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ALMOST COMPLEX TORIC MANIFOLDS AND POSITIVE LINE BUNDLES

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1. Introduction. Masuda [M] developed the theory of unitary toric manifolds which generalized the theory of toric varieties and the theory of Hamiltonian toral manifolds (toric manifolds) due to Delzant [D] in some direction. There are works of Karshon and Tolman [KT] and of Grossberg and Karshon [GK] in a similar direction. An advantage of Masuda's theory is the introduction of the notion of multi-fan attached to unitary toric manifolds. The multi-fan is essentially a simplicial complex with some extra data and equipped with a map from the set of its vertices to the second homology group $H_2(BT;\mathbf{Z})$ of the classifying space BT of the torus T acting on the manifold.

One of the main results of [M] is the multiplicity formula which relates the index of the Dirac operator twisted by a line bundle L acted on by the torus T to some data coming from the moment map of the bundle. Similar formulas are also in [KT] and [GK]. The moment map is a T-invariant map from M into the second cohomology with real coefficients $H^2(BT,\mathbf{R})$. Irreducible representations of a torus T are one dimensional. They form an abelian multiplicative group $\mathrm{Hom}(T,S^1)$. We identify $\mathrm{Hom}(T,S^1)$ with $H^2(BT,\mathbf{Z})$. So, if χ is a virtual representation of T then it can be written as

$$\chi=\sum_{u\in H^2(BT,\mathbf{Z})} m(u)\chi^u,\ \ m(u)\in\mathbf{Z},$$

where χ^u is the irreducible representation corresponding to u. The multiplicity formula identifies the integer $m(u)$ with the value $d'_L(u)$ of a degree function d'_L defined on $H^2(BT,\mathbf{R})$ minus a union of certain affine hyperplanes when χ is the index of the Dirac operator D_L twisted by the line bundle L.

In this paper we shall deal with almost complex toric manifolds satisfying some mild conditions. Given a T-line bundle we define a piecewise linear map Ψ_L from the realization of the first barycentric subdivision of the simplicial complex attached to the multi-fan

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into $H^2(BT, \mathbf{R})$ such that its image is contained in the union of hyperplanes described above. Theorem 3.5, work done jointly with Masuda, states that the winding number $\bar{d}_L(u)$ of the map Ψ_L around $u \in H^2(BT, \mathbf{R})$ coincides with the value $d_L(u)$ of the degree function d_L which is intimately related to d'_L. The same sort of statement (Theorem 3.11) holds also for the degree function of [KT]. This allows us to express the multiplicity $m(u)$ purely in terms of the multi-fan and algebraic data of the line bundle L.

We shall also define a notion of positiveness of line bundles in such a way that analogues of the Nakai criterion and the Kodaira vanishing theorem hold (Corollary 4.8 and Theorem 4.21; also see Theorem 4.19). Furthermore, it is shown (Theorem 5.4) that if the Todd genus of the manifold is equal to 1, then the above definition of positiveness is quite parallel to the classical convexity criterion of ample line bundles in the theory of toric varieties, cf. e.g. [F],[O].

The organization of the paper is as follows. In Section 2 we review Masuda's theory in a way suitable to our purpose. Section 3 is devoted to the results surrounding the map Ψ. In Section 4 we introduce the notion of positiveness of line bundles and give proof of the analogues of the Nakai criterion and the Kodaira vanishing theorem. In proving the Nakai criterion a formula in Theorem 4.2 which expresses the number $c_1(L)^n[M]$ in terms of the degree function $\bar{d}_L$ is crucial. For the proof of Theorem 4.2 we use a combinatorial formula (Lemma 4.5) concerning the volume of rational polytopes. The formula is a simple one but seems to be new. In the last section, Section 5, the case where the Todd genus equals 1 will be dealt with.

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2. Almost complex toric manifolds and multi-fan. A closed, connected 2n-dimensional almost complex manifold M acted on by a torus T will be called an almost complex toric manifold if the following conditions are satisfied.

1. The action preserves the almost complex structure.
2. If T_0 denotes the trivializer of the action, then $\dim T/T_0 = n$.
3. The fixed point set of the action (which we denote by M^T) is not empty.

We set $\overline{T} = T/T_0$. $\overline{T}$ acts effectively on M, and M^T is an isolated set.

According to [M] a closed, connected codimension 2 submanifold M_i of M will be called a characteristic submanifold if it is a fixed point set component of a certain subcircle $\overline{S}_i$ of $\overline{T}$ and $M_i \cap M^{\overline{T}} \neq \emptyset$. M_i inherits the almost complex structure from M. Let Σ_M^0 denote the set of all indices i of characteristic submanifolds M_i. We set

$$\Sigma_M^{k-1} = \{I = \{i_1, i_2, \dots i_k\}; M_I = M_{i_1} \cap \cdots \cap M_{i_k} \neq \emptyset, i_\nu \in \Sigma_M^0\}.$$

Then Σ_M^0, $\Sigma_M^1, \dots, \Sigma_M^{n-1}$ form a simplicial set Σ_M (in [M] this simplicial set was denoted by Γ_M). Note that all M_I are also almost complex toric manifolds. In the sequel we shall make the following assumption.

$$\text{All } M_I \text{ are connected and } M_I \cap M^{\overline{T}} \neq \emptyset. \tag{2.1}$$

The assumption implies in particular M_I is a point of M^T for any $(n-1)$-simplex $I \in \Sigma_M^{n-1}$.

Let ν_i be the normal bundle of M_i in M. It is a complex line bundle. Denoting by $f_i : M_i \to M$ the inclusion map we define $\xi_i \in H^2_{\overline{T}}(M;\mathbf{Z})$ by $\xi_i = f_{i*}(1)$ where $f_{i*} : H^0_{\overline{T}}(M;\mathbf{Z}) \to H^2_{\overline{T}}(M;\mathbf{Z})$ is the Gysin homomorphism of f_i. If $p \in M_i \cap M^T$, then $\nu_i|p$ is an irreducible $\overline{T}$ module, that is, $\nu_i|p \in \mathrm{Hom}(\overline{T}, S^1)$. The restriction of $\nu_i|p$ to $\overline{S}_i$ does not depend on the choice of p in $M_i \cap M^T$.

$\mathrm{Hom}(\overline{T}, S^1)$ is an abelian multiplicative group. We shall identify it with $H^2(B\overline{T};\mathbf{Z})$ by the isomorphism $\mathrm{Hom}(\overline{T}, S^1) \to H^2(B\overline{T};\mathbf{Z})$ given by $\alpha \mapsto c_1^{\overline{T}}(\alpha)$, where $c_1^{\overline{T}}$ is the equivariant first Chern class. The $\overline{T}$-module corresponding to $u \in H^2(B\overline{T};\mathbf{Z})$ is denoted by χ^u. For example, $\nu_i|p = \chi^{\xi_i|p}$.

Lemma 2.1 ([M], Lemma 1.3). *Take $I \in \Sigma_M^{n-1}$ and $p \in M_I$. Then the set $\{\xi_i|p, i \in I\}$ forms a basis of $H^2(B\overline{T};\mathbf{Z})$. In particular, $\xi_i|p$ is a primitive element.*

In a similar way there is a standard isomorphism between $\mathrm{Hom}(S^1, \overline{T})$ and $H_2(B\overline{T};\mathbf{Z})$. The embedding $\overline{S}_i \hookrightarrow \overline{T}$ determines a primitive element of $H_2(B\overline{T};\mathbf{Z})$ up to sign and hence a primitive element $v_i \in H_2(B\overline{T};\mathbf{Z})$ up to sign. The sign will be determined by requiring

$$< \xi_i|p,\ v_i >= 1$$

where $< \ >$ is the coupling between cohomology and homology. It follows easily that $\{v_i; i \in I\}$ is the dual basis of $\{\xi_i|p, i \in I\}$. The following lemma will play an important role in the sequel.

Lemma 2.2 ([M], Lemma 1.5). *There is an identity*

$$u = \sum_{i \in \Sigma_M^0} < u, v_i > \xi_i \in \hat{H}^2_{\overline{T}}(M;\mathbf{R})$$

which holds for any $u \in H^2(B\overline{T};\mathbf{R})$. Here $\hat{H}^2_{\overline{T}}(M;\mathbf{R})$ is the degree 2 part of $H^*_{\overline{T}}(M;\mathbf{R})/S$-torsion where S is the multiplicative set generated by non-zero elements in $H^2(B\overline{T};\mathbf{R})$.

The simplicial set Σ_M is equipped with a projection map $\pi : \Sigma_M^0 \to H_2(B\overline{T};\mathbf{Z})$ defined by $\pi(i) = v_i$. It induces a piecewise affine map $\pi : |\Sigma_M| \to H_2(B\overline{T};\mathbf{R})$, where $|\Sigma_M|$ is the realization of Σ_M. We shall denote by s_I the realization of $I \in \Sigma_M$ in $|\Sigma_M|$. For each I, π maps s_I injectively on the affine simplex s'_I in $H_2(B\overline{T};\mathbf{R})$ spanned by $\{v_i; i \in I\}$. Once an orientation of $H_2(B\overline{T};\mathbf{R})$ is fixed, each $(n-1)$-simplex s'_I $(I \in \Sigma_M^{n-1})$ will be given the orientation o_I defined in the following way. Fix $i \in I$. The vector v_i intersects s'_I transversally. Requiring the positive orientation of the vector v_i followed by o_I should coincide with the given orientation of $H_2(B\overline{T};\mathbf{R})$ determines o_I. This does not depend on the choice of $i \in I$.

