots, 2n\} = \bigcup_{k=1}^{n} \{i_k, j_k\}, \qquad 1 = i_1 < i_2 < \cdots < i_n, \quad i_k < j_k,$$

the limit

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{\alpha_1,\cdots,\alpha_n=1}^{N} E(b_{\alpha_1}^{(i_1)} \cdots b_{\alpha_1}^{(j_1)} \cdots b_{\alpha_n}^{(i_n)} \cdots b_{\alpha_n}^{(j_n)}) \tag{17}$$

exists in $\mathcal{C}$.

Remark. The *entanglement* is due to the non-commutativity. If $b_n^{(i)}$ commutes with $b_m^{(j)}$ for any m, n and $i \neq j$, (17) is reduced to the limit of usual ergodic averages:

$$\lim_{N\to\infty} E\left\{ \left(\frac{1}{N} \sum_{\alpha_1=1}^{N} b_{\alpha_1}^{(i_1)} b_{\alpha_1}^{(j_1)} \right) \cdots \left(\frac{1}{N} \sum_{\alpha_n=1}^{N} b_{\alpha_n}^{(i_n)} b_{\alpha_n}^{(j_n)} \right) \right\}$$

Theorem 3. *Under the assumptions of Lemma 1, suppose that the algebra $\mathcal{C}$ is the complex numbers and that the mean covariance*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\alpha=1}^{N} E(b_\alpha^{(i)} b_\alpha^{(j)}) \tag{18}$$

exists for any i, j. Then the central limit theorem holds if and only if the entangled ergodic theorem holds.

Corollary 4. *For a state E satisfying the singleton condition and the uniform boundedness of the mixed momenta (6), the central limit theorem holds if any one of the following conditions is satisfied:*

(i) *(q–commutation relations) for each $i, j \in \mathbf{N}$, $i \neq j$, there exists a complex number q_{ij} such that $b_m^{(i)} b_n^{(j)} = q_{ij} b_n^{(j)} b_m^{(i)}$ for any $m, n \in \mathbf{N}$;*

(ii) *(symmetry) $E(b_{\alpha_1}^{(i_1)} \cdots b_{\alpha_1}^{(j_1)} \cdots b_{\alpha_n}^{(i_n)} \cdots b_{\alpha_n}^{(j_n)})$ in (17) is independent of $\alpha_1, \cdots, \alpha_n$;*

(iii) *(pair partition freeness) $E(b_{\alpha_1}^{(i_1)} \cdots b_{\alpha_1}^{(j_1)} \cdots b_{\alpha_n}^{(i_n)} \cdots b_{\alpha_n}^{(j_n)}) = 0$ for any totally crossing pair partition.*

P r o o f. It is clear that any of the conditions (i), (ii), (iii) implies the existence of the limit (17) hence, by Theorem 3 the central limit theorem holds. ∎

Being based on several examples, we conjecture that the stationarity condition ensuring the usual ergodic theorem is also sufficient for the entangled ergodic theorem in general. Some indications of the proof are given in the case where one could prove a priori that only the non-crossing pair partitions are relevant in the limit. A preliminary result toward the proof of the entangled ergodic theorem in full generality, i.e. even without restriction to pair partitions, has been obtained by Liebscher [22]. The validity of the entangled ergodic theorem would imply that the usual stationarity condition is sufficient to guarantee the validity of the central limit theorem under the only assumption of singleton independence.

4. Properties of the Haagerup states. In the notations of Section 1, the two sequences $\{(g_n), (g_n^{-1})\}$ satisfy the singleton condition with respect to the Haagerup state φ_γ only when $\gamma = 0$. However, φ_γ satisfies a weak analogue of the singleton condition. When the state φ_γ under consideration is fixed, we write for simplicity

$$\tilde{g}_\alpha = g_\alpha - \gamma.$$

Obviously $\varphi_\gamma(\tilde{g}_\alpha) = 0$.

Definition 5. (i) A product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ is called *separable* at k, $1 \le k \le m$, if $\alpha_p \ne \alpha_q^*$ whenever $1 \le p \le k < q \le m$.

(ii) $\tilde{g}_{\alpha_k}$ is called a *singleton* in the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ if $\tilde{g}_{\alpha_k} \ne \tilde{g}_{\alpha_l}^*$ for any $l \ne k$.

(iii) Let $\tilde{g}_{\alpha_k}$ be a singleton in the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$. It is called *outer* if $\tilde{g}_{\alpha_p} \ne \tilde{g}_{\alpha_q}^*$ for any $p < k < q$.

(iv) A singleton $\tilde{g}_{\alpha_k}$ is called *inner* if $\tilde{g}_{\alpha_p} = \tilde{g}_{\alpha_q}^*$ for some $p < k < q$.

