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A MULTIPLICITY RESULT FOR A SYSTEM OF REAL INTEGRAL EQUATIONS BY USE OF THE NIELSEN NUMBER

ANDREI Yu. BORISOVICH

Institute of Mathematics, University of Gdańsk
Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail: matab@univ.gda.pl

ZYGFRYD KUCHARSKI

Faculty of Technical Physics and Applied Mathematics
Technical University of Gdańsk
G. Narutowicza 11/12, 80-952 Gdańsk, Poland
E-mail: zkuchar@mif.pg.gda.pl

WACŁAW MARZANTOWICZ

Faculty of Mathematics & Computer Science
A. Mickiewicz University of Poznań
Matejki 48/49, 60-769 Poznań, Poland
E-mail: marzan@amu.edu.pl

Abstract. We prove an existence and multiplicity result for solutions of a nonlinear Urysohn type equation (2.14) by use of the Nielsen and degree theory in an annulus in the function space.

1. Main scheme. Consider a family of nonlinear equations

$$x = G_\lambda(x) \tag{1.1}$$

depending continuously on the parameter $\lambda \in [0, 1]$, where $G_\lambda : X \to X$ are continuous selfmaps of a Banach space X. The homotopy G_λ is thought of as a deformation of $G_1(x) = x$ to a simpler equation $G_0(x) = x$. We look for some open path-connected subset or ANR $D \subset X$, which is invariant with respect to the maps G_λ, i.e.

$$G_\lambda(D) \subset D, \qquad \lambda \in [0, 1], \tag{1.2}$$

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and then we restrict our considerations to D under the following assumptions:

(A) The map $\widehat{G} : X \times [0,1] \to X \times [0,1]$, defined by $\widehat{G}(x,\lambda) = (G_\lambda(x),\lambda)$, is completely continuous.

(B) The set $\operatorname{Fix}(\widehat{G}, D \times [0,1])$ of fixed points of $\widehat{G}$ which belong to $D \times [0,1]$ is a compact subset of $X \times [0,1]$.

(C) The equation $x = G_0(x)$ has precisely n solutions $\{x_1^0,\dots,x_n^0\}$ in D and there exist open nieghborhoods U_j^0 $(j = 1,\dots,n)$ of x_j^0 such that

$$\begin{aligned} &U_i^0 \cap U_j^0 = \emptyset, && i \neq j, \\ &\deg(I - G_0, U_j^0, 0) \neq 0, && j = 1,\dots,n, \end{aligned} \tag{1.3}$$

where $I : X \to X$ is the identity map.

The following result can be obtained using the Nielsen fixed point theory.

THEOREM 1.1. *Assume that the conditions* (A)–(C) *are satisfied. If the fixed points* $x_1^0,\dots,x_n^0$ *of* G_0 *are in different Nielsen classes, then for each* $\lambda \in [0,1]$ *the equation* $x = G_\lambda(x)$ *has at least* n *solutions, which belong to different Nielsen classes of* G_λ.

Recall that two fixed points x_i^0 and x_j^0 belong to the same Nielsen class if there exists a continuous path w joining x_i^0 and x_j^0 such that w and its image $G_0(w)$ are homotopic in D rel end points. The Nielsen class $\{x\}$ is called essential if there exists an open neighbourhood U such that

$$\operatorname{Fix}(G_0, D) \cap U = \{x\}, \qquad \deg(I - G_0, U, 0) \neq 0. \tag{1.4}$$

The number $\mathbf{N}(G_0, D)$ of essential classes is called the Nielsen number. It is a homotopy invariant, i.e. if G_1 is homotopic to G_0 by a homotopy $G_\lambda : D \to D$ which satisfies assumptions (A)–(B), then $\mathbf{N}(G_0, D) = \mathbf{N}(G_1, D)$. Such a homotopy $G_\lambda : D \to D$ is called admissible. In our situation, the fixed points $x_1^0,\dots,x_n^0$ by (C) belong to different essential Nielsen classes and

$$\mathbf{N}(G_\lambda, D) = n \tag{1.5}$$

for each $\lambda \in [0,1]$. For more details about Nielsen classes see [K], [J], [Br3].

REMARK 1.1. If D is simply-connected, then all fixed points in D belong to the same Nielsen class. Theorem 1.1 gives a multiplicity result only for a non-simply-connected domain D.