Lemma 2.3. *$|\Sigma_M|$ is a closed pseudo-manifold. This means that every simplex is contained in some $(n-1)$-dimensional simplex, and, for each $J \in \Sigma_M^{n-2}$, s_J is the face of precisely two $(n-1)$-simplices s_I and $s_{I'}$. Moreover, if an orientation of $H_2(B\overline{T};\mathbf{R})$ is fixed and each $(n-1)$-simplex s_I is oriented so that $\pi|s_I : s_I \to s'_I$ preserves the orientation, then $\sum s_I$ is the fundamental class of $|\Sigma_M|$. This means that, if $|\Sigma_M| = \bigcup_\nu |\Sigma_M|_\nu$ is the decomposition into connected component of $|\Sigma_M|$, then $\sum_{s_I \subset |\Sigma_M|_\nu} s_I$ generates $H_{n-1}(|\Sigma_M|_\nu;\mathbf{Z})$ for each ν.*

P r o o f. Take $J \in \Sigma_M^{n-2}$. By virtue of (2.1) M_J is a connected almost complex submanifold of dimension 2 on which the torus T acts non-trivially so that it is complex projective line and has precisely two fixed points M_I and $M_{I'}$. This means that I and I' are only simplices to which J is incident, namely s_J is the face of s_I and $s_{I'}$. Moreover Masuda showed ([M], Lemma 4.4) that s_I and $s_{I'}$ lie on different sides of $s_I \cap s_{I'}$. This implies that $\pi|s_I \cup s_{I'}$ is injective and hence the orientability of $|\Sigma_M|$. ∎

LEMMA 2.4. *The degree of* $\pi : |\Sigma_M| \to H_2(B\overline{T}; \mathbf{R}) \setminus \{0\}$ *is equal to* $T[M]$, *the Todd genus of* M.

This is essentially a restatement of [M], Theorem 4.2 paraphrased by using the projection π.

LEMMA 2.5 ([M], Lemma 3.2). *The equivariant first Chern class* $c_1^{\overline{T}}(L)$ *of a complex* $\overline{T}$*-line bundle* L *over* M *can be written in the form*

$$c_1^{\overline{T}}(L) = \sum c_i \xi_i \in \hat{H}^2_{\overline{T}}(M; \mathbf{Z}).$$

NOTE. [M], Lemma 3.2 shows also that every element in $\hat{H}^2_{\overline{T}}(M; \mathbf{Z})$ is of the form $c_1^{\overline{T}}(L)$. Similarly every element of $\hat{H}^2_{\overline{T}}(M; \mathbf{R})$ can be written in the form $\sum c_i \xi_i$ with $c_i \in \mathbf{R}$.

For a complex $\overline{T}$-line bundle L with $c_1^{\overline{T}}(L) = \sum c_i \xi_i \in \hat{H}^2_{\overline{T}}(M; \mathbf{Z})$, we define the affine hyperplane F_i in $H^2(B\overline{T}; \mathbf{R})$ by

$$F_i = \{u \in H^2(B\overline{T}; \mathbf{R}); < u, v_i >= c_i\}.$$

We set $F_I = \bigcap_{i \in I} F_i$ for $I \in \Sigma_M^{k-1}$. F_I is a point for $I \in \Sigma_M^{n-1}$.

The moment map Φ_L of L is a $\overline{T}$-invariant map $\Phi_L : M \to H^2(B\overline{T}; \mathbf{R})$ uniquely determined by the complex $\overline{T}$-line bundle L. It has the following properties.

LEMMA 2.6 ([M], Lemma 6.5). $\Phi_L(M_I) \subset F_I$ *for any* $I \in \Sigma_M^{k-1}$.

We shall add the following assumption:

All isotropy subgroups of the T *action are subtori*, *and each fixed point set component of subtori contains a point in* M^T.

With this assumption the quotient space M/T becomes a compact, connected orientable manifold of dimension n with boundary. The boundary $\partial(M/T)$ is $\bigcup M_i\,/T$. Since the moment map Φ_L is T-invariant it factors through the map $\bar{\Phi}_L : M/T \to H^2(B\overline{T}; \mathbf{R})$. Using this map $\bar{\Phi}_L$, the degree function

$$d_L : H^2(B\overline{T}; \mathbf{R}) \setminus \bigcup F_i \to \mathbf{Z}$$

is defined as follows. Choose an orientation $o(\overline{T})$ of the torus $\overline{T}$, and define the orientation $o(M/T)$ of M/T by requiring that $o(\overline{T})$ followed by $o(M/T)$ should coincide with $(-1)^{n(n-1)/2}$ times that of M as an almost complex manifold. The orientation $o(\overline{T})$ also determines that of $H^2(B\overline{T}; \mathbf{R})$. Take $u \in H^2(B\overline{T}; \mathbf{R}) \setminus \bigcup F_i$. Then $d_L(u)$ is the degree of

$$H_n(M/T,\, \partial(M/T); \mathbf{Z}) \to H_n(H^2(B\overline{T}; \mathbf{R}),\, H^2(B\overline{T}; \mathbf{R}) \setminus \{u\}; \mathbf{Z}).$$

This definition does not depend on the choice of $o(\overline{T})$.

The function d_L is locally constant. There is a transition formula for the values of d_L when one moves from a component to another. Components of $H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F_i$ will be called chambers. Two chambers W_α and W_β are called adjacent if $\overline{W}_\alpha \cap \overline{W}_\beta$ has dimension $n-1$. In this case, let F_i be such that $\overline{W}_\alpha \cap \overline{W}_\beta \subset F_i$. Then $\overline{W}_\alpha \cap \overline{W}_\beta$ is the closure of a component $W_{\alpha\beta}$ of $F_i \setminus \bigcup_{F_i \neq F_j} F_j$. $W_{\alpha\beta}$ will be called a wall between W_α and W_β. Note that there may be F_j with $j \neq i, j \in \Sigma_M^0$ but $F_j = F_i$.

The transition formula is stated in the following way.

Lemma 2.7 ([KT], Remark 6.5; [M], Lemma 6.9). *Let W_α and W_β be adjacent chambers. Take points $u_\alpha \in W_\alpha$ and $u_\beta \in W_\beta$ such that the segment $\overline{u_\alpha u_\beta}$ from u_α to u_β crosses the wall $W_{\alpha\beta}$ transversally. Then*

$$d_L(u_\alpha) = d_L(u_\beta) + \sum_{F_i \supset W_{\alpha\beta}} \operatorname{sign} < u_\beta - u_\alpha, v_i > d_{L|M_i}(u_{\alpha\beta i}),$$

where $u_{\alpha\beta i} = \overline{u_\alpha u_\beta} \cap F_i$.

Here $d_{L|M_i}$ has to be understood as follows. Let $\overline{S}_i$ be the subcircle which stabilizes points in M_i as before. Set $\overline{T}_i = \overline{T}/\overline{S}_i$. Then $\overline{T}_i$ acts effectively on M_i. Take a point $p \in M_i \cap M^T$ and put $\gamma = c_i \xi_i | p \in \operatorname{Hom}(\overline{T}, S^1)$. The restriction of γ to $\operatorname{Hom}(\overline{S}_i, S^1)$ does not depend on the choice of p in $M_i \cap M^T$, and $\overline{S}_i$ acts trivially on $L\chi^{-\gamma}|M_i$. Hence $L\chi^{-\gamma}|M_i$ can be regarded as a $\overline{T}_i$-line bundle, and $d_{L\chi^{-\gamma}|M_i}$ is defined. If u lies in $F_i \setminus \bigcup_{j \neq i} F_j$ we define

$$d_{L|M_i}(u) = d_{L\chi^{-\gamma}|M_i}(u - \gamma).$$

Note that $u - \gamma$ is in $H^2(B\overline{T}_i;\mathbf{R})$. It is easy to show that $d_{L|M_i}(u)$ is well-defined independently of the choice of p.

Lemma 2.8 ([M], Theorem 3.1). *Let $K = \bigwedge^n T^*M$ be the canonical line bundle of M. The equivariant Chern class of K considered as an element of $\hat{H}^2_{\overline{T}}(M;\mathbf{Z})$ is given by*

$$c_1^{\overline{T}}(L) = -\sum \xi_i \in \hat{H}^2_{\overline{T}}(M;\mathbf{Z}).$$

Define $\Phi'_L : M \to H^2(B\overline{T};\mathbf{R})$ by

$$\Phi'_L = \Phi_L - \frac{1}{2}\Phi_K.$$

For each $i \in \Sigma_M^0$ the affine hyperplane F'_i is defined by

$$F'_i = \{u \in H^2(B\overline{T};\mathbf{R}); < u, v_i >= c_i + 1/2\}.$$

From 2.6 and 2.8 it follows that $\Phi'_L(M_i)$ is contained in F'_i. The degree function

$$d'_L : H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F'_i \to \mathbf{Z}$$

is defined by using Φ'_L in a similar way as d_L. Note that $H^2(B\overline{T};\mathbf{Z})$ is contained in $H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F'_i$.

We are now in a position to state the main result of [M]. Once a $\overline{T}$-invariant metric on M and a $\overline{T}$-invariant $U(1)$-connection of L are given, the Dirac operator D_L of the almost complex manifold M twisted by the line bundle L is defined. Its index, $\operatorname{ind} D_L$, is a $\overline{T}$-module. In the topological context it is expressed as the image $\pi_*(L)$ of L by the

Gysin homomorphism $\pi_* : K_{\overline{T}}(M) \to K_{\overline{T}}(pt) = R(\overline{T})$, where $R(\overline{T})$ is the character ring of $\overline{T}$. It is identified with the group ring of $\mathrm{Hom}(\overline{T}, S^1)$ over $\mathbf{Z}$.

