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PROJECTION OF ANALYTIC SETS AND BERNSTEIN INEQUALITIES

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I. Bernstein inequality and the number of zeroes. We first give two definitions (cf. N. Roytvarf, Y. Yomdin [R-Y]). Δ_R denotes, as usual, the closed disk of radius R, centred at 0.

DEFINITION I.1. Let $R > 0$, $0 < \alpha < 1$ and $K > 0$ be given and let f be holomorphic in a neighborhood of Δ_R. We say that f belongs to the *Bernstein class* $B^1_{R,\alpha,K}$ if

$$\frac{\max\{|f(z)|, z \in \Delta_R\}}{\max\{|f(z)|, z \in \Delta_{\alpha R}\}} \leq K.$$

Remark. The name "Bernstein class" is justified by the fact that, according to one of the classical Bernstein inequalities, any polynomial of degree d belongs to $B^1_{R,\alpha,K}$, $K = (1/\alpha)^d$ for any R and α.

DEFINITION I.2. Let a natural N, $R > 0$ and $C > 0$ be given, and let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be an analytic function in a neighborhood of $0 \in \mathbb{C}$. We say that f belongs to the *Bernstein class* $B^2_{N,R,C}$, if

$$|f_j|R^j \leq C \max\{|f_i|R^i,\, i = 0, \ldots, N\}, \quad j \geq N + 1.$$

The two classes B^1 and B^2 essentially coincide. More precisely, we have the following

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LEMMA I.3. *Let f be an element of $B^2_{N,R,C}$. Then f is analytic in an open disk $\mathring{\Delta}_R$ and for any $R' < R$, $0 < \alpha < 1$ and $K = \left(\frac{1}{\alpha}\right)^N \left[1 + (1-\alpha^N)\frac{\alpha}{1-\alpha} + C\frac{\beta}{1-\beta}\right]$, $\beta = R'/R$, f belongs to $B^1_{R',\alpha,K}$.*

Proof. The convergence of $f(z) = \sum_{k=0}^{\infty} f_k z^k$ on Δ_R is immediate. Let $m = \max\{|f(z)|, z \in \Delta_{\alpha R'}\}$. Then by the Cauchy formula, $|f_i| \le m/(\alpha R')^i$ for any i. In particular, $|f_i|R^i \le m/(\alpha R'/R)^i \le m/(\alpha R'/R)^N$ for $i = 0, \dots, N$. Hence $|f_j|R^j \le Cm/(\alpha R'/R)^N$ for any $j \ge N+1$. Now we can estimate $|f|$ on $\Delta_{R'}$ as follows:

$$\begin{aligned}\max\{|f(z)|, z \in \Delta_{R'}\} &\le \sum_{k=0}^{N} |f_k|{R'}^k + \sum_{k=N+1}^{\infty} |f_k|{R'}^k \\ &\le m\sum_{k=0}^{N}\Big(\frac{1}{\alpha R'}\Big)^k {R'}^k + \frac{Cm}{(\alpha R'/R)^N}\sum_{k=N+1}^{\infty}(R'/R)^k \\ &= m\Big(\frac{1}{\alpha}\Big)^N\Big[1 + (1-\alpha^N)\frac{\alpha}{1-\alpha} + C\frac{\beta}{1-\beta}\Big]. \ \blacksquare\end{aligned}$$

Remark. The constant K in Lemma I.3 can be chosen as $\left(\frac{1}{\alpha}\right)^N \left(1 + C\frac{\alpha\beta}{1-\alpha\beta} + C\frac{\beta}{1-\beta}\right)$ which in some cases gives a better estimate.

Conversely, if f belongs to $B^1_{R,\alpha,K}$, then it belongs to $B^2_{N,R,C}$ with $N = \log_\alpha K$ and C given explicitly through R, α, K (cf. N. Roytvarf, Y. Yomdin [R-Y], Hayman [Ha]).

A relevance of Bernstein classes to our purpose is explained by the following lemma (which is well known in different forms in various fields of complex analysis; we give a version, obtained by M. Waldschmidt [W] in relation to transcendent number theory).