For example, in the product $\tilde{g}_1\tilde{g}_2\tilde{g}_1^{-1}\tilde{g}_3\tilde{g}_2$, the second $\tilde{g}_2$ is an inner singleton and the forth $\tilde{g}_3$ and the last $\tilde{g}_2$ are outer singletons. Notice that $\tilde{g}_2$ is not a "singleton" in the sense that $\tilde{g}_2$ appears twice, cf. Definition 1.

Lemma 5. *If $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ is separable at k, then*

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = \varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k})\varphi_\gamma(\tilde{g}_{\alpha_k+1} \cdots \tilde{g}_{\alpha_m})$$

Lemma 6. *If $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ has an outer singleton, then*

$$\varphi(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = 0.$$

P r o o f. If $\tilde{g}_{\alpha_k}$ is an outer singleton, applying Lemma 5 twice we find

$$\begin{aligned} \varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) &= \varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k})\varphi_\gamma(\tilde{g}_{\alpha_{k+1}} \cdots \tilde{g}_{\alpha_m}) \\ &= \varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_{k-1}})\varphi_\gamma(\tilde{g}_{\alpha_k})\varphi_\gamma(\tilde{g}_{\alpha_{k+1}} \cdots \tilde{g}_{\alpha_m}) = 0, \end{aligned}$$

as desired. ∎

The next result is a generalization of von Waldenfels' argument [28, 29] to products with inner singletons.

LEMMA 7. *Assume that a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ has no singleton at all or has no outer singletons. Let s be the number of inner singletons in the product and let*

$$p = |\{g_j\,;\ \textit{there exist } 1 \le k, l \le m \textit{ such that } \quad \alpha_k = (j,+),\ \alpha_l = (j,-)\}|$$

Then

$$s \le m-2 \quad \textit{and} \quad p \le \frac{m-s}{2} \tag{19}$$

P r o o f. Since there is no outer singleton, there exist at least two factors $\tilde{g}_{\alpha_k}$ and $\tilde{g}_{\alpha_l}$ with $\alpha_k^* = \alpha_l$. Hence $m \ge 2$ and $s \le m-2$. If $\tilde{g}_{\alpha_l}$ is not a singleton, there exists at least one element $\tilde{g}_{\alpha_k}$ such as $\alpha_k^* = \alpha_l$ and then $j_k = j_l$ $(k \ne l)$. Therefore $2p + s \le m$. ∎

DEFINITION 6. Assume that a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ contains $s \ge 0$ inner singletons and no outer singletons. Let $\alpha_{j_1}, \cdots, \alpha_{j_s}$ be the suffices which correspond the singletons and denote the rest by $\beta_1, \cdots, \beta_{m-s}$ in order. We say that the product satisfies the condition (NCI) if $g_{\beta_1} \cdots g_{\beta_{m-s}} = e$.

LEMMA 8. *If the product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ consists only of non–crossing pair partitions and of s inner singletons then*

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = (-\gamma)^s + (-\gamma)^{s+1} P(\gamma) \tag{20}$$

where P is a polynomial. If the (NCI) condition is not satisfied then

$$\varphi_\gamma(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}) = (-\gamma)^{s+1} P(\gamma). \tag{21}$$

From Lemma 8 one can deduces the central limit theorem for the Haagerup states. For more detailed argument see [4].

THEOREM 9. *Let $NCI_m(s,\varepsilon)$ be the set of equivalence classes of products $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_m}$ with the index $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m)$, which consist of $p = (m-s)/2$ non-crossing pairs and of s inner singletons. Then,*

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\varepsilon_1} \cdots \tilde{a}_N^{\varepsilon_m}) = \sum_{s=0}^{m-2} (-\lambda)^s \cdot |NCI_m(s,\varepsilon)|. \tag{22}$$

5. Limit process. By a general theory [2] there exist an algebraic probability space $\{\mathcal{A}_\lambda, \psi_\lambda\}$ and two random variables a_λ, a_λ^+ such that

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}(\tilde{a}_N^{\varepsilon_1} \cdots \tilde{a}_N^{\varepsilon_k}) = \psi_\lambda(a_\lambda^{\varepsilon_1} \cdots a_\lambda^{\varepsilon_k}). \tag{23}$$