THEOREM 2.9 ([M], Theorem 7.2; [KT], Theorem 2). *If we write* $\operatorname{ind} D_L$ *as*

$$\operatorname{ind} D_L = \sum_{u \in H^2(B\overline{T};\mathbf{Z})} m(u)\chi^u,$$

then $m(u) = d'_L(u)$.

This finishes the review of [M].

3. Map Ψ. As we saw in the previous section the degree function d_L was defined by using the moment map of L. It is desirable to explain it by using only combinatorial data of the simplicial set Σ_M and the numbers c_i associated with $i \in \Sigma_M^0$ which describe the T-line bundle L. The aim of this section is to give such an explanation. This is a joint work with Masuda. The results can be extended to cover unitary toric manifolds. So we will only give statement of results and sketch of proof here leaving the details elsewhere.

Let Σ'_M be the first barycentric subdivision of Σ_M and $S_M = |\Sigma'_M|$ the realization of Σ'_M. The barycenter of $I \in \Sigma_M^{k-1}$ is denoted by b_I. These barycenters form the set of vertices of Σ'_M. A simplex of Σ'_M is of the form

$$(b_{I_1}, b_{I_2}, \ldots, b_{I_l}) \text{ with } I_1 \subset I_2 \subset \ldots \subset I_l.$$

The realization of $(b_{I_1}, \ldots, b_{I_l})$ in S_M will be denoted by $|b_{I_1}, \ldots, b_{I_l}|$. For each $I \in \Sigma_M^{k-1}$ we set

$$\sigma_I = \bigcup_{I_1 = I} |b_{I_1}, \ldots, b_{I_l}| \subset S_M$$

It is called the dual cell of $I \in \Sigma_M^{k-1}$. When $I = \{i\} \in \Sigma_M^0$ we simply write σ_i for σ_I. We see that $\sigma_I \subset \sigma_J$ if $I \supset J$. Also, if $I \cap I' = \emptyset$ and $I \cup I'$ is a simplex of Σ_M, then $\sigma_I \cap \sigma_{I'} = \sigma_{I \cup I'}$. The set of all dual cells $\{\sigma_I\}$ stratifies the complex S_M. In particular

$$S_M = \bigcup_{i \in \Sigma_M^0} \sigma_i.$$

Let $Lk_{\Sigma_M} I$ be the link of I in S_M for $I \in \Sigma_M^{k-1}$. It is a simplicial set whose vertices are those $j \in \Sigma_M^0$ such that $j \notin I$ and $\{j\} \cup I \in \Sigma_M^k$, and whose simplices are those $J \in \Sigma_M^{l-1}$ such that $I \cap J = \emptyset$ and $I \cup J \in \Sigma_M^{k+l-1}$.

LEMMA 3.1. *The boundary $\partial\sigma_I$ of the dual cell σ_I is the realization of a simplicial set isomorphic to the first barycentric subdivision $Lk'_{\Sigma_M} I$ of $Lk_{\Sigma_M} I$.*

Proof. The boundary is the realization of a simplicial set $\Sigma(I)$ whose simplices are those $(b_{I_1}, \ldots, b_{I_l})$ with $I \subset I_1$ but $I \neq I_1$ and $I_1 \subset \cdots \subset I_l$. The correspondence which sends each simplex $(b_{J_1}, \ldots, b_{J_l})$ of $Lk'_{\Sigma_M} I$ into $(b_{I \cup J_1}, \ldots, b_{I \cup J_l})$ is an isomorphism of simplicial sets between $Lk'_{\Sigma_M} I$ and $\Sigma(I)$. ■

LEMMA 3.2. *The simplicial set $Lk_{\Sigma_M} I$ is isomorphic to Σ_{M_I} for any $I \in \Sigma_M^{k-1}$.*

Proof. Take $i \in Lk_M^0 I$. Then $M_{I,i} = M_I \cap M_i$ is a characteristic submanifold of M_I by virtue of the assumption (2.1). The vertex map from $Lk_M^0 I$ to $\Sigma_{M_I}^0$ which sends i into (I, i) gives the desired isomorphism. ■

As an immediate corollary of Lemmas 3.1 and 3.2 we obtain

COROLLARY 3.3. *$\partial\sigma_I$ is homeomorphic to S_{M_I}.*

Given a collection $\hat{c} = \{c_i\}$ indexed by Σ_M^0 we define a map

$$\Psi_{\hat{c}} : S_M \to H^2(B\overline{T};\mathbf{R})$$

in the following way. It will be affine on each simplex $|b_{I_1}, \dots, b_{I_l}|$ of S_M. Hence it is sufficient to assign a value for each vertex b_I. We do this by descending induction on the dimension $k-1$ of $I \in \Sigma_M^{k-1}$. When $k = n$, $\Psi_{\hat{c}}(b_I)$ is determined by the equation

$$< \Psi_{\hat{c}}(b_I), v_i >= c_i \text{ for } i \in I.$$

Note that $\{v_i; i \in I\}$ is a basis of $H_2(B\overline{T};\mathbf{Z})$ by Lemma 2.1. For $I \in \Sigma_M^{k-1}$ with $0 < k < n$ we set $C_I = \{J \in \Sigma_M^k; I \subset J\}$ and define

$$\Psi_{\hat{c}}(b_I) = \frac{1}{\#C_I} \sum_{J \in C_I} \Psi_{\hat{c}}(b_J),$$

where $\#C_I$ is the cardinality of C_I. The affine hyperplanes F_i and F_I are defined in a similar manner as in Section 2. As a direct consequence of the definition we have:

LEMMA 3.4. *$\Psi_{\hat{c}}(\sigma_I) \subset F_I$ for any $I \in \Sigma_M^{k-1}$. In particular $\Psi_{\hat{c}}(S_M) \subset \bigcup F_i$.*

Let u be a point in $H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F_i$. We define $\bar{d}_{\hat{c}}(u) \in \mathbf{Z}$ as the degree of the homomorphism

$$\Psi_{\hat{c}*} : H_{n-1}(S_M;\mathbf{Z}) \to H_{n-1}(H^2(B\overline{T};\mathbf{R}) \setminus \{u\};\mathbf{Z}).$$

Note that a preferred orientation of $H_2(B\overline{T};\mathbf{R})$ determines those of S_M and $H^2(B\overline{T};\mathbf{R})$ simultaneously. Thus $\bar{d}_{\hat{c}}(u)$ is defined independently of the choice of orientations of $H_2(B\overline{T};\mathbf{R})$. When $\hat{c}$ comes from a T-line bundle L, i.e. when

$$c_1^{\overline{T}}(L) = \sum c_i\xi_i \in \hat{H}^2_{\overline{T}}(M;\mathbf{Z})$$

we write $\bar{d}_L$ for $\bar{d}_{\hat{c}}$.

THEOREM 3.5. *The function $\bar{d}_L : H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F_i \to \mathbf{Z}$ coincides with d_L.*

Before proceeding to the proof we shall make some comments concerning the function $\bar{d}_{\hat{c}}$. At this point and hereafter we shall identify $\hat{c} = \{c_i\}$ with the cohomology class $\hat{c} = \sum_i c_i\xi_i \in \hat{H}^2_{\overline{T}}(M;\mathbf{R})$.

Given an element $\gamma \in H^2(B\overline{T};\mathbf{R})$ we put

$$\hat{c}^\gamma = \{c_i'\} \text{ with } c_i' = c_i - < \gamma, v_i >$$

and

$$F_i^\gamma = \{u; < u, v_i >= c_i'\}.$$

for each i. The translation by $-\gamma$ sends the hyperplane F_i to F_i^γ.

ASSERTION 3.6. *If we regard $\hat{c} = \sum c_i\xi_i$ and $\hat{c}^\gamma = \sum c_i'\xi_i$ as elements in $\hat{H}^2_{\overline{T}}(M;\mathbf{R})$, then $\hat{c}^\gamma = \hat{c} - \gamma$. If u is in $H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F_i$, then $u - \gamma$ is in $H^2(B\overline{T};\mathbf{R}) \setminus \bigcup F_i^\gamma$, and*

$$\bar{d}_{\hat{c}}(u) = \bar{d}_{\hat{c}^\gamma}(u - \gamma).$$

P r o o f. By Lemma 2.2, $\gamma = \sum < \gamma, v_i > \xi_i$ as elements in $\hat{H}^2_T(M;\mathbf{R})$. Hence

$$\hat{c} - \gamma = \sum (c_i - < \gamma, v_i >)\xi_i = \hat{c}^\gamma.$$

Then it is easy to see that that $\Psi_{\hat{c}^\gamma}$ is the composition of $\Psi_{\hat{c}}$ and the translation by $-\gamma$. The identity above for the degree function $\bar{d}_{\hat{c}}$ follows readily from this observation. ∎

Take $i \in \Sigma^0_M$ and take a vector γ in F_i. Then F_i^γ is identified with $H^2(B\overline{T}_i;\mathbf{R}) \subset H^2(B\overline{T};\mathbf{R})$. We set

$$\hat{c}^{\gamma_i}|M_i = \{c'_j;\ j \in \Sigma_{M_i}\},$$

and define $\Psi_{\hat{c}^{\gamma_i}|M_i} : S_{M_i} \to H^2(B\overline{T}_i;\mathbf{R})$ as before. From this the degree function

$$\bar{d}_{\hat{c}^{\gamma_i}|M_i} : H^2(B\overline{T}_i;\mathbf{R}) \setminus \bigcup_{j\in\Sigma_{M_i}} H^2(B\overline{T}_i;\mathbf{R}) \cap F_j^\gamma \to \mathbf{Z}$$

is induced as before.