There are very few papers employing the Nielsen theory to nonlinear problems ([Br2], [Br3], [F], [BKM1]) .

2. Systems of equations. In this note we study a class of nonlinear systems of integral equations of Urysohn type. Using the Nielsen number we show that the discussed system has at least two non-zero solutions. The form of the integral kernel yields an a priori estimate which guarantees that the linear deformation of the original map preserves the annulus.

We will work in the Banach space $X = C[0,1] \times C[0,1]$ of pairs of continuous functions with the norm

$$x = (u, v), \qquad \|x\| = \overline{u} + \overline{v}, \tag{2.1}$$

where $\overline{u} = \max |u(t)|$ and $\overline{v} = \max |v(t)|$.

In $C[0,1]$ we consider two closed cones of positive and of negative continuous functions, respectively:

$$C^{+}[0,1] = \{u(t) : u(t) \geqslant 0\}, \qquad C^{-}[0,1] = \{u(t) : u(t) \leqslant 0\}. \tag{2.2}$$

We will also use the set

$$C^{\pm}[0,1] = C^{+}[0,1] \ \cup \ C^{-}[0,1]. \tag{2.3}$$

Definition 2.1. By the *annulus* in the Banach space $X = C[0,1] \times C[0,1]$ we shall understand the set

$$A_c = C^{\pm}[0,1] \times C^{\pm}[0,1] - \{(0,0)\}. \tag{2.4}$$

Lemma 2.1. *The set A_c is a path-connected ANR and the fundamental group of A_c is isomorphic to the group of integer numbers, i.e.*

$$\pi_1(A_c) \simeq \mathbb{Z}. \tag{2.5}$$

Proof. Consider the two-dimensional subspace of pairs of constant functions in X

$$E^2 = \{(c_1, c_2) : c_i \in \mathbb{R}\}. \tag{2.6}$$

We denote by E^2_0 this plane with the point $(0,0)$ deleted. Notice that $E^2_0 \subset A_c$. Moreover, we have the deformation retraction $\rho : A_c \times [0,1] \to A_c$ defined by the formula

$$\rho(u, v, \lambda) = (\lambda u + (1-\lambda)\operatorname{sign} u \cdot \overline{u}, \lambda v + (1-\lambda)\operatorname{sign} v \cdot \overline{v}), \tag{2.7}$$

such that

$$\rho(A_c, 1) = A_c, \qquad \rho(A_c, 0) = E^2_0. \tag{2.8}$$

Therefore, we have

$$\pi_1(A_c) = \pi_1(E^2_0) = \pi_1(S^1) = \mathbb{Z}. \ \blacksquare \tag{2.9}$$

Next, consider a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(c_1, c_2) = (\delta_1 c_2^{\beta}, \delta_2 c_1^{\alpha}), \tag{2.10}$$

where α and β are positive rational numbers, m and n are relatively prime, $c^{\frac{n}{m}} = (\operatorname{sign} c \cdot |c|^{\frac{1}{m}})^n$ by definition, and δ_1, δ_2 are nonzero. Note that $g(\mathbb{R}^2_0) \subset \mathbb{R}^2_0$, where $\mathbb{R}^2_0$ is $\mathbb{R}^2$ with the point $(0,0)$ deleted.

Lemma 2.2. *For given positive rational numbers $\alpha = \frac{n_1}{m_1}$, $\beta = \frac{n_2}{m_2}$ such that $\alpha \cdot \beta \neq 1$, and for $\delta_1, \delta_2 \in \{-1, +1\}$ define the continuous map $g : \mathbb{R}^2_0 \to \mathbb{R}^2_0$ by (2.10). Then the fixed point set of g is compact, and the degree of g is given by the formula*

$$\deg(g) = -\delta_1 \delta_2 \left(\frac{1 - (-1)^{n_1}}{2}\right)\left(\frac{1 - (-1)^{n_2}}{2}\right), \tag{2.11}$$

and consequently the Nielsen number

$$\mathbf{N}(g, \mathbb{R}^2_0) = |1 - \deg(g)| \in \{0, 1, 2\}. \tag{2.12}$$

PROOF. The first part follows from the fact that the degree is multiplicative. Since for $\alpha = \frac{n}{m}$ we have $\deg(x^\alpha) = 0$ or 1 depending on whether n is even or odd, the second part of the statement is a property of the Nielsen number of a selfmap of S^1, or equivalently of $\mathbb{R}^2_0$. ∎

REMARK 2.1. If $\delta_1 = \delta_2$ and both n_1, n_2 are odd numbers, then $\mathbf{N}(g, \mathbb{R}^2_0) = 2$.