LEMMA I.4. *Let $R > 0$, and $0 < \alpha < 1$ be given and let f be holomorphic in a neighborhood of Δ_R. Then the number of zeroes of f in $\mathring{\Delta}_{\alpha R}$ does not exceed*

$$\frac{\mathrm{Log}\big(\max\{|f(z)|, z \in \Delta_R\}/\max\{|f(z)|, z \in \Delta_{\alpha R}\}\big)}{\mathrm{Log}\big[(1+\alpha^2)/2\alpha\big]}.$$

In other words, for an element f of $B^1_{R,\alpha,K}$,

$$\#\{f^{-1}(0) \cap \Delta_{\alpha R}\} \le \frac{\mathrm{Log}\, K}{\mathrm{Log}\big[(1+\alpha^2)/2\alpha\big]}.$$

Frequently, in the theory of differential equations, we deal with analytic developments $f(z) = \sum_{k=0}^{\infty} f_k z^k$ where f_k is defined inductively by an expression which involves the preceding coefficients. So it can often be shown that f belongs to a certain Bernstein class B^2. Combining the above results, we can estimate the number of zeroes of the functions in B^2 as follows:

PROPOSITION I.5. *Let f be an element of $B^2_{N,R,C}$. Then for any $R'' < R$, the number of zeroes of f in $\mathring{\Delta}_{R''}$ does not exceed*

$$N \cdot \min_{\{\alpha, (R''/R) < \alpha < 1\}} \frac{1 + \mathrm{Log}\big(1 + (1-\alpha^N)\frac{\alpha}{1-\alpha} + C\frac{\gamma}{1-\gamma}\big)/\mathrm{Log}(1/\alpha)}{1 + \mathrm{Log}\big((1+\alpha^2)/2\big)/\mathrm{Log}(1/\alpha)},$$

where $\gamma = R''/\alpha R < 1$.

Proof. For any α, $R''/R < \alpha < 1$, let $R' = R''/\alpha$. Then by Lemma I.3, f belongs to $B^1_{R',\alpha,K}$, with $K = \left(\frac{1}{\alpha}\right)^N \left[1 + (1-\alpha^N)\frac{\alpha}{1-\alpha} + C\frac{\gamma}{1-\gamma}\right]$, where $\gamma = R''/\alpha R = R'/R$. Hence, by Lemma I.4 the number of zeroes of f on $\mathring{\Delta}_{R''} = \mathring{\Delta}_{\alpha R'}$ is bounded by

$$\frac{\operatorname{Log}(1/\alpha)^N\left(1 + (1-\alpha^N)\frac{\alpha}{1-\alpha} + C\frac{\gamma}{1-\gamma}\right)}{\operatorname{Log}\left((1+\alpha^2)/2\alpha\right)} = N \cdot \frac{1 + \operatorname{Log}\left(1 + (1-\alpha^N)\frac{\alpha}{1-\alpha} + C\frac{\gamma}{1-\gamma}\right)/\operatorname{Log}(1/\alpha)}{1 + \operatorname{Log}((1+\alpha^2)/2)/\operatorname{Log}(1/\alpha)}.$$

Since the value of α between (R''/R) and 1 or, equivalently, the value of R', $R > R' > R''$ can be chosen arbitrarily, the proposition follows. ∎

COROLLARY I.6. *Let f be an element of $B^2_{N,R,C}$. Then*

1) *For $R'' = R/4$, the number of zeroes of f on $\mathring{\Delta}_{R''}$ does not exceed $N\log_{5/4}(4+2C)$.*
2) *For $R'' = R/2\max(C,2)$, the number of zeroes of f on $\mathring{\Delta}_{R''}$ is at most $20N$.*
3) *For $R'' = Re^{-(10N+2)}/\max(C,2)$, this number is at most N.*

Proof. To prove 1), take $\alpha = \frac{1}{2}$. Then $\gamma = \frac{1}{2}$ and

$$\#\{f^{-1}(0) \cap \Delta_{R''}\} \le N \cdot \frac{1 + \operatorname{Log}(1 + (1 - \left(\frac{1}{2}\right)^N) + C)/\operatorname{Log} 2}{1 + \operatorname{Log}\frac{5}{8}/\operatorname{Log} 2} \le N\log_{5/4}(4+2C).$$

In 2) we also choose $\alpha = \frac{1}{2}$. Then $\gamma = 1/\max(C,2)$, and we get

$$\#\{f^{-1}(0) \cap \Delta_{R''}\} \le N \cdot \frac{1 + \operatorname{Log}\left(1 + (1 - \left(\frac{1}{2}\right)^N) + 2\right)/\operatorname{Log} 2}{1 + \operatorname{Log}\frac{5}{8}/\operatorname{Log} 2} \le 20N.$$

Finally, for $R'' = Re^{-(10N+2)}/\max(C,2)$, we put $\alpha = e^{-10N}$; then $\gamma = 1/e^2\max(C,2)$, and $\#\{f^{-1}(0) \cap \Delta_{R''}\} \le N \cdot (1 + (2/3N))$. Since the number of zeroes is an integer, this yields $\#\{f^{-1}(0) \cap \Delta_{R''}\} \le N$.

Remark. By taking into account the Bernstein inequality, the last conclusion is strong enough to prove that the number of zeroes of a polynomial of degree d does not exceed d.