ASSERTION 3.7. *If we regard $\hat{c}$ as an element of $\hat{H}^2_T(M;\mathbf{R})$ then $\hat{c}^{\gamma_i}|M_i$ is nothing but the restriction of $\hat{c}^\gamma$ to M_i. In particular, in case $\hat{c}$ comes from a T-line bundle L and γ lies in $H^2(B\overline{T};\mathbf{Z}) \cap F_i$, $\hat{c}^{\gamma_i}|M_i$ coincides with $\bar{d}_{L\chi^{-\gamma}|M_i}$.*

P r o o f. We see that $\xi_j|M_i = 0$ if $M_i \cap M_j = \emptyset$ by definition of ξ_j. Hence

$$\hat{c}|M_i = \sum_{j\in Lk^0_{\Sigma_M}\{i\}} c_j\xi_j|M_i + c_i\xi_i|M_i.$$

Moreover $\xi_j|M_i$ belongs to $\hat{H}^2_{T_i}(M_i;\mathbf{Z}) \subset \hat{H}^2_T(M_i;\mathbf{Z})$ because $M_i \cap M_j$ is a characteristic submanifold of M_i for $j \in Lk^0_{\Sigma_M}\{i\}$ under the assumption (2.1).

On the other hand, by Lemma 2.2

$$\gamma = \sum_{j\in Lk^0_{\Sigma_M}\{i\}} < \gamma, v_j > \xi_j|M_i + < \gamma, v_i > \xi_i|M_i.$$

Since $< \gamma, v_i >= c_i$ we obtain

$$\hat{c}^{\gamma_i}|M_i = \hat{c}|M_i - \gamma = \sum_{j\in Lk^0_{\Sigma_M}\{i\}} (c_j - < \gamma, v_j >)\xi_j|M_i = \sum_{j\in Lk^0_{\Sigma_M}\{i\}} c'_j\xi_j|M_i. \ ∎$$

Here is an analogue of Lemma 2.7:

LEMMA 3.8. *Let W_α and W_β be adjacent chambers and u_α and u_β be such points that the segment $\overline{u_\alpha u_\beta}$ crosses the wall $W_{\alpha\beta}$ transversally. Then*

$$\bar{d}_{\hat{c}}(u_\alpha) = \bar{d}_{\hat{c}}(u_\beta) + \sum_{F_i\supset W_{\alpha\beta}} sign < u_\beta - u_\alpha, v_i > \bar{d}_{\hat{c}^{\gamma_i}|M_i}(u_{\alpha\beta i} - \gamma_i)$$

where γ_i is any vector in F_i.

If Lemma 3.8 is admitted for a moment, then Theorem 3.5 can be deduced as follows. Lemma 2.7, Lemma 3.8 and Assertion 3.7 show that d_L and $\bar{d}_L$ satisfy the same transition formula. Moreover, if $u \in H^2(B\overline{T};\mathbf{R})$ sits far away, then $d_L(u) = 0 = \bar{d}_L(u)$. Therefore d_L and $\bar{d}_L$ coincide everywhere.

Lemma 3.8 itself follows from

LEMMA 3.9. *Let W be a chamber and $u \in W$. Let r be a generic ray starting from u with direction vector β, i.e. $r = \{u + \beta t; t \geq 0\}$. Then*

$$\bar{d}_{\hat{c}}(u) = \sum_{i:F_i \cap r \neq \emptyset} sign < \beta, v_i > \bar{d}_{\hat{c}^{\gamma_i}|M_i}(F_i \cap r \ - \gamma_i).$$

We here state the most crucial fact for proving Lemma 3.9. We identify Σ_{M_i} with $Lk_{\Sigma_M}\{i\}$ by Lemma 3.2, and then identify S_{M_i} with $\partial\sigma_i$ by Lemma 3.1. This identification extends to an identification of the cone CS_{M_i} over S_{M_i} with σ_i, the vertex o of the cone being identified with $b_i \in \sigma_i$. Let $\gamma_i \in F_i$. Then $\Psi_{\hat{c}^{\gamma_i}|M_i}$ is extended to a map $\tilde{\Psi}_{\hat{c}^{\gamma_i}|M_i} : \sigma_i \to H^2(B\overline{T}_i; \mathbf{R})$ sending o to $\Psi_{\hat{c}}(b_i) - \gamma_i$. With this understood, we have

ASSERTION 3.10. $\tilde{\Psi}_{\hat{c}^{\gamma_i}|M_i}(x) + \gamma_i = \Psi_{\hat{c}}(x)$ *for* $x \in \sigma_i$.

The rest of details are rather routine and will be given elsewhere.

Finally we shall indicate the relation between the degree function of [KT] and $\bar{d}_{\hat{c}}$. Note that the Lie algebra $L(\overline{T})$ of $\overline{T}$ is canonically identified with $H_2(B\overline{T}; \mathbf{R})$ and the dual $L(\overline{T})^*$ with $H^2(B\overline{T}; \mathbf{R})$. Let ω be a $\overline{T}$-invariant closed 2-form over M which admits a moment map, i.e. a map $\Phi_\omega : M \to L(\overline{T})^* = H^2(B\overline{T}; \mathbf{R})$ satisfying

$$d < \Phi_\omega, v >= -i(\underline{v})\omega$$

where $\underline{v}$ is the vector field (infinitesimal action) associated with $v \in L(\overline{T})$. The cohomology class $[\omega] \in H^2(M; \mathbf{R})$ comes down from some $[\hat{\omega}] \in H^2_{\overline{T}}(M; \mathbf{R})$ as was proved in [AB]. The class $[\hat{\omega}]$ is uniquely determined modulo the image of $H^2_{\overline{T}}(pt; \mathbf{R}) = H^2(B\overline{T}; \mathbf{R})$. The image of $[\hat{\omega}]$ in $\hat{H}^2_{\overline{T}}(M; \mathbf{R})$ can be written in the form

$$\hat{c}_\omega = \sum c_i \xi_i, \ c_i \in \mathbf{R}.$$

Karshon and Tolman defined a degree function

$$d_\omega : H^2(B\overline{T}; \mathbf{R}) \setminus \bigcup F_i \to \mathbf{Z}$$

using the moment map Φ_ω, and they showed similar formulas as Lemma 3.8 and Lemma 3.9. Then a similar argument as in the proof of Theorem 3.5 yields

THEOREM 3.11. $d_\omega = \bar{d}_{\hat{c}_\omega}$.

4. Positive line bundles. We keep the assumption on almost complex toric manifolds made in Section 2. We are interested in giving a combinatorial interpretation of $c_1(L)^n[M]$ for a T-line bundle over M. Note that $c_1(L)$ comes from the equivariant class $c_1^T(L) \in H^2_T(M; \mathbf{Z})$.

We begin with an observation concerning the evaluation on the fundamental class. Consider the digram

$$H^*(M) \xleftarrow{p} H^*_T(M) \xrightarrow{q} \hat{H}^*_T(M),$$

where $\hat{H}^*_T$ denotes the quotient by S-torsion classes as in Section 2 and Section 3, the coefficient group is the integers $\mathbf{Z}$ or the reals $\mathbf{R}$, and p and q are obvious maps.

LEMMA 4.1. *Let M be an oriented closed T-manifold. For any $x_1, \ldots, x_l \in H^*_T(M)$, the evaluation $p(x_1) \cdots p(x_l)[M]$ depends only on $q(x_1), \ldots, q(x_l)$.*

P r o o f. Consider the commutative diagram:

$$\begin{array}{ccccc} & & H_T^*(M) & \xrightarrow{p} & H^*(M) \\ & & {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ H^*(BT) & = & H_T^*(pt) & \xrightarrow{p} & H^*(pt) \end{array}$$

where pt is one point and π_* is the Gysin homomorphism. We have

$$p(x_1)\cdots p(x_l)[M] = \pi_* p(x_1\cdots x_l) = p\pi_*(x_1\cdots x_l).$$

If $y_1,\ldots,y_l$ are S-torsion elements, then

$$p(x_1+y_1)\cdots p(x_l+y_l)[M] = p\pi_*(x_1\cdots x_l) + p\pi_*(y),$$

with y an S-torsion element. Since π_* is an $H_T^*(pt)$-module map, $\pi_*(y)\in H_T^*(pt)$ is also an S-torsion element, and hence $\pi_*(y)=0$. ∎

Hereafter we will not distinguish $H_T^*(M)$ and $\hat{H}_T^*(M)$ as far as the evaluation on the fundamental class is concerned.

In Section 3 we considered $\overline{T}$-line bundles. In this section we consider more generally T-line bundles. If L is a T-line bundle it is easy to see that there exists an element $\delta\in H^2(BT;\mathbf{Z})$ such that the bundle $L\chi^{-\delta}$ comes from a $\overline{T}$-line bundle. If

$$c_1^{\overline{T}}(L\chi^{-\delta}) = \sum c_i\xi_i \in \hat{H}^2_{\overline{T}}(M;\mathbf{Z}),\ c_i\in\mathbf{Z},$$

then

$$c_1^T(L) = \sum c_i\xi_i + \delta \in \hat{H}^2_T(M;\mathbf{Z}).$$

Note that we regard $\hat{H}^*_{\overline{T}}(M)$ as embedded in $\hat{H}^*_T(M)$.