For $\nu = L, R$ let

$$\Gamma(\mathbf{C})_\nu = \mathbf{C} \oplus \bigoplus_{n=1}^{\infty} \mathbf{C}^{\otimes n} \quad \left(= \bigoplus_{n=0}^{\infty} \mathbf{C} \right)$$

denote two copies of the full Fock spaces over $\mathbf{C}$ with free creations a_ν^+ and free annihilation a_ν. Let $\mathcal{H} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}_{m,n}$ be the free product $\Gamma(\mathbf{C})_L * \Gamma(\mathbf{C})_R$, that is, the (m,n)-particle space $\mathcal{H}_{m,n}$ is the complex linear span of the set of vectors $\{a_{\nu_1}^+ \cdots a_{\nu_k}^+ \Phi\}$ which satisfy the following conditions:

$$|\{j \mid \nu_j = L\}| = m, \qquad |\{j \mid \nu_j = R\}| = n,$$

and the scalar product is given by

$$\left\langle a^+_{\nu_1}\cdots a^+_{\nu_k}\Phi,\ a^+_{\nu'_1}\cdots a^+_{\nu'_l}\Phi\right\rangle_{\mathcal{H}} = \begin{cases} 1, & \text{if} \quad (\nu_1,\cdots,\nu_k)=(\nu'_1,\cdots,\nu'_l), \\ 0, & \text{otherwise.}\end{cases}$$

The actions of the creation operators

$$L^+ := a^+_L * 1 : \mathcal{H}_{m,n} \to \mathcal{H}_{m+1,n}\ ; \quad R^+ := 1 * a^+_R : \mathcal{H}_{m,n} \to \mathcal{H}_{m,n+1}$$

are given respectively by

$$\begin{aligned} L^+ a^+_{\nu_1}\cdots a^+_{\nu_k}\Phi &= a^+_L a^+_{\nu_1}\cdots a^+_{\nu_k}\Phi \\ R^+ a^+_{\nu_1}\cdots a^+_{\nu_k}\Phi &= a^+_R a^+_{\nu_1}\cdots a^+_{\nu_k}\Phi \end{aligned}$$

and the action of the annihilation

$$L = a_L * 1 : \mathcal{H}_{m,n} \to \mathcal{H}_{m-1,n}\ ; \quad R = 1 * a_R : \mathcal{H}_{m,n} \to \mathcal{H}_{m,n-1}$$

is given by

$$La^+_{\nu_1}\cdots a^+_{\nu_k}\Phi = \begin{cases} a^+_{\nu_2}\cdots a^+_{\nu_k}\Phi, & \text{if } \nu_1 = L \text{ and } k\ge 2, \\ \Phi, & \text{if } \nu_1 = L \text{ and } k = 1, \\ 0, & \text{otherwise,}\end{cases}$$

$$Ra^+_{\nu_1}\cdots a^+_{\nu_k}\Phi = \begin{cases} a^+_{\nu_2}\cdots a^+_{\nu_k}\Phi, & \text{if } \nu_1 = R \text{ and } k\ge 2, \\ \Phi, & \text{if } \nu_1 = R \text{ and } k = 1, \\ 0, & \text{otherwise.}\end{cases}$$

Let $P:\mathcal{H}\to\mathcal{H}$ be the orthogonal projection onto $\mathcal{H}^{\perp}_{0,0}$. Put

$$A^-_\lambda = L^+ + R - \lambda P, \qquad A^+_\lambda = L + R^+ - \lambda P,$$

where $\lambda \ge 0$ is a constant.

THEOREM 10. *The limit process* $(a^+_\lambda, a^-_\lambda, \psi_\lambda)$ *is represented on* $\mathcal{H}$. *That is, all its correlations* (*23*) *are given by*

$$\psi_\lambda(a^{\varepsilon_1}_\lambda\cdots a^{\varepsilon_m}_\lambda) = \langle \Phi, A^{\varepsilon_1}_\lambda\cdots A^{\varepsilon_m}_\lambda\Phi\rangle_{\mathcal{H}}\,.$$

Proof. In Theorem 9 we have seen that the ψ_λ-correlators are completely determined by the cardinalities of the sets NCI_m. We thus need only to establish a bijective correspondence between NCI_m-partitions associated with $a^{\varepsilon_1}_\lambda\cdots a^{\varepsilon_m}_\lambda$ and terms in the expansion of

$$\langle \Phi, A^{\varepsilon_1}_\lambda\cdots A^{\varepsilon_m}_\lambda\Phi\rangle = \sum_{B^{\varepsilon_1}_{\nu_1},\cdots,B^{\varepsilon_m}_{\nu_m}} \langle \Phi, B^{\varepsilon_1}_{\nu_1}\cdots B^{\varepsilon_m}_{\nu_m}\Phi\rangle,$$

