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CURVATURES OF CONFLICT SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. This article extends to three dimensions results on the curvature of the *conflict curve* for pairs of convex sets of the plane, established by Siersma [3]. In the present case a *conflict surface* arises, equidistant from the given convex sets. The Gaussian, mean curvatures and the location of umbilic points on the *conflict surface* are determined here. Initial results on the Darbouxian type of umbilic points on conflict surfaces are also established. The results are expressed in terms of the principal directions and on the curvatures of the borders of the given convex sets.

1. Introduction. Let A_1 and A_2 be closed non-empty sets in Euclidean space R^3, which is endowed with the standard orientation and the distance

$$d(p,q) = |p-q| = \langle p-q, p-q\rangle^{1/2},$$

where $\langle\ ,\ \rangle$ is the standard Euclidean inner product in R^3.

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The *conflict set* $C(A_1, A_2)$ between A_1 and A_2 is defined by

$$C(A_1, A_2) = \{p;\, d(p, A_1) = d(p, A_2)\},$$

where $d(p, A) = \inf\{d(p, q);\, q \in A\}$. The set $C(A_1, A_2)$ is also viewed as the common boundary between the *territory* of A_1 relative to A_2, defined by

$$\mathrm{Terr}(A_1, A_2) = \{p;\, d(p, A_1) < d(p, A_2)\},$$

and $\mathrm{Terr}(A_2, A_1)$, which is the *territory* of A_2 relative to A_1.

The set $C(A_1, A_2)$ is also called the *bisector* or the *equidistant set* between A_1 and A_2 [7].

In this paper we consider only the case where A_i are convex closed sets, with disjoint interiors and smooth regular boundaries $B_i = \partial A_i$ of class C^k, $k \geq 2$. Assume that the surfaces B_i are *oriented* by their unit normal vector fields N_i, pointing towards the *interior* of A_i.

To each point p on the closed set $E_i = R^3 \setminus \mathrm{Int}(A_i)$, we associate its projection $\Pi_i(p) = p_i$ onto B_i, characterized by

$$d(p, A_i) = \langle \Pi_i(p) - p, N_i(\Pi_i(p))\rangle.$$

From the tubular neighborhood properties and the convexity hypothesis, it follows that Π_i is a retraction of class C^{r-1} of a neighborhood of E_i onto B_i; see [6], Vol. 1. Therefore $C = C(A_1, A_2)$, is defined implicitly by the *zero* level set of the function

$$c(p) = d(p, A_2) - d(p, A_1),$$

which is regular of class C^{r-1} and satisfies

$$\nabla c(p) = N_1(\Pi_1(p)) - N_2(\Pi_2(p)).$$

It is clear that this regularity property fails if the convexity and disjointness hypotheses are not imposed.

In this paper the conflict surface C will be *oriented* by the unit normal N, along ∇c, i.e. pointing from A_2 towards A_1:

$$N = |N_1(\Pi_1(p)) - N_2(\Pi_2(p))|^{-1}[N_1(\Pi_1(p)) - N_2(\Pi_2(p))].$$

To simplify the notation write $\nu(V) = |V|^{-1}V$ for the *normalization* of a nonvanishing vector V. Therefore,

$$N = \nu(N_1(\Pi_1(p)) - N_2(\Pi_2(p))).$$

A positive moving frame attached to C is therefore given by $\{T_1, T_2, N\}$, where:

$$\begin{aligned} T_1 &= \nu\big(N_1(\Pi_1(p)) + N_2(\Pi_2(p))\big),\\ T_2 &= \nu\big(N_1(\Pi_1(p)) \wedge N_2(\Pi_2(p))\big). \end{aligned}$$

Notice that the vector fields T_i are singular at points p of the closed set $M = M(A_1, A_2)$, where $N_1(\Pi_1(p)) + N_2(\Pi_2(p)) = 0$, which occurs when the distance from C to A_i,

$$r(p) = d(p, A_1) = d(p, A_2),$$

is minimal, assuming the value $d_m = \frac{1}{2}d(A_1, A_2)$. Under the condition of *strict convexity* of B_i at points of the sets $\Pi_i(M)$, M reduces to a unique point p_m.