More generally we consider $\hat{c}\in\hat{H}^2_T(M;\mathbf{R})$ of the form

$$\hat{c} = \sum c_i\xi_i + \delta,\ c_i\in\mathbf{R},\ \delta\in H^2(BT;\mathbf{R}).$$

Set

$$\hat{c}^\delta = \sum c_i\xi_i \in \hat{H}^2_{\overline{T}}(M;\mathbf{R}),$$

and define

$$F_i^\delta = \{u\in H^2(B\overline{T};\mathbf{R}); < u, v_i >= c_i\}.$$

Chambers with respect to F_i^δ are denoted by $W_\alpha^\delta,\ W_\beta^\delta$ and so on, and walls by $W_{\alpha\beta}^\delta$ and so on. $F_i,\ W_\alpha,\ W_\alpha,\ W_{\alpha\beta}$ will denote the subsets of $H^2(B\overline{T};\mathbf{R})+\delta$ obtained from $F_i^\delta,\ W_\alpha^\delta,\ W_\beta^\delta,\ W_{\alpha\beta}^\delta$ by the translation by δ. For $u_\alpha\in W_\alpha$ we define

$$\bar{d}_{\hat{c}}(u_\alpha) = \bar{d}_{\hat{c}^\delta}(u-\delta).$$

Since $\bar{d}_{\hat{c}}$ is constant on W_α we put $\bar{d}_{\hat{c}}(W_\alpha)=\bar{d}_{\hat{c}}(u_\alpha)$. When $\hat{c}=c_1^T(L)$ we write $\bar{d}_L$ for $\bar{d}_{\hat{c}}$ and $\bar{d}_L(W_\alpha)$ for $\bar{d}_{\hat{c}}(W_\alpha)$.

The lattice $H^2(B\overline{T};\mathbf{Z})$ in $H^2(B\overline{T};\mathbf{R})$ defines a measure on $H^2(B\overline{T};\mathbf{R})$. If $e_1,\ldots,e_n$ is a basis of $H^2(B\overline{T};\mathbf{Z})$, then the parallelotope spanned by $0,e_1,\cdots,e_n$ has measure 1. The translation by δ transfers the measure on $H^2(B\overline{T};\mathbf{R})$ over $H^2(B\overline{T};\mathbf{R})+\delta$. The volume of a subset A of $H^2(B\overline{T};\mathbf{R})$ or $H^2(B\overline{T};\mathbf{R})+\delta$ is denoted by $|A|$.

THEOREM 4.2. *Let $\hat{c}$ be as above. Then*

$$p(\hat{c})^n[M] = n!\sum \bar{d}_{\hat{c}}(W_\alpha)|W_\alpha|,$$

where $[M]$ is the fundamental class of M, and the sum ranges over all chambers W_α. Since $\bar{d}_{\hat{c}}(W_\alpha) = 0$ for all unbounded chambers the above sum has a meaning.

COROLLARY 4.3. *Let L be a T-line bundle over M with*

$$c_1^T(L) = \sum c_i\xi_i + \delta \in \hat{H}_T^2(M;\mathbf{Z}),\ \delta \in H^2(BT;\mathbf{Z}).$$

Then

$$c_1(L)^n[M] = n!\sum \bar{d}_L(W_\alpha)|W_\alpha|,$$

REMARK 4.4. Theorem 4.2 can be regarded as a combinatorial counterpart of Theorem 1 in [KT]. Karshon and Tolman expressed the push-forward $(\Phi_\omega)_*\omega^n$ of the Liouville measure by a formula corresponding to Theorem 4.2.

For the proof of Theorem 4.2 we use a purely combinatorial lemma. To state the lemma we need some preliminaries. Let V be a real vector space with a lattice N in V. Let V^* be the dual space of V and N^* the dual lattice of N in V^*. The lattice N^* defines a measure on V^* as above. Let v be a primitive vector in N. Then $N_v^* = \{u \in N^*; < u, v >= 0\}$ is a lattice in $V_v^* = \{u \in V^*; < u, v >= 0\}$, and hence determines a measure in V_v^*. If c is a real number, then we transfer the measure on V_v^* to the affine hyperplane $F_v = \{u \in V^*; < u, v >= c\}$ by translation by a vector lying in F_v. The volume of a subset $B \subset F_v$ is also denoted by $|B|$.

Let $v_1, \dots, v_d$ be primitive vectors in N and $\hat{c} = (c_1, \dots, c_d)$ a sequence of real numbers. Define

$$F_i = \{u; u \in V^*,\ < u, v_i >= c_i\}.$$

Chambers and walls with respect to these affine hyperplanes F_i are defined as in Section 3. Let D be the union of the closures of certain bounded chambers. The boundary ∂D of D is the union of the closures of certain walls:

$$\partial D = \bigcup D_{i_k},\ D_{i_k} \subset F_{i_k}.$$

Let u_{i_k} be a vector which points outward from D and crosses D_{i_k} transversally. We set

$$\epsilon_{i_k} = sign < u_{i_k}, v_{i_k} > .$$

LEMMA 4.5. $n|D| = \sum \epsilon_{i_k} c_{i_k}|D_{i_k}|$.

Proof. Let $x_1, \dots, x_n$ be a coordinate system of V^* with respect to a basis $e_1, \dots, e_n$ of N^*. Define an $(n-1)$-form ω on V^* by

$$\omega = \sum(-1)^{j-1}x_j dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n.$$

The form ω is $SL(n, \mathbf{R})$-invariant so that it does not depend on the choice of integral basis $e_1, \dots, e_n$. By the Stokes theorem

$$\int_{\partial D} \omega = n\int_D dx_1 \cdots dx_n = n|D|,$$

where V^* is oriented by $dx_1 \wedge \cdots \wedge dx_n$. On the other hand

$$\int_{\partial D} \omega = \sum_{i_k} \int_{D_{i_k}} \omega.$$

Once i_k is fixed, we can take $e_1, \ldots, e_n$ so that $x_1 = v_{i_k}$ on D_{i_k}. Then ω takes the form $\omega = x_1 dx_2 \wedge \ldots \wedge dx_n$ on D_{i_k}. Comparing the orientations of F_i determined by $dx_2 \wedge \cdots \wedge dx_n$ and of $D_{i_k} \subset \partial D$ we obtain

$$\int_{D_{i_k}} \omega = \epsilon_{i_k} c_{i_k} |D_{i_k}|.$$

Hence $n|D| = \sum \epsilon_{i_k} c_{i_k} |D_{i_k}|$. ∎

Proof of Theorem 4.2. Clearly we may assume that $\hat{c}$ lies in $\hat{H}^2_{\overline{T}}(M; \mathbf{R})$, so that $\hat{c} = \sum c_i \xi_i$. Proof will proceed by induction on n.

First consider the case $n = 1$. In this case M is isomorphic to the projective line with non-trivial torus action, and M^T consists of two points $p_1,\ p_2$. Moreover $v_1,\ v_2$ satisfy $v_1 + v_2 = 0$. Define $u_1,\ u_2 \in H^2(B\overline{T}; \mathbf{R})$ by

$$< u_1, v_1 >= c_1,\ < u_2, v_2 >= c_2$$

i.e.

$$< u_1, v_1 >= c_1,\ < u_2, v_1 >= -c_2.$$

Then $F_1 = \{u_1\}$ and $F_2 = \{u_2\}$. If $u_1 = u_2$ i.e. $c_1 + c_2 = 0$ then there is no bounded chamber. If $u_1 \neq u_2$ then there is a unique bounded chamber W and $\bar{d}_{\hat{c}}(W) = sign(c_1 + c_2)$ and $|W| = |c_1 + c_2|$. Hence $\bar{d}_{\hat{c}}(W)|W| = c_1 + c_2$.

On the other hand

$$\hat{c} = c_1 \xi_1 + c_2 \xi_2.$$

But, by Lemma 2.2,

$$u_2 =< u_2, v_1 > \xi_1 + < u_2, v_2 > \xi_2 = c_2(-\xi_1 + \xi_2).$$

Hence $\hat{c} = (c_1 + c_2)\xi_1 + u_2 \in H^2_{\overline{T}}(M; \mathbf{R})$ and $p(\hat{c}) = (c_1 + c_2)p(\xi_1)$. Since ξ_1 is the Poincaré dual of a point in M, we have

$$p(\xi_1)[M] = 1 \text{ and } p(\hat{c})[M] = c_1 + c_2.$$

Thus

$$p(\hat{c})[M] = \bar{d}_{\hat{c}}(W)|W|.$$

Now suppose $n > 1$. For the sake of simplicity we write $\hat{c}^n[M]$ for $p(\hat{c})^n[M]$ and so on. We have

$$\hat{c}^n[M] = (\hat{c}^{n-1} \sum_i c_i \xi_i)[M] = \sum_i c_i \hat{c}^{n-1}[M_i]$$

since ξ_i is the Poincaré dual of M_i in M. By the inductive assumption we get

$$\hat{c}^n[M] = (n-1)! \sum_i c_i \sum_\alpha \bar{d}_{\hat{c}|M_i}(W_{i\alpha})|W_{i\alpha}|. \tag{4.1}$$

Here $\bar{d}_{\hat{c}|M_i}(W_{i\alpha})$ has the following meaning. We define γ_i and $\hat{c}^{\gamma_i}|M_i$ as in Section 3. Note that $p(\hat{c}^{\gamma_i}|M_i) = p(\hat{c}|M_i)$. The $\{W_{i\alpha}\}$ are chambers in $H^2(B\overline{T}_i; \mathbf{R})$ with respect to $\hat{c}^{\gamma_i}|M_i$. We write $\bar{d}_{\hat{c}|M_i}(W_{i\alpha})$ for $\bar{d}_{\hat{c}^{\gamma_i}|M_i}(W_{i\alpha})$. As in Section 3 this has an invariant meaning independent of γ_i. If one translates $W_{i\alpha}$ by the vector γ_i, then the image is a connected component of $F_i \setminus \bigcup_{F_j \neq F_i} F_j$. We also denote this component by $W_{i\alpha}$.

As a subset of F_i the closure $\overline{W}_{i\alpha}$ of $W_{i\alpha}$ is the union of the closures of certain walls. The closure of a wall $W_{\beta\beta'}$ will be denoted by $D_{\beta\beta'}$. When we particularly want to regard $D_{\beta\beta'}$ as sitting in F_i we use the notation $D_{\beta\beta' i}$. To proceed further, the following observation is in order.