where $B^-_R = L^+$, $B^-_L = R$, $B^+_R = R^+$, $B^+_L = L$ and $B^-_0 = B^+_0 = -\lambda P$. In a product $B^{\varepsilon_1}_{\nu_1}\cdots B^{\varepsilon_m}_{\nu_m}$, we call $(B^{\varepsilon_p}_{\nu_p}, B^{\varepsilon_q}_{\nu_q})$ $(p<q)$ a *pair* if $B^{\varepsilon_p}_{\nu_p} = L$ and $B^{\varepsilon_q}_{\nu_q} = L^+$ or $B^{\varepsilon_p}_{\nu_p} = R$ and $B^{\varepsilon_q}_{\nu_q} = R^+$. If $B^{\varepsilon_p}_{\nu_p} = -\lambda P$ we call it a *singleton*. From the definition of $\mathcal{H}, A^+_\lambda, A_\lambda$ we

see easily that $\langle\Phi, B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}\Phi\rangle \neq 0$ if and only if $B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}$ forms a non-crossing pair partition with s inner singletons $(0 \le s \le m-2)$. In this case,

$$\langle\Phi, B_{\nu_1}^{\varepsilon_1}\cdots B_{\nu_m}^{\varepsilon_m}\Phi\rangle = (-\lambda)^s.$$

Therefore we obtain the desired bijective correspondence. ■

6. Functional central limit theorem for the Haagerup state. In general, a central limit theorem is extended in a canonical manner to a functional central limit theorem (or invariance principle) from which the corresponding process is derived, see e.g., [26]. Given a sequence $\{b_i\}$ of random variables, for the functional central limit theorem we consider

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{\lfloor Nt\rfloor} b_i = \frac{1}{\sqrt{N}}\int_0^\infty \sum_{i=1}^{\lfloor Nt\rfloor} \chi_{(i-1,i)}(s) b_i ds,$$

which is in the limit $N \to \infty$ equivalent to

$$\frac{1}{\sqrt{N}}\int_0^\infty \chi_{[0,Nt]}(s)\sum_{i=1}^\infty \chi_{(i-1,i)}(s) b_i ds = \frac{1}{\sqrt{N}}\sum_{i=1}^\infty b_i \int_{i-1}^{i} \chi_{[0,Nt]}(s) ds$$
$$= \frac{1}{\sqrt{N}}\sum_{i=1}^\infty b_i \int_{i-1}^{i} \chi_{[0,t]}\left(\frac{s}{N}\right) ds.$$

Thus, we consider more generally

$$\frac{1}{\sqrt{N}}\sum_{i=1}^\infty b_i \int_{i-1}^{i} f\left(\frac{s}{N}\right) ds,$$

where f is a suitable test function.

Going back to our case, we put

$$S_N^{(\varepsilon)}(f) = \sum_{i=1}^\infty \tilde{g}_i^\varepsilon \int_{i-1}^{i} f\left(\frac{t}{N}\right) dt, \qquad \varepsilon = \pm 1,$$

where f is an $\mathbf{R}$-valued continuous function with compact support. Then we calculate the mixed momenta:

$$\varphi_\gamma\left(\frac{S_N^{(\epsilon_1)}(f_1)}{\sqrt{N}}\cdots\frac{S_N^{(\epsilon_m)}(f_m)}{\sqrt{N}}\right) =$$
$$= \frac{1}{(\sqrt{N})^m}\sum_{i_1,\cdots,i_m=1}^\infty \varphi_\gamma(\tilde{g}_{i_1}^{\epsilon_1}\cdots\tilde{g}_{i_m}^{\epsilon_m})\int_{i_1-1}^{i_1} f_1\left(\frac{t_1}{N}\right)dt_1\cdots\int_{i_m-1}^{i_m} f_m\left(\frac{t_m}{N}\right)dt_m, \tag{24}$$

where $\varepsilon_j = \pm 1$ and f_j is a continuous function with compact support, $j = 1, 2, \cdots, m$. In view of the uniform bound $\|f_j\|_{L^1} \le C$ we apply the arguments in Section 4 (only non-crossing pair partitions with inner singletons contribute to the limit). Then, in the limit (24) is equivalent to

$$\frac{1}{(\sqrt{N})^m}\sum_{s=0}^{m-2}(-\gamma)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)}\ \sum_{\substack{i_{\omega(1)},\cdots,i_{\omega(s)}\\ \text{distinct}}} \int_{i_{\omega(1)}-1}^{i_{\omega(1)}} f_{\omega(1)}\left(\frac{t}{N}\right)dt\cdots\int_{i_{\omega(s)}-1}^{i_{\omega(s)}} f_{\omega(s)}\left(\frac{t}{N}\right)dt$$