II. Projection of analytic sets. We introduce now the algebra $\mathbb{C}(R)$ as follows:

$$f(\boldsymbol{x}, z) = f_{\boldsymbol{x}}(z) = f(x_1, \ldots, x_n; z) = \sum_{k=0}^{\infty} z^k f_k(x_1, \ldots, x_n)$$

belongs to $\mathbb{C}(R)$ if there are (α, β) so that the coefficients $f_k(x_1, \ldots, x_n)$ $(k = 0, 1, \ldots)$ are polynomials in $\boldsymbol{x} = (x_1, \ldots, x_n)$ of degree less than $\alpha k + b$ and if $\sum_{k=0}^{\infty} R^k |f_k| < \infty$. The norm $|f_k|$ of the polynomial f_k is (for instance) the sum of the absolute value of its coefficients.

DEFINITION II.1. We define the *Bautin ideal* I of $f(\boldsymbol{x}, z)$ as the ideal generated by the coefficients $f_k(x_1, \ldots, x_n)$. The *Bautin index* is the minimal integer N so that $f_0(x_1, \ldots, x_n)$, …, $f_N(x_1, \ldots, x_n)$ generate this ideal I.

We fix a total ordering $<$ on $\mathbb{N}^n$ so that:

$$\alpha \in \mathbb{N}^n,\ \beta \in \mathbb{N}^n,\ \beta \neq 0, \text{ then } \alpha < \alpha + \beta;$$
$$\alpha \in \mathbb{N}^n,\ \beta \in \mathbb{N}^n,\ \gamma \in \mathbb{N}^n, \alpha < \gamma \text{ then } \alpha + \beta < \gamma + \beta.$$

There are several possible choices of such an ordering. For instance, we can choose $<$ as follows: Let $C(\alpha) = \sum_{i=1}^n \alpha_i$, $\alpha < \beta$ if $C(\alpha) < C(\beta)$ or if $C(\alpha) = C(\beta)$ and there exists k, $1 \leq k \leq n$, such that $\alpha_j = b_j$ for $j < k$ and $\alpha_k < \beta_k$.

Given a polynomial $f = \sum_{A \in \mathbb{N}^n} f_A x^A$, and the total ordering on $\mathbb{N}^n$, we denote by $\exp f$ the largest exponent A such that $f_A \neq 0$. Let $\exp I = \{A \in \mathbb{N}^n, A = \exp f, f \in I\}$. There is a unique minimal set $E = \{E_1, E_2, \ldots, E_d\}$ such that for every element β of $\exp I$, there is an element ε in E and an element α in $\mathbb{N}^n$ such that $\beta = \varepsilon + \alpha$.

If the ordering of the multi-indices is given, we call a set of elements $\{h_1, \ldots, h_d\}$ of I such that $\exp h_k = E_k$ a *standard basis* (or *Gröbner basis*) of the ideal I.

Let $\rho = \max\{|h_i - x^{E_i}|, i = 1, \ldots, d\}$.

THEOREM II.2. *There exists $C > 0$, depending on f, such that the function $\sum_{k=0}^{\infty} |f_k| z^k$ belongs to $B^2_{N,R/(1+\rho)^{n\alpha},C}$.*

Proof. Let $h_1(x_1, \ldots, x_n)$, $\ldots$, $h_d(x_1, \ldots, x_n)$ be a Gröbner basis of the ideal I. Moreover, since $f_0, \ldots, f_N$ generate I, we have $h_j = \sum_{i=0}^N \phi_i^j f_i$. Let $C_1 = \max_{i,j} |\phi_i^j|$.

From classical estimates on division of polynomials by an ideal, we obtain that there is $C_2 > 0$ such that for any element h of the ideal I,

$$h = \sum_{j=1}^d g_j h_j, \text{ with } |g_j| \leq C_2 |h| (1+\rho)^{n \deg h}.$$

Several generalizations of these crucial estimates have been produced in the setting of analytic coefficients (cf. [Br], [H], [Ga]). Hence,

$$h = \sum_{i=0}^N g_i' f_i, \text{ with } |g_i'| \leq d C_1 C_2 |h| (1+\rho)^{n \deg h}.$$

In particular, for any $j \geq N+1$ we have

$$f_j = \sum_{i=0}^N g_i'^j f_i, \text{ with } |g_i'^j| \leq d C_1 C_2 |f_j| (1+\rho)^{n(\alpha j + \beta)} \leq d C_1 C_2 C_3 ((1+\rho)^{n\alpha}/R)^j,$$

since for an element $f(\boldsymbol{x}, z) = \sum_{k=0}^\infty z^k f_k(x_1, \ldots, x_n)$ of $\mathbb{C}(R)$, there is a constant C_3 such that $|f_j| \leq C_3$. Denote $d C_1 C_2 C_3 (1+\rho)^{n\beta}$ by C_4. This yields

$$|f_j| \leq \sum_{i=0}^N |g_i'^j|\,|f_i| \leq C_4 ((1+\rho)^{n\alpha}/R)^j \sum_{i=0}^N |f_i|.$$

We obtain:

$$|f_j| R' j \leq C_4 (N+1) \max\{|f_i|, i = 0, \ldots, N\} \leq C_5 \max\{|f_i| R'^i, i = 0, \ldots, N\},$$

where $C_5 = C_4 \max(1/R'^N, 1)$, $R' = R/(1+\rho)^{n\alpha}$.

