HOMOTOPY AND GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 45
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

INTRODUCTION

This volume of the Banach Center Publications presents the results obtained by the participants of the workshop "Homotopy and Geometry" held at the Banach Center in June 9–13, 1997. The workshop was organized and sponsored by two institutions, the Stefan Banach International Mathematical Center and the Mathematical Institute of the University of Wrocław.

The use of the soft techniques of topology has a long history as a powerful tool in the world of hard geometry. In particular, in recent years, the subject of symplectic topology has arisen from important and interesting connections between symplectic geometry and algebraic topology. The list of spectacular results in this new area contains many applications of the homotopical approach. For instance, we have the following.

(1) The problem of constructing symplectic non-Kähler manifolds (i.e. the Thurston–Weinstein problem) has been approached through a beautiful mixture of geometry and topology. Constructions usually require the hardest of geometry while the ability to distinguish Kähler from merely symplectic often hinges on topological information. In particular, symplectic (non-toral) nilmanifolds (themselves objects of both geometric and algebraic universes) are found never to be Kähler because their rational homotopy theoretic minimal models are never formal—a quality possessed by all Kähler manifolds. Other conditions, such as the Hard Lefschetz condition on Kähler manifolds, are also homotopical conditions arising from combinations of topology and (complex) geometry and may also be used to tell symplectic from Kähler.

(2) Arnold's conjecture on fixed points of Hamiltonian diffeomorphisms has been a major motivation for the development of symplectic geometry (and symplectic topology). Floer's breakthrough in a general approach to the conjecture was, of course, to use analysis to construct a homology theory which reflected homotopical properties of a symplectic manifold with a certain type of flow. Earlier, Conley and Zehnder had shown that hard analysis could be mixed wonderfully with such homological invariants as cuplength to prove the conjecture for tori.

(3) Group actions on symplectic manifolds have proved important from the very beginning of the study of Hamiltonian systems since Hamiltonian actions allow for a reduction of order in the system. The very condition of being Hamiltonian (for a circle action say) is a homological condition, so perhaps it is not surprising that topology enters the subject in a fundamental way. In particular, Kirwan's theorem that a Hamiltonian

action causes the Serre spectral sequence of the associated Borel fibration to collapse to the E_2 level is a beautiful example of the interplay between group actions, symplectic geometry and topology.

Geometric applications of homotopy theory are often quite powerful and yield new insights into many interesting problems. The idea of the Banach Center workshop of which these Proceedings are a record was to bring together mathematicians working in both the "soft" and "hard" worlds of topology and geometry so that the boundaries between these subjects might be discovered and explored. A focal point for the conference was symplectic topology, but other topics, with sometimes rather indirect and even conjectural links to the central theme, were also discussed.

This volume is organized as follows. Each section is devoted to a single topic which is introduced by an article which serves the dual roles of being an introduction and a survey of the (homotopical) approach to the topic together with being a presentation of the original results of the author. Other papers in a section normally contain the original results of the participants of the workshop. There are five sections:

(1) Homotopy theory, critical points and Arnold's conjecture
(2) Homotopy theory of group actions in symplectic geometry
(3) Rational homotopy theory
(4) Homotopy and singularities
(5) Other topics

Critical points and Arnold's conjecture. The first section of these Proceedings is devoted to areas surrounding the famous conjecture of Arnold that the number of fixed points of an exact (or Hamiltonian) symplectomorphism is at least the minimal number of critical points of functions on the manifold. Since the minimal number of critical points is bounded from below by the Lusternik–Schnirelmann category, and so also by the cuplength of the manifold (in the general case and by the sum of Betti numbers and torsion numbers in the non-degenerate case), the conjecture has often been rephrased in terms of these topological invariants. The opening article by Kaoru Ono introduces the subject and surveys the recent progress in this area. In particular, Ono sketches the basics of Floer homology and his own work with Fukaya extending the construction of Floer to handle the non-degenerate situation. Several related topics involving Lusternik–Schnirelmann (LS) category and homotopical dynamics are also presented here. The paper of Bryden–Zvengrowski really concerns the cohomology algebra of certain Seifert manifolds, but a straightforward application of their calculations is the determination of the LS category of these manifolds via the cuplength estimate mentioned above. As such, these manifolds should be added to the toolkit of mathematicians interested in category. The article of Cornea outlines his approach to understanding certain gradient-like flows in terms of homotopical quantities such as category, cone-length and Spanier–Whitehead duality. Yuli Rudyak describes in his paper certain new invariants related to LS category which have proven tremendously successful in attacking problems old and new. In particular, Rudyak outlines his unique approach to the Arnold conjecture (in its original critical point form) in terms of stable homotopy and these new category invariants.