$$\times \sum_{\substack{i_{\alpha(j)}\notin\{\omega(1),\cdots,\omega(s)\}\\ \text{distinct}}} \prod_{j=1}^{p} \int\limits_{i_{\alpha(j)}-1}^{i_{\alpha(j)}} \int\limits_{i_{\alpha(j)}-1}^{i_{\alpha(j)}} f_{\alpha(j)}\left(\frac{t_{\alpha(j)}}{N}\right) f_{\beta(j)}\left(\frac{t_{\beta(j)}}{N}\right) dt_{\alpha(j)}dt_{\beta(j)}$$

$$+O\left(\frac{1}{\sqrt{N}}\right), \tag{25}$$

where $p=(m-s)/2$ and

$$NCI_m(s,\epsilon) = \left\{ \begin{array}{c} (\alpha,\beta,\omega) = (\alpha(1),\cdots,\alpha(p),\beta(1),\cdots,\beta(p),\omega(1),\cdots,\omega(s)); \\ \{\alpha(1),\cdots,\alpha(p),\beta(1),\cdots,\beta(p),\omega(1),\cdots,\omega(s)\} = \{1,\cdots,m\}, \\ \alpha(j)<\beta(j),\ \alpha(j)<\alpha(j+1),\ \omega(j)<\omega(j+1),\ \varepsilon_{\alpha(j)} = -\varepsilon_{\beta(j)}, \\ \text{for each } l \text{ there exists } j \text{ such that } \alpha(j)<\omega(l)<\beta(j) \end{array} \right\}.$$

In (25), the indices $i_{\alpha(j)}$'s and $i_{\omega(j)}$'s are different each other. But again by the uniform boundedness of f_j's, one obtains, for instance,

$$\sum_{i_{\omega(1)}\notin\{i_{\alpha(1)},\cdots,i_{\alpha(p)},i_{\omega(2)},\cdots,i_{\omega(s)}\}} \int\limits_{i_{\omega(1)}-1}^{i_{\omega(1)}} f_{\omega(1)}\left(\frac{t}{N}\right)dt$$

$$= \int\limits_0^\infty f_{\omega(1)}\left(\frac{t}{N}\right)dt + O\left(\frac{1}{N}\right) = N\int\limits_0^\infty f_{\omega(1)}(s)ds + O\left(\frac{1}{N}\right)$$

and

$$\sum_{i_{\alpha(1)}\notin\{i_{\alpha(2)},\cdots,i_{\alpha(p)},i_{\omega(1)},\cdots,i_{\omega(s)}\}} \int\limits_{i_{\alpha(1)}-1}^{i_{\alpha(1)}} \int\limits_{i_{\alpha(1)}-1}^{i_{\alpha(1)}} f_{\alpha(1)}\left(\frac{t_1}{N}\right) f_{\beta(1)}\left(\frac{t_2}{N}\right) dt_1dt_2 =$$

$$= \sum_{i=1}^\infty \int\limits_{i-1}^{i}\int\limits_{i-1}^{i} f_{\alpha(1)}\left(\frac{t_1}{N}\right) f_{\beta(1)}\left(\frac{t_2}{N}\right) dt_1dt_2 + O\left(\frac{1}{N^2}\right)$$

$$= N^2\sum_{i=1}^\infty \int\limits_{(i-1)/N}^{i/N}\int\limits_{(i-1)/N}^{i/N} f_{\alpha(1)}(s_1)f_{\beta(1)}(s_2)ds_1ds_2 + O\left(\frac{1}{N^2}\right).$$

Recall that $\gamma = O(1/\sqrt{N})$. Then (25) becomes

$$\frac{1}{(\sqrt{N})^m}\sum_{s=0}^{m-2}(-\gamma)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} N\int\limits_0^\infty f_{\omega(1)}(s)ds\cdots N\int\limits_0^\infty f_{\omega(s)}(s)ds$$

$$\times \sum_{i_{\alpha(1)},\cdots,i_{\alpha(p)}=1}^{\infty} \prod_{j=1}^{p} N^2 \int\limits_{(i_{\alpha(j)}-1)/N}^{i_{\alpha(j)}/N}\int\limits_{(i_{\alpha(j)}-1)/N}^{i_{\alpha(j)}/N} f_{\alpha(j)}(s_{\alpha(j)})f_{\beta(j)}(s_{\beta(j)})ds_{\alpha(j)}ds_{\beta(j)}$$

$$+O\left(\frac{1}{\sqrt{N}}\right). \tag{26}$$

Lemma 11. *Let f_1, f_2 be continuous functions with compact supports. Then,*

$$\lim_{N\to\infty} N\sum_{i=1}^\infty \int\limits_{(i-1)/N}^{i/N}\int\limits_{(i-1)/N}^{i/N} f_1(s_1)f_2(s_2)ds_1ds_2 = \int\limits_0^\infty f_1(s)f_2(s)ds.$$