This proves Theorem II.2 with $C = C_5$. ■

THEOREM II.3. *Let f be an element of $\mathbb{C}(R)$. Assume that there exist α, β so that the coefficients $f_k(x_1, \ldots, x_n)$ are homogeneous of degree $\alpha k + \beta$. Then for all $\boldsymbol{x}$, the series $f(\boldsymbol{x}; z) = \sum_{k=0}^{\infty} z^k f_k(x)$ belongs to $B^2_{N, R'/|\boldsymbol{x}|^\alpha, C|\boldsymbol{x}|^\beta}$.*

Write

$$f_j(\boldsymbol{x}) = \sum_{i=0}^{N} g_i'^j(\boldsymbol{x}) f_i(\boldsymbol{x}).$$

Denote by $\mathrm{hom}_i(p(x))$ the homogeneous component of degree i of a polynomial $p(x)$. We obtain

$$f_j(\boldsymbol{x}) = \sum_{i=0}^{N} \mathrm{hom}_{\alpha(j-i)+\beta}(g_i'^j(\boldsymbol{x})) f_i(\boldsymbol{x}).$$

Then we get

$$|\mathrm{hom}_{\alpha(j-i)+\beta}(g_i'^j(x))| \le |g_i'^j|\, |\boldsymbol{x}|^{\alpha(j-i)+b} \le dC_1C_2C_3(1/R')^j |\boldsymbol{x}|^{\alpha(j-i)+\beta},$$

and this yields

$$|f_j(\boldsymbol{x})| \le \sum_{i=0}^{N} dC_1C_2C_3(1/R')^j |\boldsymbol{x}|^{\alpha(j-i)+\beta} |f_i(\boldsymbol{x})| \le C_4 |\boldsymbol{x}|^\beta (|\boldsymbol{x}|^\alpha/R')^j \sum_{i=0}^{N} |f_i(\boldsymbol{x})|/|\boldsymbol{x}|^{\alpha i}.$$

Hence,

$$|f_j(\boldsymbol{x})|(R'/|\boldsymbol{x}|^\alpha)^j \le C|\boldsymbol{x}|^\beta \max\{|f_i(\boldsymbol{x})|(R'/|\boldsymbol{x}|^\alpha)^i, i = 0, \ldots, N\}.$$

THEOREM II.4. *Let f be an element of $\mathbb{C}(R)$, N, C be as above. Let $R'(\boldsymbol{x}) = \frac{1}{4}R'/|\boldsymbol{x}|^\alpha$, $R''(\boldsymbol{x}) = (R/|x|^\alpha)/2\max(C|\boldsymbol{x}|^\beta, 2)$, $R^*(\boldsymbol{x}) = (R/|\boldsymbol{x}|^\alpha)e^{-(10N+2)}/\max(C|\boldsymbol{x}|^\beta, 2)$. Then for any $\boldsymbol{x}$, the function $f_{\boldsymbol{x}}(z)$ can have on the disks $\mathring{\Delta}_{R'(x)}$, $\mathring{\Delta}_{R''(x)}$, $\mathring{\Delta}_{R^*(x)}$, at most $N \log_{5/4}(4 + 2C|\boldsymbol{x}|^\beta)$, $20N$ and N zeroes, respectively.*

In the article [F-Y], we followed a different presentation based on the use of the norm "maximum on a polydisc". This allows to handle more general data (f may have analytic coefficients) but it is necessary to use priviliged neighborhoods.

The Łojasiewicz inequality appears closely related to the subject. We take the opportunity of this Symposium to mention briefly the connection and postpone its developments to further studies.

The Łojasiewicz inequality entails a constant K and an exponent δ so that

$$K\Big(\sum_{k=0}^{\infty} f_k^2(\boldsymbol{x})\Big)^\delta \le \sum_{i=0}^{N} f_i^2(\boldsymbol{x}).$$

It yields the inequality

$$|f_j(\boldsymbol{x})| \le C \max\{|f_i(\boldsymbol{x})|^{1/\delta}, i = 0, \ldots, N\}, \quad j \ge N+1.$$

Going back to the Jensen inequality, we obtain the same type of bound for the number of zeroes (cf. Lemma I.4). The only change is that this bound gets multiplied by $1/\delta$.

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