Remark 2.1 has a simple geometrical sense. The fixed points of g are given as solutions of the system

$$(c_1, c_2) = (\delta_1 c_2^\beta, \delta_2 c_1^\alpha). \tag{2.13}$$

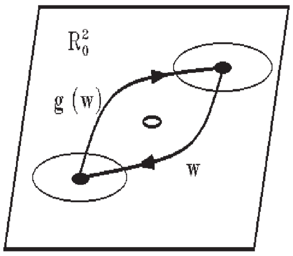

Fig. 1

If $\delta_1 = \delta_2$ and n_1, n_2 are odd, then (2.13) has two solutions:

$$\begin{aligned} &(-1,-1) \quad \text{and} \quad (+1,+1) \quad \text{if } \delta_1 = \delta_2 = 1, \\ &(-1,+1) \quad \text{and} \quad (+1,-1) \quad \text{if } \delta_1 = \delta_2 = -1, \end{aligned}$$

which are different essential Nielsen classes (see Fig. 1).

We are in a position to formulate our main theorem.

Consider the following system of two nonlinear real integral equations:

$$\begin{cases} u(t) = \displaystyle\int_0^1 K_1(t, s, u(s), v(s)) v^\beta(s) ds, \\ v(t) = \displaystyle\int_0^1 K_2(t, s, u(s), v(s)) u^\alpha(s) ds, \end{cases} \tag{2.14}$$

where α and β are positive rational numbers, $u^{\frac{n}{m}} = (\operatorname{sign} u \cdot |u|^{\frac{1}{m}})^n$ by definition. System (2.14) is equivalent to the operator equation $x = G(x)$, where the operator $G : X \to X$ is defined by the formula

$$G(u, v) = \Big(\int_0^1 K_1(\dots) v^\beta(s) ds, \int_0^1 K_2(\dots) u^\alpha(s) ds \Big), \tag{2.15}$$

and hence is a completely continuous.

THEOREM 2.1. *Suppose (2.14) satisfies the following assumptions:*

1) $K_i(t, s, u, v) \in C^1([0,1]^2 \times \mathbb{R}^2)$ *for* $i = 1, 2$;
2) $\underline{K} \leqslant |K_i(t, s, u, v)| \leqslant \overline{K}$ *for all* $(t, s, u, v) \in [0,1]^2 \times \mathbb{R}^2$, *where* $0 < \underline{K} \leqslant 1 \leqslant \overline{K}$;
3) $\alpha = \frac{n_1}{m_1}, \beta = \frac{n_2}{m_2} \in \mathbb{Q}_+$ *and* $\alpha\beta \neq 1$.

Then the operator $G : A_c \to A_c$ (see (2.15)) is well defined, the set $\mathrm{Fix}(G, A_c)$ *is compact, the Nielsen number* $\mathbf{N}(G, A_c)$ *is well defined and*

$$\mathbf{N}(G, A_c) = \mathbf{N}(g, \mathbb{R}_0^2), \tag{2.16}$$

where $g : \mathbb{R}_0^2 \to \mathbb{R}_0^2$ is the map defined in (2.10) with $\delta_i = \mathrm{sign} K_i$. Consequently, the system (2.1) has at least 2 non-zero solutions if $\delta_1 = \delta_2$ and n_1, n_2 are odd.

Proof. Deform the system (2.14) to a simpler system

$$\begin{cases} u(t) = \displaystyle\int_0^1 \delta_1 v^\beta(s)ds, \\ v(t) = \displaystyle\int_0^1 \delta_2 u^\alpha(s)ds, \end{cases} \tag{2.17}$$

which is equivalent to the operator equation $x = G_0(x)$, where $G_0 : X \to X$ is defined by

$$G_0(u, v) = \Big(\int_0^1 \delta_1 v^\beta(s)ds, \int_0^1 \delta_2 u^\alpha(s)ds \Big). \tag{2.18}$$