The proof is easy. By this lemma the limit of (26) as $N \to \infty$ becomes

$$\lim_{N\to\infty} \frac{1}{(\sqrt{N})^m} \sum_{s=0}^{m-2} (-\gamma)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} N \int_0^\infty f_{\omega(1)}(s)ds \cdots N \int_0^\infty f_{\omega(s)}(s)ds$$

$$\times N \int_0^\infty f_{\alpha(1)}(s) f_{\beta(1)}(s)ds \cdots N \int_0^\infty f_{\alpha(p)}(s) f_{\beta(p)}(s)ds$$

$$= \sum_{s=0}^{m-2} (-\lambda)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} \prod_{i=1}^{s} \int_0^\infty f_{\omega(i)}(s)ds \prod_{j=1}^{p} \int_0^\infty f_{\alpha(j)}(s) f_{\beta(j)}(s)ds.$$

Consequently,

THEOREM 12. *For $j = 1, 2, \cdots, m$ let $f_j : \mathbf{R} \to \mathbf{R}$ be a continuous function with compact support. Then one has*

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}} \left(\frac{S_N^{(\epsilon_1)}(f_1)}{\sqrt{N}} \cdots \frac{S_N^{(\epsilon_m)}(f_m)}{\sqrt{N}} \right)$$
$$= \sum_{s=0}^{m-2} (-\lambda)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} \prod_{i=1}^{s} \int_0^\infty f_{\omega(i)}(s)ds \prod_{j=1}^{p} \int_0^\infty f_{\alpha(j)}(s) f_{\beta(j)}(s)ds.$$

The above is a functional central limit theorem. We now put $S_{N,t}^{(\epsilon)}(f) = S_N^{(\epsilon)}(\chi_{[0,t]}f)$. By modifying the above argument, we obtain

THEOREM 13. *For continuous functions f_j, $j = 1, 2, \cdots, m$, with compact supports, we have*

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}} \left(\frac{S_{N,t_1}^{(\epsilon_1)}(f_1)}{\sqrt{N}} \cdots \frac{S_{N,t_m}^{(\epsilon_m)}(f_m)}{\sqrt{N}} \right)$$
$$= \sum_{s=0}^{m-2} (-\lambda)^s \sum_{(\alpha,\beta,\omega)\in NCI_m(s,\epsilon)} \prod_{i=1}^{s} \langle 1, f_{\omega(i)} \rangle_{t_{\omega(i)}} \prod_{j=1}^{(m-s)/2} \langle f_{\alpha(j)}, f_{\beta(j)} \rangle_{\min\{t_{\alpha(j)}, t_{\beta(j)}\}}$$

where

$$\langle f, g \rangle_t = \int_0^t f(s) g(s) ds.$$

Now we have the Fock representation of this process. Let $\mathcal{H}$ be the Fock space introduced in Section 6, and $\mathcal{K} = L^2(\mathbf{C})$. Using the notations in Section 5, put

$$A_{\lambda,t}^-(f) = L^+ \otimes \chi_{[0,t]} f + R \otimes \chi_{[0,t]} f - \lambda \langle 1, f \rangle_t P,$$
$$A_{\lambda,t}^+(f) = L \otimes \chi_{[0,t]} f + R^+ \otimes \chi_{[0,t]} f - \lambda \langle 1, f \rangle_t P.$$

Then by Theorem 10 and Theorem 13, we have

THEOREM 14. *The limit process $(a_t^+, a_t^-, \psi_\lambda)$ is represented on $\mathcal{H} \otimes \mathcal{K}$, and its all correlators are given by*

$$\psi_\lambda(a_{t_1}^{\varepsilon_1}(f_1) \cdots a_{t_m}^{\varepsilon_m}(f_m)) = \left\langle \Phi, A_{\lambda,t_1}^{\varepsilon_1}(f_1) \cdots A_{\lambda,t_m}^{\varepsilon_m}(f_m) \Phi \right\rangle_{\mathcal{H}\otimes\mathcal{K}}.$$

7. Singleton independence. We are led to the following

DEFINITION 7. Let $\mathcal{A}$ be a $*$-algebra and let $S = \{g_j, g_j^* \,;\, j \in \mathbf{N}\}$ be a countable subset of $\mathcal{A}$. Assume we are given a family of states φ_γ, $\gamma \geq 0$, on $\mathcal{A}$ such that $\varphi_\gamma(g_\alpha) = \gamma$ for any g_α, where $\alpha = (j, \varepsilon)$ and $g_\alpha = g_j^\varepsilon$. Then the sequence $\{g_j\}$ is called to be *singleton independent* with respect to φ_γ if

$$|\varphi_\gamma(g_{\alpha_1} \cdots g_{\alpha_k})| \leq \gamma c_k |\varphi(g_{\alpha_1} \cdots \hat{g}_{\alpha_s} \cdots g_{\alpha_k})|, \tag{27}$$

whenever α_s is a singleton for $(\alpha_1, \cdots, \alpha_k)$.