Suppose that F_i and $F_{i'}$ with $i \neq i'$ both contain $D_{\beta\beta'}$. It means in particular that $F_i = F_{i'}$ as hyperplanes. Then, either $v_i = v_{i'}$ and $c_i = c_{i'}$ or $v_i = -v_{i'}$ and $c_i = -c_{i'}$, because v_i and $v_{i'}$ are primitive vectors in $H_2(B\overline{T}; \mathbf{Z})$. Therefore the number

$$c_{\beta\beta'} = c_i \, sign < u_{\beta'} - u_\beta, v_i >, \; u_\beta \in W_\beta, \; u_{\beta'} \in W_{\beta'},$$

depends only on W_β and $W_{\beta'}$. Note that

$$c_{\beta\beta'} = -c_{\beta'\beta}.$$

Now, in view of (4.1), Theorem 4.2 will follow from

ASSERTION 4.6.

$$\sum_{i,\alpha} c_i \bar{d}_{\hat{c}|M_i}(W_{i\alpha})|W_{i\alpha}| = n \sum_\beta \bar{d}_{\hat{c}}(W_\beta)|W_\beta|.$$

Proof. When $D_{\beta\beta' i}$ is contained in $W_{i\alpha}$, we set $\bar{d}_{\hat{c}|M_i}(D_{\beta\beta' i}) = \bar{d}_{\hat{c}|M_i}(W_{i\alpha})$. Then

$$\sum_{i,\alpha} c_i \bar{d}_{\hat{c}|M_i}(W_{i\alpha})|W_{i\alpha}| = \sum_{\beta,\beta',i} c_i \bar{d}_{\hat{c}|M_i}(D_{\beta\beta' i})|D_{\beta\beta' i}| \tag{4.2}$$

where the summation ranges over all $i \in \Sigma_M^0$ and all walls $W_{\beta\beta'}$ lying in F_i. The right hand side of (4.2) can be put in the form

$$= \sum_{\beta,\beta',i} sign < u_{\beta\beta'}, v_i > \bar{d}_{\hat{c}|M_i}(D_{\beta\beta' i}) c_{\beta\beta'} |D_{\beta\beta'}| \tag{4.3}$$

where $u_{\beta\beta'} = u_{\beta'} - u_\beta$. By the degree transition formula (Lemma 3.8)

$$\sum_i sign < u_{\beta\beta'}, v_i > \bar{d}_{\hat{c}|M_i}(D_{\beta\beta' i}) = \bar{d}_{\hat{c}}(W_\beta) - \bar{d}_{\hat{c}}(W_{\beta'}).$$

Putting this in (4.3) and noting that $c_{\beta\beta'} = -c_{\beta'\beta}$, (4.3) is transformed into

$$\sum_{\beta,\beta'} \bar{d}_{\hat{c}}(W_\beta) c_{\beta\beta'} |D_{\beta\beta'}|.$$

But, if one fixes β, then Lemma 4.5 tells us that

$$\sum_{\beta'} c_{\beta\beta'} |D_{\beta\beta'}| = n|W_\beta|.$$

Hence the right hand side of (4.2) is finally transformed into

$$n \sum_\beta \bar{d}_{\hat{c}}(W_\beta)|W_\beta|.$$

This finishes the proof of Assertion 4.6 and hence Theorem 4.2. ∎

Corollary 4.3 suggests the following definition of positiveness of T-line bundles over M. More generally, in view of Theorem 4.2, we define the positiveness of $\hat{c} = \sum c_i \xi_i + \delta \in$

$\hat{H}_T^2(M;\mathbf{R})$, $\delta \in H^2(BT;\mathbf{R})$, in the following way. In case $\hat{c} = c_1^T(L)$ and $\hat{c}$ is positive we call the T-line bundle L positive. When $n = 1$, using the notations in the proof of Theorem 4.2, we call $\hat{c}$ positive if $c_1 + c_2 > 0$. This condition is equivalent to the condition $\hat{c}[M] > 0$ by Theorem 4.2.

Suppose $n > 1$. We proceed by induction on the dimension n. We say that $\hat{c}$ is positive if the following conditions are satisfied:

1. $\hat{c}|M_I$ is positive for all $I \in \Sigma_M^{k-1},\ 1 \leq k \leq n$.
2. $F_i \neq F_j$ for $i \neq j$.
3. One can go out from any chamber to an unbounded chamber by successively passing walls in positive direction.

Here we say that one can pass in positive direction from a chamber W_α to an adjacent chamber W_β if $< u_\beta - u_\alpha, v_i >> 0$ for all F_i containing the wall $W_{\alpha\beta}$, where $u_\alpha \in W_\alpha$ and $u_\beta \in W_\beta$.

Proposition 4.7. *Suppose that $\hat{c}$ is positive. Then*

1. *$\bar{d}_{\hat{c}}(W_\alpha) \geq 0$ for all chambers W_α.*
2. *$\hat{c}^k[M_I] > 0$ for all $k,\ 1 \leq k \leq n$ and all $I \in \Sigma_M^{n-k-1}$.*

When $k = n$, (2) means $\hat{c}^n[M] > 0$.

In view of Theorem 3.11 and/or Theorem 3.5 we can restate Proposition 4.7 in terms of $d_{\hat{c}}$ and/or d_L. For example

Corollary 4.8. *Let L be a positive T-line bundle over M. Then*

1. *$d_L(W_\alpha) \geq 0$ for all chambers W_α.*
2. *$c_1(L)^k[M_I] > 0$ for all $k,\ 1 \leq k \leq n$ and all $I \in \Sigma_M^{n-k-1}$.*

Proof of Proposition 4.7. Proof will proceed by induction. First we show that $\bar{d}_{\hat{c}}(W_\alpha) \geq 0$ for all chambers W_α. In the case $n = 1$ we have $\bar{d}_{\hat{c}}(W_\alpha) = sign(c_1 + c_2) > 0$ for the unique bounded chamber W_α and $\bar{d}_{\hat{c}}(W_\beta) = 0$ for unbounded chambers W_β as was shown in the proof of Theorem 4.2. In particular $\hat{c}[M] = c_1 + c_2 > 0$ in this case.

Suppose $n > 1$. The inductive assumption implies $\bar{d}_{\hat{c}|M_i}(W_{\beta\beta'}) \geq 0$ for all F_i and for all walls $W_{\beta\beta'} \subset F_i$. Then, from the condition (3) of positiveness and the degree transition formula it follows that $\bar{d}_{\hat{c}}(W_\alpha) \geq 0$ for all chambers W_α since $\bar{d}_{\hat{c}}(W_\beta) = 0$ for faraway unbounded chambers W_β. Theorem 4.2 then yields

$$\hat{c}^n[M] > 0.$$

For $I \in \Sigma_M^{n-k-1}$ with $1 \leq k < n$, $\hat{c}|M_I$ is positive by the condition (1) of positiveness. Thus, by inductive assumption, $\hat{c}^k[M_I] > 0$. ∎

Remark 4.9. As indicated at the end of Section 3 the equivariant cohomology class $\hat{c}$ can be thought of as representing a closed 2-form admitting moment map over M. In this context positive classes may be thought of as representing certain closed 2-forms participating in the positiveness property of symplectic forms.

Remark 4.10. Corresponding to Remark 4.9 positive T-line bundles have property (2) of Corollary 4.8 analogous to ample line bundles over toric manifold. However the converse statement of Corollary 4.8 is not true in general. In section 5, it will be shown, under the additional assumption $T[M]=1$, that the converse is also true. In this sense Corollary 4.8 may be regarded as an analogue of the Nakai criterion of ampleness of line bundles.

From the definition of positiveness we readily have the following

PROPOSITION 4.11. *If $\hat{c}$ is positive, then $r\hat{c}$ is also positive for any positive real number r. If $\{\epsilon_i\}$ is a collection of sufficiently small real numbers indexed by Σ_M^0 and $\hat{c}'=\hat{c}+\sum_i \epsilon_i\xi_i$, then $\hat{c}'$ is also positive.*

We shall discuss some numerical necessary conditions for $\hat{c}$ to be positive. To begin with, assume $n\geq 2$ and take an $(n-2)$-simplex $J\in\Sigma_M^{n-2}$. Then $Lk_{\Sigma_M}J$ consists of two points i and i'.

LEMMA 4.12. *v_i and $v_{i'}$ are related by an equation of the following form*:

$$v_i+v_{i'}=\sum_{j\in J}a_jv_j,\ \ a_j\in\mathbf{Z}.$$

Proof. Let $J=\{j_1,\dots,j_{n-1}\}$. $v_i,v_{j_1},\dots,v_{j_{n-1}}$ and $v_{i'},v_{j_1},\dots,v_{j_{n-1}}$ are both bases of $H_2(B\overline{T};\mathbf{Z})$. Therefore they are related by a linear relation

$$(v_{i'},v_{j_1},\dots,v_{j_{n-1}})=(v_i,v_{j_1},\dots,v_{j_{n-1}})\begin{pmatrix} -1 & 0 & \cdots & & 0\\ a_{j_1} & 1 & 0 & \cdots & 0\\ & & \ddots & & \\ & & & \ddots & \\ a_{j_{n-1}} & 0 & \cdots & 0 & 1\end{pmatrix}.$$

Here -1 comes because v_i and $v_{i'}$ sit on different sides of $s_J=|v_{j_1},\dots,v_{j_{n-1}}|$ by [M], Lemma 4.4, cf. also Lemma 2.3. ∎

LEMMA 4.13. *Under the same situation as in Lemma 4.12 we have*:

$$\begin{aligned}&\xi_i[M_J]=\xi_{i'}[M_J]=1,\\ &\xi_j[M_J]=-a_j\,,\quad j\in J,\\ &\xi_k[M_J]=0\,,\quad k\neq i,i',\ k\notin J\,.\end{aligned}$$

Proof (Due to Masuda). $p(\xi_i|M_J),\ p(\xi_{i'}|M_J)\in H^2(M_J;\mathbf{Z})$ are the Poincaré dual in M_J of a point as is easily seen. Therefore

$$\xi_i[M_J]=\xi_{i'}[M_J]=1.$$

If $k\neq i,i',\ k\notin J$, then $M_k\cap M_J=\emptyset$. Therefore $\xi_k|M_J=0$ and $\xi_k[M_J]=0$.