Recall that *strict convexity* of B_i at p_i means that DN_i is an automorphism of the tangent space TB_i, with the usual identification of the tangent space TB_i at p_i with that of the unit sphere TS^2, at $N_i(p_i)$. In terms of the *principal curvatures* $k_1^i \leq k_2^i$, which are the eigenvalues of $-DN_i$, the condition of strict convexity (resp. convexity) means that both are positive (resp. non-negative). In terms of the *Gaussian curvature*

$$\mathcal{K}^i = k_1^i.k_2^i = \det(-DN_i),$$

this means that $\mathcal{K}^i > 0$ (resp. $\mathcal{K}^i \geq 0$). Recall also that the *mean curvature* of B_i is given by

$$\mathcal{H}^i = \frac{1}{2}\,\mathrm{trace}(-DN_i) = \frac{1}{2}(k_1^i + k_2^i).$$

The expression

$$\mathcal{U}^i = \frac{1}{2}\big(k_2^i - k_1^i\big) = \sqrt{(\mathcal{H}^i)^2 - \mathcal{K}^i},$$

also called *skew curvature*, whose zeros locate the *umbilic points* of the surface, will be also of later of use here.

Theorem 1 of this paper establishes expressions for the Gaussian and mean curvatures of the surface $C = C(A_1, A_2)$. These results can be regarded as the starting step for the investigation of the *principal configuration* of C [4]. Recall that this configuration is defined by the principal curvatures, the *umbilic points*, the fields of principal directions (which are the eigenspaces of $-DN_i$), and their integral curves (including the periodic principal lines). The location of umbilic points and the slope of principal directions on $C\backslash M$ is established in Corollary 3. Initial results on the Darbouxian type of the umbilic points on $M = M(A_1, A_2)$ are established in Section 3.

A complete study of the dependence on B_i of the principal configuration on conflict surfaces will not be carried out in this work. However, the curvature formulas of this paper are expressed in terms of some elements of the *principal configurations* of the surfaces B_i.

Global topological properties of conflict sets, such as their connectivity, have been studied in [7]. The basic smoothness and curvature expressions of the conflict curve for convex sets in the plane have been established in [3], where interesting connections with Euclidean geometry of conics can also be found.

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2. Curvatures of conflict surfaces. Let $p \in C\backslash M$ be such that $p_i = \Pi_i(p)$ are non-umbilic points in B_i. Let $\{E_1^i, E_2^i, N_i\}$ be positive principal frames on B_i around p_i. This means that

$$DN_i.E_j^i = -k_j^i E_j^i,$$

where $k_1^i < k_2^i$, $i = 1, 2$, are the principal curvatures of B_i. Denote by α_i the angle between the vectors E_1^i and $F_1^i = \nu(D\Pi_i(p)T_1)$. Write

$$\tau_g^i = \tau_g^i(\alpha_i) = (k_2^i - k_1^i)\sin\alpha_i \cos\alpha_i$$

for the *geodesic torsion* along the unit vector F_1^i on B_i.

Write

$$k_n^i = k_n^i(\alpha_i) = k_1^i \cos^2(\alpha_i) + k_2^i \sin^2(\alpha_i),$$

which, by Euler's formula, is the normal curvature of B_i in the direction of $F_1^i = \nu(D\Pi_i(p)T_1)$.

Similarly, the normal curvature of B_i in the direction of $F_2^i = \nu(D\Pi_i(p)T_2)$ is given by

$$k_{n\perp}^i = k_{n\perp}^i(\alpha_i) = k_n^i\Big(\alpha_i + \frac{\pi}{2}\Big) = k_2^i \sin^2(\alpha_i) + k_1^i \cos^2(\alpha_i).$$

Denote by A_i^r the convex set of points at distance $r \geq 0$ from A_i; its border is the surface B_i^r obtained by moving each point p_i on B_i to $\Sigma^r(p_i) = p_i - rN_i(p_i)$. By restriction, Π_i defines the diffeomorphism $(\Sigma^r)^{-1}$ of B_i^r onto B_i and, by composition, it defines the retraction $\Pi_i^r = \Pi_i \circ (\Sigma^r)^{-1}$ onto B_i^r.

The principal frames $\{E_1^i, E_2^i, N_i\}$ on B_i are parallel translated along the normals to principal frames on B_i^r. This follows from the fact that Σ^r preserves the principal direction fields as well as the umbilic points. The principal curvatures however change into $k_j^i(r) = \frac{k_j^i}{1+rk_j^i}$, [6], Vol. 3.