Homotopy and group actions in symplectic geometry. The second section treats aspects of the homotopy theory of group actions on symplectic manifolds. The opening article, written by John Oprea, is an exposition of the homotopical approach to symplectic circle actions as well as a first step to extending the approach beyond the symplectic world. The boundaries between geometry and homotopy theory are explored in this section by reformulating many notions from symplectic geometry in homotopical terms. While Oprea's paper points out how symplectic circle action results may arise in a wider homotopy theoretic context, the papers of Allday and Hattori show how some symplectic theorems are essentially geometric. Thus the undiscovered country between geometry and topology becomes a bit clearer here. Also included in this section is an article by Hattori on geometric work extending the notion of toric manifold with positive line bundle to the almost complex world.

Rational homotopy and geometry. Since the development of Sullivan's approach in the 1970's, rational homotopy theory has proven to be a powerful tool in geometry. Perhaps this was to be expected because Sullivan based his theory on the construction of certain (rational polynomial) forms akin to the differential forms of de Rham theory. Nevertheless, the methods of rational homotopy have been unexpectedly effective in areas ranging from the closed geodesic problem to the construction of non-Kähler symplectic manifolds. The paper of Greg Lupton serves as an introduction to the whole subject of rational homotopy theory since it deals with many of the tools and techniques used in the subject. For instance, Lupton describes the rational models of fibrations, the important property of formality (possessed by Kähler manifolds) and the interesting class of elliptic spaces as well as some rational numerical invariants (e.g. rational cup-length and the invariant of Graham Toomer) which have proven especially useful in critical point theory and symplectic geometry. In the same framework of Sullivan's approach, the papers of Cordero *et al.* and Fernandez *et al.* also explore the boundaries between homotopy theory, complex geometry and symplectic geometry. Cordero *et al.* build upon the fundamental work of Neisendorfer and Taylor to study the Sullivan theory of the Dolbeault complex of forms for certain nilmanifolds and derive consequences such as degeneration of the associated Fröhlicher spectral sequences. Fernandez *et al.* describe a type of cohomology called *coeffective cohomology* which may be associated to a symplectic (or almost contact) manifold and which provides a new tool in the study of symplectic versus Kähler. Finally, in some sense, we come full circle in the paper of Garvin *et al.* where de Rham's identification of form cohomology with singular cohomology is generalized (following Miller) to any coefficients. Applications of this generalization await.

Homotopy, singularities and characteristic classes. One of the true starting points for the intrusion of topology into geometry was the beautiful Gauss–Bonnet theorem relating total Gauss curvature to the Euler characteristic. Of course, now we see vast generalizations of this theorem in Chern–Weil theory and the subject of characteristic classes. In this fourth section, we see these generalizations in action. Indeed, the paper of Daniel Lehmann outlines a method for studying (non-isolated) singularities on certain singular varieties in terms of generalized Milnor numbers *obtained as obstructions to the* *higher dimensional Gauss–Bonnet theorem* $\chi(V) = c_n(V) \cap [V]$. The paper of Čadek and Vanžura uses characteristic classes in their modern role as obstructions; in this case, to the existence of certain G-structures on bundles. In his paper, Kubarski focuses on a formulation and applications of the characteristic homomorphism for flat bundles (akin to that of Chern–Weil theory) in the context of Lie algebroids (over a foliated manifold say).

Other topics. Just as mathematics often develops far in advance of application, so the discovery of new results in algebraic topology often predates uses in geometry. With this in mind, this section presents several papers of a homotopical flavor with no *immediate* geometric context. For instance, the paper of Arkowitz deals with properties of homology decompositions, the Eckmann–Hilton duals of Postnikov towers. While Postnikov towers have found a place in 'geometry' (e.g. the development of rational homotopy and its applications), it remains to be seen whether homology decompositions have a similar role. The paper of Hubbuck mixes together K-theory and number theory to obtain a vanishing result for the stable Hurewicz homomorphism (the unstable version of which has found a home in the hypotheses of symplectic geometry).