Consider a linear homotopy $x = G_\lambda(x)$, $\lambda \in [0, 1]$, connecting $G = G_1$ with G_0, which is defined by

$$G_\lambda = \lambda G_1 \ + \ (1 - \lambda)G_0. \tag{2.19}$$

Explicitly, we have the equations

$$\begin{cases} u(t) = \displaystyle\int_0^1 (\lambda K_1(t, s, u(s), v(s)) + (1 - \lambda)\delta_1)v^\beta(s)ds, \\ v(t) = \displaystyle\int_0^1 (\lambda K_2(t, s, u(s), v(s)) + (1 - \lambda)\delta_2)u^\alpha(s)ds, \end{cases} \tag{2.20}$$

thus the operator $G_\lambda : X \to X$ is of the form

$$G_\lambda(u, v) = \Big(\int_0^1 \widetilde{K}_1(t, s, u(s), v(s), \lambda)v^\beta(s)ds, \int_0^1 \widetilde{K}_2(t, s, u(s), v(s), \lambda)u^\alpha(s)ds \Big), \tag{2.21}$$

where the kernels $\widetilde{K}_1, \widetilde{K}_2$ are given by the right side of (2.20).

Let us verify conditions (A)–(C) for the family (2.21).

The map $\widehat{G} : X \times [0, 1] \to X \times [0, 1]$, defined by $\widehat{G}(x, \lambda) = (G_\lambda(x), \lambda)$, is completely continuous. This follows from the smoothness of $\widetilde{K}_1, \widetilde{K}_2$ (see the first assumption of Theorem 2.1), from $G_\lambda : (C[0, 1])^2 \to (C^1[0, 1])^2$ and from the existence of a completely continuous embedding $i : (C^1[0, 1])^2 \to (C[0, 1])^2$.

The set A_c is an ANR in X and

$$G_\lambda(A_c) \subset A_c \tag{2.22}$$

for each $\lambda \in [0, 1]$. This follows from assumption 2 of Theorem 2.1.

For the proof that the set $\mathrm{Fix}(\widehat{G}, A_c \times [0, 1])$ is a compact subset of $X \times [0, 1]$ we need the following lemma.

LEMMA 2.3. *Suppose that there exist two constants* $0 < r < R$ *such that for every pair* $(x, \lambda) \in A_c \times [0,1]$ *which satisfies* $x = G_\lambda(x)$ *we have*

$$r \leqslant \|x\| \leqslant R. \tag{2.23}$$

Then the set $\mathrm{Fix}(\widehat{G}, A_c \times [0,1])$ *is a compact subset in* $X \times [0,1]$.

Obviously, the set $\mathrm{Fix}(\widehat{G}, X \times [0,1])$ of all fixed points is closed The set $(A_c \cup \{0\}) \times [0,1]$ is closed by its definition (see (2.4)). From the lower a priori estimate $0 < r \leqslant \|x\|$ it follows that the set $\mathrm{Fix}(\widehat{G}, A_c \times [0,1])$ is closed, too. Its boundedness follows from the upper a priori estimate (see (2.23)). The completely continuous map $\widehat{G}$ sends bounded sets to relatively compact sets. Consequently, $\mathrm{Fix}(\widehat{G}, A_c \times [0,1])$ is compact.

Proof of the lower and upper a priori estimate. Let $x = (u, v) \in A_c$ be a solution of the system (2.20) for $\lambda \in [0,1]$. Observe that the kernels $\widehat{K}_1$ and $\widehat{K}_1$ are bounded independently of $\lambda \in [0,1]$:

$$|\widetilde{K}_i(\ldots)| = |\lambda K_i(\ldots) + (1-\lambda)\delta_i| = \lambda |K_i(\ldots)| + (1-\lambda), \tag{2.24}$$

$$\underline{K} \leqslant |\widetilde{K}_i(\ldots)| \leqslant \overline{K}. \tag{2.25}$$

We shall use the following notations:

$$\begin{aligned} \overline{u} &= \max |u(t)|, \qquad \underline{u} = \min |u(t)|, \\ \overline{v} &= \max |v(t)|, \qquad \underline{v} = \min |v(t)|, \end{aligned} \tag{2.26}$$

for $t \in [0,1]$, and

$$A = \int_0^1 |u(s)|^\alpha ds, \qquad B = \int_0^1 |v(s)|^\beta ds. \tag{2.27}$$