The above expressions for τ_g^i, k_n^i and $k_{n\perp}^i$ on B_i, being defined in terms of principal curvatures, can be obviously modified to be valid on B_i^r and denoted respectively by $\tau_g^i(r)$, $k_n^i(r)$ and $k_{n\perp}^i(r)$. For instance:

$$\begin{aligned}
\tau_g^i(r) &= \tau_g^i(r,\alpha_i) = (k_2^i(r) - k_1^i(r)) \sin\alpha_i \cos\alpha_i,\\
k_n^i(r) &= k_1^i(r)\cos^2(\alpha_i) + k_2^i(r)\sin^2(\alpha_i),\\
k_{n\perp}^i(r) &= k_2^i(r)\sin^2(\alpha_i) + k_1^i(r)\cos^2(\alpha_i).
\end{aligned}$$

Denote by ϕ the angle between N_i and T_1. For future reference, notice that

$$\sin\phi = \frac{1}{2}\big|N_1(\Pi_1(p)) - N_2(\Pi_2(p))\big| = \langle F_1^1, T_1\rangle = \langle F_1^2, T_1\rangle.$$

Theorem 1. *Let* $\mathcal{K}^i = \mathcal{K}^i(r) = k_1^i(r)k_2^i(r)$, $\mathcal{H}^i = \mathcal{H}^i(r) = \frac{1}{2}(k_1^i(r) + k_2^i(r))$ *and* $\mathcal{U}^i = \mathcal{U}^i(r) = \frac{1}{2}(k_2^i(r) - k_1^i(r))$, *where* $k_j^i(r) = \frac{k_j^i}{1+rk_j^i}$.

a) *The Gaussian curvature of the conflict surface C is given by*

$$\mathcal{K} = \frac{1}{2}\Big[\frac{1}{2}(\mathcal{K}^1 + \mathcal{K}^2) - \mathcal{H}^1\mathcal{H}^2 + \cos(2(\alpha_2 - \alpha_1))\mathcal{U}^1\mathcal{U}^2\Big].$$

b) *The mean curvature of the conflict surface C is given by*

$$\mathcal{H} = \Big(\frac{1+\sin^2\phi}{4\sin^2\phi}\Big)\Big[(\mathcal{H}^1 - \mathcal{H}^2) + \Big(\frac{\cos^2\phi}{1+\sin^2\phi}\Big)\big(\cos(2\alpha_1)\mathcal{U}^1 - \cos(2\alpha_2)\mathcal{U}^2\big)\Big].$$

The proof of this theorem will follow from the next proposition.

Proposition 2. *With the above notation, at points of* $C\backslash M$

$$\begin{aligned}
DN.T_1 &= \frac{\sin\phi}{2}\big[k_n^2(r) - k_n^1(r)\big]T_1 + \frac{1}{2}\big[\tau_g^2(r) - \tau_g^1(r)\big]T_2,\\
DN.T_2 &= \frac{1}{2}[\tau_g^2(r) - \tau_g^1(r)]T_1 + \frac{1}{2\sin\phi}\big[k_{n\perp}^2(r) - k_{n\perp}^1(r)\big]T_2.
\end{aligned}$$

P r o o f. The conclusion follows from the calculation of the inner products in

$$DN(T_1) = \langle DN(T_1), T_1\rangle T_1 + \langle DN(T_1), T_2\rangle T_2,$$
$$DN(T_2) = \langle DN(T_2), T_1\rangle T_1 + \langle DN(T_2), T_2\rangle T_2.$$

Differentiation of N gives:

$$DN = \frac{DN_1.D\Pi_1^r - DN_2.D\Pi_2^r}{2\sin\phi} + 2D\big[(\sin\phi)^{-1}\big].\big[N_1(\Pi_1^r(p)) - N_2(\Pi_2^r(p))\big].$$

This shows that for the present analysis the contribution of the second term is null.

The basis $\{T_1, T_2\}$ on C projects along Π_i^r onto the orthonormal bases $\{F_1^i, F_2^i\}$ on B_i^r. Calculation shows that

$$\{F_1^1 = v(N_2 - \langle N_1, N_2\rangle N_1),\ F_2^1 = T_2\},$$
$$\{F_1^2 = v(-N_1 + \langle N_1, N_2\rangle N_2),\ F_2^2 = T_2\}.$$