Open problems. The conference spawned many questions and problems relating homotopy and geometry. Many of these are mentioned in individual papers. Just a few of the problems not mentioned in these papers are discussed below.

1. (Y. Rudyak) As mentioned in Rudyak's paper, the strict category weight $\mathrm{swgt}(u)$ is equal to the maximum k such that the cohomology class u is in $\mathrm{Ker}(p_k^* : H^*(X) \to H^*(P_k X))$. Thus, the kernels of the p_k^*'s are important to understanding this new and useful estimator of Lusternik–Schnirelmann category. The question then is: *is it possible to determine* $\mathrm{Ker}(p_k^* : H^*(X) \to H^*(P_k X))$? Of course, it seems that the case $k = 2$ is unreasonably effective—at least in symplectic geometry—so even this kernel would be interesting to determine. There is some hope here since, as mentioned by Rudyak, $P_2(X) = \Sigma\Omega X$.

2. (Y. Rudyak) From Rudyak's paper, we also see the importance of understanding the answer to, *when does a degree 1 map* $f : M \to N$ *imply*

$$\mathrm{cat}(M) \geq \mathrm{cat}(N) \quad \textit{or} \quad \mathrm{cat}(f) = \mathrm{cat}(M)\ ?$$

3. (J. Oprea) The methods of rational homotopy theory have been applied rather successfully to aspects of symplectic geometry and very successfully to Kähler manifolds—in simply connected cases. Rational homotopy may be extended to nilpotent spaces (i.e. spaces whose fundamental group is nilpotent and whose fundamental group acts nilpotently on the higher homotopy groups), so *it is important to know when symplectic* (*or even Kähler*) *manifolds are nilpotent*. Of course, there are many nilmanifolds (which are nilpotent $K(\pi, 1)$'s) which are symplectic (but non-Kähler), so we should avoid talking about these or products of them. Are there other nilpotent non-simply connected symplectic manifolds? Note that, in Bob Gompf's construction of symplectic manifolds

realizing arbitrary finitely presented groups, it is typically the case that π_2 contains a free $\mathbb{Z}\pi_1$-module—ruling out nilpotence.

4. (G. Lupton and J. Oprea) Deligne–Griffiths–Morgan–Sullivan kicked off rational homotopy theory by showing that Kähler manifolds are formal; that is, their rational homotopy types are determined by their rational cohomology algebras. *Do simply connected manifolds share this property? That is, are all simply connected closed symplectic manifolds formal?* There are not so many of these creatures about, so a first step to understanding whether or not this question is meaningful would be to analyze the rational homotopy types of McDuff's blow-ups (the first examples of simply connected non-Kähler symplectic manifolds) and Gompf's simply connected examples.

5. (H. Baues) In a more homotopical vein, given a torus T^{2n}, does there exist a space (or manifold?) M^{2n} and a map $f : T^{2n} \to M$ such that

1. M has cells in even degrees only,
2. $f_* : H_*(T^{2n}) \to H_*(M)$ is an isomorphism in even degrees?

Of course, the first step might be to model things rationally and then try to lift to the integral world, but even this is not so easy.

Thanks. The editors express their sincere thanks to all the participants of the workshop and especially to their co-organizers: Yves Felix, Daniel Lehmann and Peter Zvengrowski, whose help in bringing the conference to fruition cannot be overestimated. Also, we are indebted to the Banach Center and to the Scientific Council and its President, Professor Friedrich Hirzebruch, for their support of this workshop. We hope that this volume will stimulate new and further interest in the rapidly developing interactions between topology and geometry. Also, we would like to say a special thanks to all of the referees for the papers in this volume. Refereeing is a usually thankless (literally!) task, but every referee should know that his or her comments served to improve every single paper here.

Finally, we would like to thank the technical staff of the Banach Center, especially Mrs. Grażyna Pieścik-Bojarska, for their continual, friendly and essential help during the conference.

May 1998

John Oprea
Aleksy Tralle