From (2.27) and (2.26) we get

$$\underline{u}^\alpha \leqslant A \leqslant \overline{u}^\alpha, \qquad \underline{v}^\beta \leqslant B \leqslant \overline{v}^\beta. \tag{2.28}$$

From (2.20), (2.24) and (2.27) we get

$$\underline{K} B \leqslant \underline{u} \leqslant \overline{u} \leqslant \overline{K} B, \qquad \underline{K} A \leqslant \underline{v} \leqslant \overline{v} \leqslant \overline{K} A. \tag{2.29}$$

From (2.27) and (2.29) we get

$$(\underline{K} B)^\alpha \leqslant A \leqslant (\overline{K} B)^\alpha, \qquad (\underline{K} A)^\beta \leqslant B \leqslant (\overline{K} A)^\beta, \tag{2.30}$$

$$\begin{aligned} \underline{K}^{\,\alpha(\beta+1)} \, A^{\,\alpha\beta} \leqslant A \leqslant \overline{K}^{\,\alpha(\beta+1)} \, A^{\,\alpha\beta}, \\ \underline{K}^{\,\beta(\alpha+1)} \, B^{\,\alpha\beta} \leqslant B \leqslant \overline{K}^{\,\beta(\alpha+1)} \, B^{\,\alpha\beta}. \end{aligned} \tag{2.31}$$

Case **I)** $0 < \alpha\beta < 1$. Then

$$\underline{K}^{\,\frac{\alpha(\beta+1)}{1-\alpha\beta}} \leqslant A \leqslant \overline{K}^{\,\frac{\alpha(\beta+1)}{1-\alpha\beta}}, \qquad \underline{K}^{\,\frac{\beta(\alpha+1)}{1-\alpha\beta}} \leqslant B \leqslant \overline{K}^{\,\frac{\beta(\alpha+1)}{1-\alpha\beta}}, \tag{2.32}$$

$$\underline{K}^{\,\frac{\beta+1}{1-\alpha\beta}} \leqslant \overline{u} \leqslant \overline{K}^{\,\frac{\beta+1}{1-\alpha\beta}}, \qquad \underline{K}^{\,\frac{\alpha+1}{1-\alpha\beta}} \leqslant \overline{v} \leqslant \overline{K}^{\,\frac{\alpha+1}{1-\alpha\beta}}. \tag{2.33}$$

Case **II)** $1 < \alpha\beta$. Then

$$\overline{K}^{\,\frac{\alpha(\beta+1)}{1-\alpha\beta}} \leqslant A \leqslant \underline{K}^{\,\frac{\alpha(\beta+1)}{1-\alpha\beta}}, \qquad \overline{K}^{\,\frac{\beta(\alpha+1)}{1-\alpha\beta}} \leqslant B \leqslant \underline{K}^{\,\frac{\beta(\alpha+1)}{1-\alpha\beta}}, \tag{2.34}$$

$$\underline{K}\, \overline{K}^{\,\frac{\beta(\alpha+1)}{1-\alpha\beta}} \leqslant \overline{u} \leqslant \overline{K}\, \underline{K}^{\,\frac{\beta(\alpha+1)}{1-\alpha\beta}}, \qquad \underline{K}\, \overline{K}^{\,\frac{\alpha(\beta+1)}{1-\alpha\beta}} \leqslant \overline{v} \leqslant \overline{K}\, \underline{K}^{\,\frac{\alpha(\beta+1)}{1-\alpha\beta}}. \tag{2.35}$$

The last two inequalities give lower and upper a priori estimates for $\|x\|$, where $\|x\| = \overline{u} + \overline{v}$. Therefore, the compactness of $\mathrm{Fix}(\widehat{G}, A_c \times [0,1])$ follows from Lemma 2.3.

We verified conditions (A)–(C) for the homotopy $x = G_\lambda(x)$ and so we have proved that this homotopy is admissible. Finally, we have to calculate the Nielsen number for the correspondence G_0.