Therefore,

$$D\Pi_i^r(T_1) = \sin\phi F_1^i \quad \text{and} \quad D\Pi_i^r(T_2) = F_2^i.$$

The bases E^i and F^i are related by

$$F_1^i = \cos(\alpha_i)E_1^i + \sin(\alpha_i)E_2^i, \quad F_2^i = -\sin(\alpha_i)E_1^i + \cos(\alpha_i)E_2^i,$$

$$\begin{aligned}
DN_i.D\Pi_i^r(T_1) &= \sin\phi DN_i\big(\cos(\alpha_i)E_1^i + \sin(\alpha_i)E_2^i\big)\\
&= \sin\phi\big[\cos(\alpha_i)DN_i(E_1^i) + \sin(\alpha_i)DN_i(E_2^i)\big]\\
&= \sin\phi\big[-k_1^i(r)\cos(\alpha_i)E_1^i - k_2^i(r)\sin(\alpha_i)E_2^i\big]\\
&= \sin\phi\big\{-k_1^i(r)\cos(\alpha_i)[\cos(\alpha_i)F_1^i - \sin(\alpha_i)F_2^i]\\
&\qquad - k_2^i(r)\sin(\alpha_i)[\cos(\alpha_i)F_1^i + \sin(\alpha_i)F_2^i]\big\}\\
&= \sin\phi\big\{[-k_1^i(r)\cos^2(\alpha_i) - k_2^i(r)\sin^2(\alpha_i)]F_1^i\\
&\qquad + [(k_1^i(r) - k_2^i(r))\sin(\alpha_i)\cos(\alpha_i)]F_2^i\big\}\\
&= \sin\phi\big\{[-k_n^i(r)]F_1^i + [-\tau_g^i(r)]F_2^i\big\}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
DN_i.D\Pi_i^r(T_2) &= DN_i\Big(\cos\Big(\alpha_i + \frac{\pi}{2}\Big)E_1^i + \sin\Big(\alpha_i + \frac{\pi}{2}\Big)E_2^i\Big)\\
&= \big[k_1^i(r)\sin(\alpha_i)E_1^i - k_2^i(r)\cos(\alpha_i)E_2^i\big]\\
&= \big\{k_1^i(r)\sin(\alpha_i)[\cos(\alpha_i)F_1^i - \sin(\alpha_i)F_2^i]\\
&\qquad - k_2^i(r)\cos(\alpha_i)[\sin(\alpha_i)F_1^i + \cos(\alpha_i)F_2^i]\big\}\\
&= \big\{k_1^i(r)\sin(\alpha_i)\cos(\alpha_i)F_1^i - k_1^i(r)\sin^2(\alpha_i)F_2^i\\
&\qquad - k_2^i(r)\cos(\alpha_i)\sin(\alpha_i)F_1^i - k_2^i(r)\cos^2(\alpha_i)F_2^i\big\}\\
&= \big\{[(k_1^i(r) - k_2^i(r))\sin(\alpha_i)\cos(\alpha_i)]F_1^i\\
&\qquad - [k_1^i(r)\sin^2(\alpha_i) + k_2^i(r)\cos^2(\alpha_i)]F_2^i\big\}\\
&= \big\{[-\tau_g^i(r)]F_1^i - [k_{n\perp}^i(r)]F_2^i\big\}.
\end{aligned}$$

Therefore,

$$DN(T_1) = \frac{\sin\phi}{2\sin\phi}\big\{[-k_n^1(r)F_1^1 - \tau_g^1(r)F_2^1] - [-k_n^2(r)F_1^2 - \tau_g^2(r)F_2^2]\big\}$$

$$= \frac{1}{2}\big\{[k_n^2(r)F_1^2 - k_n^1(r)F_1^1] + [\tau_g^2(r)]F_2^2 - [\tau_g^1(r)F_2^1]\big\},$$

$$DN(T_2) = \frac{1}{2\sin\phi}\big\{[-\tau_g^1(r)]F_1^1 - [k_{n\perp}^1(r)]F_2^1 + [\tau_g^2(r)]F_1^2 + [k_{n\perp}^2(r)]F_2^2\big\}$$

$$= \frac{1}{2\sin\phi}\big\{[k_{n\perp}^2(r)F_2^2 - k_{n\perp}^1(r)F_2^1] + [\tau_g^2(r)]F_1^2 - [\tau_g^1(r)]F_1^1\big\}.$$

Performing the inner products, taking into account that

$$\sin\phi = \cos\Big(\frac{\pi}{2} - \phi\Big) = \langle F_1^1, T_1\rangle = \langle F_1^2, T_1\rangle,$$

finishes the proof. ∎

Remark 1. Notice that the proof of the last proposition uses only the principal frames at points (and not on open sets) of B^i. Therefore the calculations for DN hold also at points of $C\backslash M$ whose projections on either one (or both) of the surfaces B^i are umbilic points, in that case the corresponding expressions $\tau_g^i(r)$ vanish. Proposition 4 will deal with a preliminary analysis of the case of points on M, where T_1 and the angles α_i are not defined.