We put $\xi_J=\prod_{j\in J}\xi_j\in\hat{H}_{\overline{T}}^{2(n-1)}(M;\mathbf{Z})$. For any $u\in H^2(B\overline{T};\mathbf{R})$, we have $u\xi_J[M]=0$. But

$$u\xi_J[M]=\sum_k <u,v_k>\xi_k\xi_J[M]=\sum_k <u,v_k>\xi_k[M_J]$$

by Lemma 2.2. Hence we get

$$0=<u,v_i>+<u,v_{i'}>+\sum_{j\in J}<u,v_j>\xi_j[M_J].$$

Then, by Lemma 4.12,

$$0 = \sum_{j \in J} < u, v_j > (a_j + \xi_j[M_J]).$$

Since this holds for any u, it follows that $\xi_j[M_J] = -a_j$ for all $j \in J$. ■

As an immediate consequence of Lemma 4.13 we obtain

COROLLARY 4.14. *Suppose* $n \geq 2$. *Then*

$$\hat{c}[M_J] = c_i + c_{i'} - \sum_{j \in J} c_j a_j$$

for any $J \in \Sigma_M^{n-2}$.

As a direct consequence of Corollary 4.14 and Proposition 4.7 we obtain

PROPOSITION 4.15. *Suppose* $n \geq 2$. *Then* $\hat{c}[M_J] > 0$ *if and only if*

$$c_i + c_{i'} - \sum_{j \in J} c_j a_j > 0$$

for any $J \in \Sigma_M^{n-2}$. *In particular if* $\hat{c}$ *is positive, then*

$$c_i + c_{i'} - \sum_{j \in J} c_j a_j > 0 \text{ for all } J \in \Sigma_M^{n-2}.$$

We keep the same situation as above and put

$$I = \{i\} \cup J \text{ and } I' = \{i'\} \cup J \in \Sigma_M^{n-1}.$$

Put also $u_I = \Psi_{\hat{c}}(b_I)$ and $u_{I'} = \Psi_{\hat{c}}(b_{I'})$. They are determined by

$$< u_I, v_i >= c_i, \qquad < u_I, v_j >= c_j \text{ for } j \in J,$$
$$< u_{I'}, v_{i'} >= c_{i'}, \qquad < u_{I'}, v_j >= c_j \text{ for } j \in J.$$

LEMMA 4.16.

$$< u_I - u_{I'}, v_i >=< u_{I'} - u_I, v_{i'} >= c_i + c_{i'} - \sum_{j \in J} a_j c_j.$$
$$< u_I - u_{I'}, v_j >= 0 \text{ for } j \in J.$$

P r o o f.

$$< u_I - u_{I'}, v_i >= c_i - < u_{I'}, v_i >= c_i - < u_{I'}, -v_{i'} + \sum a_j v_j >= c_i + c_{i'} - \sum a_j c_j.$$

Similar computations yield the other equalities. ■

COROLLARY 4.17. *Suppose* $\hat{c}[M_J] > 0$. *Then*

$$< u_I - u_{I'}, v_i >=< u_{I'} - u_I, v_{i'} > > 0.$$

In particular $< u_{I'}, v_i > < c_i$. *Also* $< \Psi_{\hat{c}}(b_J), v_i > < c_i$.

P r o o f. For the third inequality note that $\Psi_{\hat{c}}(b_J) = \frac{1}{2}(u_I + u_{I'})$. ■

Recall that s_I is the realization of $I \in \Sigma_M$ in $|\Sigma_M|$, and σ_i is the dual cell in S_M corresponding to $i \in \Sigma_M^0$. Two $(n-1)$-simplices I, I' will be called adjacent if $I \cap I'$ belongs to Σ_M^{n-2}.

LEMMA 4.18. *Suppose that* $n \geq 2$ *and fix* $I \in \Sigma_M^{n-1}$. *Let* $I_1, \ldots, I_l$ *be the totality of* $(n-1)$*-simplices such that* I *and* I_ν *are adjacent. Put* $J_\nu = I \cap I_\nu \in \Sigma_M^{n-2}$. *If* $\hat{c}[M_\nu] > 0$ *for all* ν, $1 \leq \nu \leq l$, *then*

$$< \Psi_{\hat{c}}(v), v_i > = c_i \qquad \text{if } \ v \in s_I \cap \sigma_i,$$
$$< \Psi_{\hat{c}}(v), v_i > < c_i \qquad \text{if } \ v \notin s_I \cap \sigma_i,$$

for any $i \in I$.

Proof. Proof will proceed by induction on n. By Lemma 3.4, $< \Psi_{\hat{c}}(v), v_i >= c_i$ if $v \in \sigma_i$. Therefore it suffices to show that $< \Psi_{\hat{c}}(v), v_i > < c_i$ if $x \notin s_I \cap \sigma_i$. For $i \in I$, let $J \in \Sigma_M^{n-2}$ and $I' \in \Sigma_M^{n-1}$ be defined by $\{i\} \cup J = I$ and $I \cap I' = J$.

Consider first the case $n = 2$. If $n = 2$, $J = \{j\}$ and s_I is divided into two parts $s_I \cap \sigma_i = |b_i, b_I|$ and $s_I \cap \sigma_j = |b_j, b_I|$. By Corollary 4.17, $< \Psi_{\hat{c}}(b_j), v_i > < c_i$. Let $v \in s_I \cap \sigma_j = |b_j, b_I|$. Then v is of the form

$$v = tb_j + (1-t)b_I, \ 0 \leq t \leq 1.$$

Therefore $< \Psi_{\hat{c}}(v), v_i >= t\Psi_{\hat{c}}(b_j) + (1-t)c_i < c_i$ unless $t = 0$.

Suppose $n > 2$. M_J is contained in M_j for each $j \in J$. By the inductive assumption applied on M_j, $< \Psi_{\hat{c}}(x_j), v_i > < c_i$ if $x_j \in s_{I-\{j\}}$ but $x_j \notin s_{I-\{j\}} \cap \sigma_i$. We also have $< \Psi_{\hat{c}}(b_J), v_i > < c_i$ by Corollary 4.17. Since every $v \in s_I$ with $v \notin s_I \cap \sigma_i$ can be written as a linear combination with non-negative coefficients of the x_j as above and b_J, the conclusion follows. ∎

Let L be a T-line bundle. Then there is an element $\delta \in H^2(BT; \mathbf{Z})$ such that $L' = L\chi^{-\delta}$ comes from a $\overline{T}$-line bundle, and

$$c_1^T(L) = c_1^{\overline{T}}(L') + \delta = \sum c_i \xi_i + \delta$$

as remarked at the beginning of Section 4. Moreover we have $\operatorname{ind} D_L = (\operatorname{ind} D_{L'})\chi^\delta$.

THEOREM 4.19. *Let* L *be a positive line bundle over* M. *If we write*

$$\operatorname{ind} D_L = \sum_{u \in H^2(B\overline{T}; \mathbf{Z})} m(u)\chi^{u+\delta},$$

then $m(u) \geq 0$. *Moreover* $m(u_I) > 0$ *for all* $I \in \Sigma_M^{n-1}$, *where* $u_I = \Psi_{L'}(b_I)$.

Proof. We may assume that $\delta = 0$ from the start. By Theorem 2.9 and Theorem 3.11,

$$m(u) = \bar{d}_{\hat{c}'}(u) \text{ for } u \in H^2(B\overline{T}; \mathbf{Z}), \tag{4.4}$$

where $\hat{c}' = \sum (c_i + \frac{1}{2})\xi_i$. However, it is easy to see that $\bar{d}_{\hat{c}'}$ with $\hat{c}' = \sum (c_i + r)\xi_i$ also satisfies the same identity (4.4) so far as $0 < r < 1$. We take r small enough. Then, by Proposition 4.11, $\hat{c}'$ is positive since $c_1^{\overline{T}}(L)$ is positive by assumption. Therefore

$$m(u) = \bar{d}_{\hat{c}'}(u) \geq 0$$

by Proposition 4.7.

Next we shall show $\bar{d}_{\hat{c}'}(u_I) > 0$ by induction on n. When $n = 1$, Σ_M^0 consists of two points b_1, b_2 and $< u_1 - u_2, v_1 >= c_1 + c_2 > 0$, where $u_i = \Psi_L(b_i)$. Therefore u_1 and u_2 are contained in the unique bounded chamber for $\hat{c}'$. Hence $\bar{d}_{\hat{c}'}(u_i) > 0$ for $i = 1, 2$.