Note that the image of $G_0 : X \to X$ is the two-dimensional space of constant functions

$$E^2 = \{(c_1, c_2) : c_i \in \mathbb{R}\} \tag{2.36}$$

and thus all its fixed points belong to this plane. Moreover, $G_0(A_c) \subset E_0^2$, where E_0^2 is the punctured plane. The map g defined by (2.10) is the restriction of G_0 to the plane E^2. Finally, we have

$$\mathbf{N}(G_\lambda, A_c) = \mathbf{N}(G_0, A_c) = \mathbf{N}(g, \mathbb{R}_0^2) \tag{2.37}$$

and by Lemma 2.2 we know when this Nielsen number is non-zero. ∎

3. Multidimensional system of integral equations. Consider a system of $2n$ nonlinear integral equations of Urysohn type:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 K_{11}(t,s,x(s)) v_1^{\beta_1}(s)ds, \\ v_1(t) = \displaystyle\int_0^1 K_{12}(t,s,x(s)) u_1^{\alpha_1}(s)ds, \\ \dots \\ u_n(t) = \displaystyle\int_0^1 K_{n1}(t,s,x(s)) v_n^{\beta_n}(s)ds, \\ v_n(t) = \displaystyle\int_0^1 K_{n2}(t,s,x(s)) u_n^{\alpha_n}(s)ds, \end{cases} \tag{3.1}$$

where $x = (u_1, v_1, \dots, u_n, v_n) \in \mathbb{R}^{2n}$.

We assume that the following conditions are satisfied for all $i = 1, \dots, n$ and $j = 1, 2$:

1) $K_{ij}(t,s,x) \in C^1([0,1]^2 \times \mathbb{R}^{2n})$;

2) $\underline{K}^{ij} \leqslant |K_{ij}(t,s,x)| \leqslant \overline{K}_{ij}$ for all $(t,s,x) \in [0,1]^2 \times \mathbb{R}^{2n}$, where $0 < \underline{K}^{ij} \leqslant 1 \leqslant \overline{K}_{ij}$; (3.2)

3) $\alpha_i = n_{i1}/m_{i1},\ \beta_i = n_{i2}/m_{i2} \ \in\ \mathbb{Q}_+$ and $\alpha_i\beta_i \neq 1$.

We shall use the following notation:

$$X = (C[0,1])^{2n}, \qquad A_c^n = A_c \times \dots \times A_c, \qquad (\mathbb{R}_0^2)^n = \mathbb{R}_0^2 \times \dots \times \mathbb{R}_0^2. \tag{3.3}$$

The system (3.1) is equivalent to the operator equation $x = G(x)$, where the operator $G : X \to X$ is defined similarly as in (2.15). The map G is completely continuous and $G(A_c^n) \subset A_c^n$. Note that the system (3.1) has a trivial solution $x_0 = (0, \dots, 0)$.

Theorem 3.1. *Suppose that system (3.1) satisfies conditions 1–3 of (3.2). Then the set* $\mathrm{Fix}(G, A_c^n)$ *is compact. The Nielsen number* $\mathbf{N}(G, A_c^n)$ *is well defined and*

$$\mathbf{N}(G, A_c^n) = \mathbf{N}(g, (\mathbb{R}_0^2)^n), \tag{3.4}$$

where

$$g : \mathbb{R}_0^2 \times \ldots \times \mathbb{R}_0^2 \to \mathbb{R}_0^2 \times \ldots \times \mathbb{R}_0^2 \tag{3.5}$$

is the map given by

$$g(u_1, v_1, \ldots, u_n, v_n) = (\delta_{11} v_1^{\beta_1}, \delta_{12} u_1^{\alpha_1}, \ldots, \delta_{n1} v_n^{\beta_n}, \delta_{n2} u_n^{\alpha_n}) \tag{3.6}$$

with $\delta_{ij} = \operatorname{sign} K_{ij}$ *independent of* (t, s, x).

As in Theorem 2.1, the proof is based on the linear homotopy

$$x = G_\lambda(x), \qquad G_\lambda = \lambda G_0 + (1 - \lambda) G_1, \qquad \lambda \in [0, 1], \tag{3.7}$$

to a simpler system

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 \delta_{11} v_1^{\beta_1}(s) ds, \\ v_1(t) = \displaystyle\int_0^1 \delta_{12} u_1^{\alpha_1}(s) ds, \\ \ldots \\ u_n(t) = \displaystyle\int_0^1 \delta_{n1} v_n^{\beta_n}(s) ds, \\ v_n(t) = \displaystyle\int_0^1 \delta_{n2} u_n^{\alpha_n}(s) ds, \end{cases} \tag{3.8}$$

which is equivalent to the operator equation $x = G_0(x)$. Note that the corresponding operator $G_0 : X \to X$ has a finite-dimensional image in the subspace of constant functions and its restriction is the map g (see (3.5) and (3.6)). The technique of the proof of Theorem 3.1 is analogous to the proof of Theorem 2.1.