The case of $\gamma = 0$ is related to the usual singleton condition. Condition (27) and boundedness (6) imply that

$$|\varphi_\gamma(g_{\alpha_1} \cdots g_{\alpha_m})| \leq C_m \gamma^s \tag{28}$$

whenever $g_{\alpha_1} \cdots g_{\alpha_k}$ has s singletons.

Conditions (27), (28) are easily verified for the Haagerup states. Other examples are found in the unitary representations of the free groups [14]. By specializing a parameter of spherical functions associated with representations of the principal series, we obtain a family of positive definite functions:

$$\psi_N(x) = \left(1 + |x|\frac{N-1}{N}\right)(2N-1)^{-|x|/2}, \qquad x \in F_N,$$

where F_N is the free group on N generators. This state satisfies the singleton independence. In fact, one sees that

$$\psi_N = \left(1 + \frac{N-1}{N}\gamma\frac{\partial}{\partial\gamma}\right)\varphi_\gamma,$$

where φ_γ is a Haagerup state with $\gamma = 1/\sqrt{2N-1}$. Suppose that $g = g_{\alpha_1} \cdots g_{\alpha_k}$ has s singletons. Then $\varphi_\gamma(g) = \gamma^t$ with some $t \geq s$ and $\psi_N(g) = \gamma^s P(\gamma)$ where P is a polynomial. Since $\psi_N(g_j) = a = \sqrt{2N-1}/N \geq \gamma$, the singleton independence $|\psi_N(g)| \leq C_k a^s$ holds.

As before, we put

$$S_N^{(\varepsilon)} = \sum_{j=1}^{N} \tilde{g}_j^\varepsilon, \qquad \varepsilon = \pm 1,$$

and, for fixed $k \in \mathbf{N}$ and $\varepsilon_1, \cdots, \varepsilon_k \in \{\pm 1\}$ we consider the product

$$S_N^{(\varepsilon_1)} \cdots S_N^{(\varepsilon_k)} = \sum_{j_1, \cdots, j_k = 1}^{N} \tilde{g}_{j_1}^{\varepsilon_1} \cdots \tilde{g}_{j_k}^{\varepsilon_k} = \sum_{j_1, \cdots, j_k} \tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}.$$

Put $I_k = \{(1, \varepsilon_1), \cdots, (k, \varepsilon_k)\}$ and consider α as a function $\alpha : I_k \to \{1, \cdots, N\}$. For given α put $p = |\alpha(I_k)|$. We denote by $\alpha(I_k) = \{\overline{\alpha}_1, \cdots, \overline{\alpha}_p\}$ its range (with $\overline{\alpha}_i \neq \overline{\alpha}_j$) and put

$$S_j = \alpha^{-1}(\overline{\alpha}_j), \qquad j = 1, \cdots, p,$$

$$\mathcal{P}_{k,p} = \{(S_1, \cdots, S_p)\,;\, \text{partition of } I_k \text{ of cardinality } p\},$$

$$[S_1, \cdots, S_p] = \{\alpha\,;\, \alpha|_{S_j} = \alpha(S_j) = \text{ const. and } \alpha(S_i) \neq \alpha(S_j) \text{ if } i \neq j\}.$$

With these notations our goal is to study the large N asymptotics of the rescaled expectation values

$$\varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{(\varepsilon_1)}}{\sqrt{N}}\cdots\frac{S_N^{(\varepsilon_k)}}{\sqrt{N}}\right)=N^{-k/2}\sum_{p=1}^{k}\sum_{(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}}\sum_{\alpha\in[S_1,\cdots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}). \tag{29}$$

LEMMA 15. *Given $s=0,1,\cdots,k$, denote*

$$\mathcal{P}^s_{k,p}=\{(S_1,\cdots,S_p)\text{ which have exactly } s \text{ singletons}\},$$

where a singleton of $(S_1,\cdots,S_p)$ stands for S_i with $|S_i|=1$. Then it holds that $p\le(k+s)/2$. Moreover, if $p<(k+s)/2$ then

$$\lim_{N\to\infty}N^{-k/2}\sum_{(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}}\sum_{\alpha\in[S_1,\cdots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})=0.$$

Proof. For $(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}$ we have

$$k=\sum_{j=1}^{p}|S_j|=\sum_{\{j\in\{1,\cdots,p\},|S_j|\ge2\}}|S_j|+s\ge2(p-s)+s=2p-s.$$

Then, in view of the boundedness of the mixed momenta (6), we see that the sum is dominated by a constant times of

$$N^{-(k+s)/2}|\mathcal{P}^s_{k,p}|\frac{\lambda^s}{p!}N^p\to0. \ \blacksquare$$

We see from Lemma 15 that the non-trivial contribution to the limit of (29) comes from those partitions $(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}$ satisfying $p=(k+s)/2$, that is, $k=2p-s$.