Suppose $n > 1$. From the inductive assumption it follows that $\bar{d}_{\hat{c}'|M_i}(u_I) > 0$ on F_i for any $i \in I$, where $F_i = \{u; u \in H^2(B\overline{T}; \mathbf{R}), < u, v_i >= c_i\}$ as before. Let $J \in \Sigma_M^{n-2}$ and I' be defined as in the proof of Lemma 4.18. Then $< u_I - u_{I'}, v_i > > 0$ by Corollary 4.17. Take $\rho > 0$ such that $\rho < u_I - u_{I'}, v_i >= r$. Then $u_I + \rho(u_I - u_{I'})$ is in $F_i' = \{u; < u, v_i >= c_i + r\}$, and $\bar{d}_{\hat{c}'|M_i}(u_I + \rho(u_I - u_{I'})) > 0$ since $\bar{d}_{\hat{c}'|M_i}$ is invariant by translation. Then the degree transition formula applied to the chamber containing u_I and the wall containing $u_I + \rho(u_I - u_{I'})$ yields $\bar{d}_{\hat{c}'}(u_I) > 0$. ■

As an immediate corollary of Theorem 4.19 we obtain

COROLLARY 4.20. *If L is positive, then* $\operatorname{ind} D_L$ *is a non-trivial honest T-module and* dim $\operatorname{ind} D_L \geq \#\Sigma_M^{n-1}$.

THEOREM 4.21. *Let L be a positive T-line bundle over M and K the canonical bundle of M. Then* $\operatorname{ind} D_{K\otimes L}$ *is an honest T-module.*

Proof. As in the proof of Theorem 4.19 we assume that $c_1^T(L) = \sum c_i\xi_i$ from the start. We write $\operatorname{ind} D_{K\otimes L} = \sum_{H^2(B\overline{T};\mathbf{Z})} m(u)\chi^u$. Since $c_1^T(K) = -\sum \xi_i$ by Lemma 2.8,

$$c_1^T(K \otimes L) = \sum (c_i - 1)\xi_i \in \hat{H}^2_{\overline{T}}(M; \mathbf{Z}).$$

Let $\epsilon > 0$ be small enough. Put $r = 1 - \epsilon$ and $\hat{c}' = c_1^T(K \otimes L) + r\sum \xi_i = \sum (c_i - \epsilon)\xi_i$. Then $\hat{c}'$ is positive by Proposition 4.11, and $m(u) = \bar{d}_{\hat{c}'}(u) \geq 0$ for $u \in H^2(BT; \mathbf{Z})$. ■

REMARK 4.22. Theorem 4.21 can be thought of as an analogue of the Kodaira vanishing theorem. For details see [H2].

5. The case $T[M] = 1$. In this section we show that some aspects of positive line bundles or more generally of positive classes $\hat{c}$ resemble those of ample line bundles in the theory of toric varieties. We keep the assumptions made on almost complex toric manifolds. Recall that the Todd genus $T[M]$ of M equals the degree of the projection $|\Sigma_M| \to H_2(B\overline{T}; \mathbf{R}) \setminus \{0\}$. The first observation is

LEMMA 5.1. *$T[M_I] \leq T[M]$ for all $I \in \Sigma_M^{k-1}$, $1 \leq k \leq n$. In particular, if $T[M] = 1$, then $T[M_I] = 1$ for all I.*

Proof. It is sufficient to show that $T[M_i] \leq T[M]$ for all $i \in \Sigma_M^0$. Take a generic circle subgroup $\overline{S} \subset \overline{T}$ such that $M^{\overline{S}} = M^{\overline{T}}$. Then we know that

$$T[M] = \#\{p \in M^{\overline{S}}; \text{all } \overline{S} \text{ weights at } p \text{ are positive}\},$$

see e.g. [H1]. Take $\overline{S}$ near $\overline{S}_i$ such that its projection on $\overline{T}_i = \overline{T}/\overline{S}_i$ is still generic. Then

$$T[M_i] = \#\{p \in M_i^{\overline{S}}; \text{all } \overline{S} \text{ weights of } M_i \text{ at } p \text{ are positive}\}.$$

Let $v \in H_2(B\overline{T}; \mathbf{Z})$ be the primitive element corresponding to the embedding $\overline{S} \subset \overline{T}$ and lying near v_i. If p is in $M_i^{\overline{S}}$, then the $\overline{S}$ weights of M at p are the union of the $\overline{S}$ weights of M_i at p and $< \xi_i|p, v >$. But $< \xi_i|p, v >$ is close to $< \xi_i|p, v_i >= 1$. Hence $< \xi_i|p, v >= 1$. Therefore, if all the $\overline{S}$ weights of M_i at p are positive, then all the $\overline{S}$ weights of M at p are positive. This implies $T[M_i] \leq T[M]$. ■

Hereafter we assume $T[M] = 1$. We consider the following condition (P) for $\hat{c} \in \hat{H}^2_{\overline{T}}(M;\mathbf{R})$:

1. $< \Psi_{\hat{c}}(b_I), v_i > \le c_i$ for all $I \in \Sigma_M^{n-1}$ and $i \in \Sigma_M^0$.
2. The $u_I = \Psi_{\hat{c}}(u_I)$ are different to each other.

It is easy to see that the above condition (1) is equivalent to:

(1′) $< \Psi_{\hat{c}}(b_I), v_i > \le c_i$ for all $I \in \Sigma_M^{k-1}$, $1 \le k \le n$ and $i \in \Sigma_M^0$.

Let CS_M denote the cone over S_M with the vertex b_0. We define a map $\tilde{\Psi}_{\hat{c}} : CS_M \to H^2(B\overline{T};\mathbf{R})$ by

$$\tilde{\Psi}_{\hat{c}}(tb_0 + (1-t)v) = tu_0 + (1-t)\Psi_{\hat{c}}(v)$$

where $u_0 = \frac{1}{\#\Sigma_M^{n-1}} \sum_{I \in \Sigma_M^{n-1}} u_I$, the barycenter of $\{u_I; I \in \Sigma_M^{n-1}\}$. Put

$$D_{\hat{c}} = \{u \in H^2(B\overline{T};\mathbf{R}); < u, v_i > \le c_i \text{ for all } i \in \Sigma_M^0\}.$$

It is a convex set if it is not empty.

Lemma 5.2. *The condition* (*P*) *is equivalent to*: Im $\tilde{\Psi}_{\hat{c}} = \tilde{\Psi}_{\hat{c}}(CS_M) = D_{\hat{c}}$ *and the* u_I *are different to each other. In this case*

$$\bar{d}_{\hat{c}}(u) = \begin{cases} 1, & u \in \textit{interior } D_{\hat{c}}, \\ 0, & u \in \textit{outside } D_{\hat{c}}. \end{cases}$$

Proof. This is clear since $D_{\hat{c}}$ is the convex hull of $\{u_I; I \in \Sigma_M^{n-1}\}$. The statement for $\bar{d}_{\hat{c}}$ follows readily by induction on n and the degree transition formula. ∎

Since $T[M] = 1$, the complex S_M (identified with $|\Sigma_M|$) is embedded in $H_2(B\overline{T};\mathbf{R})$ via π, and the vertex of the cone CS_M can be taken at the origin $0 \in H_2(B\overline{T};\mathbf{R})$. Let $\angle s_I$ and $\angle \sigma_i$ denote the cone over s_I and σ_i respectively with the vertex at 0. We define a function $\psi_{\hat{c}} : H_2(B\overline{T};\mathbf{R}) \to \mathbf{R}$ by

$$\psi_{\hat{c}}(v) = < u_I, v > \text{ if } v \in \angle s_I,\ I \in \Sigma_M^{n-1}.$$

It is easy to see that $\psi_{\hat{c}}$ is well-defined.

Lemma 5.3. *$\hat{c}$ satisfies the condition* (*P*) *if and only if the function $\psi_{\hat{c}}$ is strictly lower convex.*

This means that $\psi_{\hat{c}}$ is lower convex in the usual sense and $u_I \ne u_{I'}$ for $I \ne I'$. This is a familiar fact in the theory of toric varieties. We refer the reader to [O], Lemma 2.12. Here we remark that, in order to check the strict lower continuity of $\psi_{\hat{c}}$, it is sufficient to do it on each $\angle \sigma_i$ since it is linear on each $\angle s_I$ and the boundaries of s_I are covered by $\{\sigma_i\}$.

Theorem 5.4. *Assume $T[M] = 1$. Then the following condition are equivalent for $\hat{c} \in \hat{H}^2_{\overline{T}}(M;\mathbf{R})$*:

1. *$\hat{c}$ is positive.*
2. *$\hat{c}[M_J] > 0$ for all $J \in \Sigma_M^{n-2}$.*
3. *$\hat{c}$ satisfies the condition* (*P*).

Moreover, if $\hat{c}$ is positive then the image of $\tilde{\Psi}_{\hat{c}}$ coincides with $D_{\hat{c}}$.

Proof. When $n = 1$, it is easy to check the equivalence of the above two conditions as in the proof of Proposition 4.2.

Suppose $n > 1$. The implication (1) $\Rightarrow$ (2) was proved in Proposition 4.7. The implication (3) $\Rightarrow$ (1) is easy in view of Lemma 5.2. Assume $\hat{c}[M_J] > 0$ for all $J \in \Sigma_M^{n-2}$. Then Lemma 4.18 implies that $\psi_{\hat{c}}$ is strictly lower convex on each $\angle\sigma_i$. Therefore it is strictly lower convex on the whole $H_2(B\overline{T}; \mathbf{R})$ as remarked after Lemma 5.3. Thus $\hat{c}$ satisfies the condition (P) by Lemma 5.3, and the image of $\tilde{\Psi}_{\hat{c}}$ coincides with $D_{\hat{c}}$ by Lemma 5.2. ∎

As an immediate corollary of Theorem 5.4, Theorem 4.19 and Lemma 5.2 we obtain

Corollary 5.5. *Assume $T[M] = 1$. Then a T-line bundle L is positive if and only if $c_1(L)[M_J] > 0$ for all $J \in \Sigma_M^{n-2}$. In this case the non-zero multiplicities $m(u)$ are all equal to* 1.

Remark 5.6. Corollary 5.5 is an analogue of the so-called toric Nakai criterion. See [O], Theorem 2.18; cf. also Remark 4.10.

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