Now we give an application of Theorem 3.1.

Example 3.1. Consider a system of three pairs of nonlinear integral equations

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 (1 + \sin^2[t v_1^3(s) + u_3^2(s)]) v_1^7(s) ds, \\ v_1(t) = \displaystyle\int_0^1 (3 + \cos[t u_2(s)]) u_1^5(s) ds, \\ u_2(t) = \displaystyle\int_0^1 (1 + t^2 + s^4) v_2^3(s) ds, \\ v_2(t) = \displaystyle\int_0^1 (3 + t \sin[u_2^4(s)]) u_2^5(s) ds, \\ u_3(t) = \displaystyle\int_0^1 \ln(0.1 + ts/2) v_3^9(s) ds, \\ v_3(t) = \displaystyle\int_0^1 \arctan(2 + u_1^2(s) + t^3 + v_3^4(s)) u_3^4(s) ds, \end{cases} \tag{3.9}$$

where $x = (u_1, v_1, u_2, v_2, u_3, v_3) \in \mathbb{R}^6$ and

$$X = (C[0, 1])^6, \qquad A_c^3 = A_c \times A_c \times A_c, \qquad (\mathbb{R}_0^2)^3 = \mathbb{R}_0^2 \times \mathbb{R}_0^2 \times \mathbb{R}_0^2. \tag{3.10}$$

We reduce the system (3.9) to a finite-dimensional equation $x = g(x)$, where

$$g : \mathbb{R}^6 \to \mathbb{R}^6, \quad g((\mathbb{R}^2_0)^3) \subset (\mathbb{R}^2_0)^3, \tag{3.11}$$

and g is defined by the formula

$$g(u_1, v_1, u_2, v_2, u_3, v_3) = (v_1^7, u_1^5, v_2^3, u_2^5, -v_3^9, u_3^4). \tag{3.12}$$

The equation $x = g(x)$ has 4 solutions in $(\mathbb{R}^2_0)^3$:

$$\begin{aligned} x_1 &= (+1, +1, +1, +1, -1, +1), \\ x_2 &= (-1, -1, +1, +1, -1, +1), \\ x_3 &= (-1, -1, -1, -1, -1, +1), \\ x_4 &= (+1, +1, -1, -1, -1, +1), \end{aligned} \tag{3.13}$$

which belong to different Nielsen classes.

Finally, we have a multiplicity result:

$$\mathbf{N}(G, A_c^n) = \mathbf{N}(g, (\mathbb{R}^2_0)^3) = 4 \tag{3.14}$$

yields that the system (3.9) has at least 4 non-zero solutions.

There is a direct approach to equations (2.1) and (3.1), based on the following theorem.

Theorem 3.2. *Let the conditions* (A)–(C) *be satisfied. Assume that there exist subdomains D_j $(j = 1, \ldots, n)$ in D such that*

$$\begin{aligned} &D_i \cap D_j = \emptyset, \quad i \neq j, \quad x_j^0 \in D_j, \\ &G_\lambda(D_j) \subset D_j, \quad \mathrm{Fix}(G_\lambda, D) \cap \partial D_j = \emptyset, \end{aligned} \tag{3.15}$$

for all $j = 1, \ldots, n$ and $\lambda \in [0,1]$. Then, the equation $x = G_\lambda(x)$ has at least one solution in each subdomain D_j $(j = 1, \ldots, n)$ for each $\lambda \in [0,1]$.

The proof of Theorem 1.1 is based on the following property of degree:

$$\deg(I - G_\lambda, D_j, 0) = \deg(I - G_0, U_j^0, 0) \neq 0. \tag{3.16}$$

Remark 3.1. In the case of the system (2.1) the interior of the annulus A_c may be written as a union of 4 open isolated cones, two of them invariant with respect to the operator G. In the case of the system (3.1) the interior of the annulus A_c^n may be written as a union of 4^n open isolated cones, some of them invariant with respect to the operator G. The technique of a priori estimates and degree property (3.16) may be applied in every invariant cone independently.

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