LEMMA 16. *Assume that $k=2p-s$ holds. Then for any $(S_1,\cdots,S_p)\in\mathcal{P}^s_{k,p}$, it holds that $|S_j|=1$ or $|S_j|=2$ for all j.*

Proof. Suppose otherwise, say, $|S_1|\ge3$. Then we have

$$\begin{aligned}k&=3+\sum_{j\ge2,|S_j|\ge2}|S_j|+s\ge3+2(p-s-1)+s\\&=3+2p-2s-2+s=2p-s+1,\end{aligned}$$

which is incompatible with $k=2p-s$. $\blacksquare$

Suppose that a partition $(S_1,\cdots,S_p)$ of $\{1,\cdots,k\}$ has s singletons and $|S_j|=1$ or 2 for $j=1,\cdots,p$. We denote by $(\tilde{S}_1,\cdots,\tilde{S}_{p-s})$ the set of all S_j's with $|S_j|=2$ and say that $(\tilde{S}_1,\cdots,\tilde{S}_{p-s})$ is the pair partition associated to $(S_1,\cdots,S_p)$. The pair partition associated to a 2–1 map $\beta:\{1,\cdots,2p\}\to\{1,\cdots,p\}$ will be called *negligible* if

$$|\varphi_\gamma(g_{\beta_1}\cdots g_{\beta_{2p}})|\le c\gamma. \tag{30}$$

LEMMA 17. *Suppose that φ_γ satisfies condition (30). Fix $s=0,\cdots,k$ and let $\tilde{\mathcal{P}}_{k,1,2,s}$ denote the set of all partitions $(S_1,\cdots,S_p)$ with s singletons such that $|S_j|=1$ or 2 and such that the associated pair partition is negligible. Then*

$$\lim_{N\to\infty}N^{-k/2}\sum_{(S_1,\cdots,S_p)\in\tilde{\mathcal{P}}_{k,1,2,s}}\sum_{\alpha\in[S_1,\cdots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})=0. \tag{31}$$

Proof. Iterating (27), we see that the sum (31) is majorized by

$$CN^{-(k+s)/2} \sum_{(S_1,\cdots,S_p)\in\tilde{\mathcal{P}}_{k,1,2,s}} \sum_{\alpha\in[S_1,\cdots,S_p]} |\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1}\cdots\tilde{g}_{\beta_{k-s}})|, \tag{32}$$

where $(\beta_1, \cdots, \beta_{k-s})$ is obtained from $(\alpha_1, \cdots, \alpha_k)$ by removing the singletons. Since the pair partition associated to $(S_1, \cdots, S_p)$ is negligible, and (30) implies

$$|\varphi_\gamma(\tilde{g}_{\beta_1}\cdots\tilde{g}_{\beta_{k-s}})| \le c \cdot \frac{\lambda}{\sqrt{N}}$$

and the sum (32) is majorized by a constant times

$$cN^{-(k+s)/2}|\tilde{\mathcal{P}}_{k,1,2,s}| \cdot \frac{\lambda}{\sqrt{N}} \cdot N^p. \tag{33}$$

Since $p = (k+s)/2$ by Lemma 15, (33) is dominated by $c/\sqrt{N} \to 0$. ∎

Summing up, we come to

THEOREM 18. *Keeping the notations in Definition 7, suppose that the states φ_γ satisfy conditions (27) and (30) for $\gamma \in [0, \bar{\gamma}]$, $\bar{\gamma} > 0$. Then it holds that*

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{(\varepsilon_1)}}{\sqrt{N}}\cdots\frac{S_N^{(\varepsilon_k)}}{\sqrt{N}}\right) = \lim_{N\to\infty} N^{-k/2} \sum_{1\le s\le k} \sum_{\alpha}{}' \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}), \tag{34}$$

where $\sum'_\alpha$ means that α runs over the non-negligible pair partitions with s singletons.

Remark. The existence of the limit (34) is guaranteed by conditions of the same type as in Corollary 4. Condition (27) is easily verified for the Haagerup states. In that case the negligible partitions are nothing but the crossing ones. Other examples will be considered elsewhere.

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