Proof of Theorem 1.

a) The Gaussian curvature of C is given by $\det(-DN)$. Therefore, taking into account that

$$k_n^2(r) = \mathcal{H}^2 - \mathcal{U}^2\cos 2\alpha_2, \quad k_n^1(r) = \mathcal{H}^1 - \mathcal{U}^1\cos 2\alpha_1, \quad k_{n\perp}^2(r) = \mathcal{H}^2 + \mathcal{U}^2\cos 2\alpha_2,$$
$$k_{n\perp}^1(r) = \mathcal{H}^1 + \mathcal{U}^1\cos 2\alpha_1, \quad \tau_g^2(r) - \tau_g^1(r) = \mathcal{U}^2\sin 2\alpha_2 - \mathcal{U}^1\sin 2\alpha_1$$
$$\text{and} \quad (\mathcal{U}^i)^2 = (\mathcal{H}^i)^2 - \mathcal{K}^i,$$

we obtain

$$\begin{aligned}\mathcal{K} &= \frac{1}{4}\big\{[k_n^2(r) - k_n^1(r)][k_{n\perp}^2(r) - k_{n\perp}^1(r)] - [\tau_g^2(r) - \tau_g^1(r)]^2\big\}\\ &= \frac{1}{4}\big\{(\mathcal{H}^2 - \mathcal{U}^2\cos 2\alpha_2) - (\mathcal{H}^1 - \mathcal{U}^1\cos 2\alpha_1)][(\mathcal{H}^2 + \mathcal{U}^2\cos 2\alpha_2)\\ &\qquad - (\mathcal{H}^1 + \mathcal{U}^1\cos 2\alpha_1)] - [\mathcal{U}^2\sin 2\alpha_2 - \mathcal{U}^1\sin 2\alpha_1]^2\big\}\\ &= \frac{1}{2}\Big[\frac{1}{2}(\mathcal{K}^1 + \mathcal{K}^2) - \mathcal{H}^1\mathcal{H}^2 + \cos(2(\alpha_2 - \alpha_1))\mathcal{U}^1\mathcal{U}^2\Big].\end{aligned}$$

b) The mean curvature is given by $\mathrm{trace}(-\frac{1}{2}DN)$.

From the relations

$$k_n^2(r) - k_n^1(r) = \mathcal{H}^2 - \mathcal{H}^1 + \mathcal{U}^1\cos 2\alpha_1 - \mathcal{U}^2\cos 2\alpha_2,$$
$$\text{and} \;\; k_{n\perp}^2(r) - k_{n\perp}^1(r) = \mathcal{H}^2 - \mathcal{H}^1 + \mathcal{U}^2\cos 2\alpha_2 - \mathcal{U}^1\cos 2\alpha_1,$$

we deduce using also the expressions in part a) that

$$\begin{aligned}\mathcal{H} &= -\frac{1}{4}\Big\{\sin\phi[k_n^2(r) - k_n^1(r)] + \frac{1}{\sin\phi}[\kappa(r, k_{n\perp}^2) - \kappa(r, k_{n\perp}^1)]\Big\}\\ &= \Big(\frac{1+\sin^2\phi}{4\sin^2\phi}\Big)\Big[(\mathcal{H}^1 - \mathcal{H}^2) + \Big(\frac{\cos^2\phi}{1+\sin^2\phi}\Big)(\cos(2\alpha_1)\mathcal{U}^1 - \cos(2\alpha_2)\mathcal{U}^2)\Big].\end{aligned}$$ ∎

COROLLARY 3.
1) *The point* $p \in C\backslash M$ *is umbilic if and only if* $\Upsilon = \Psi = 0$, *where*

$$\begin{aligned}
\Upsilon &= [\tau_g^2(r) - \tau_g^1(r)] = \mathcal{U}^2 \sin 2\alpha_2 - \mathcal{U}^1 \sin 2\alpha_1,\\
\Psi &= \sin^2\phi[k_n^2(r) - k_n^1(r)] - [k_{n\perp}^2(r) - k_{n\perp}^1(r)]\\
&= (\mathcal{H}^2 - \mathcal{H}^1)\cos^2\phi + (1+\sin^2\phi)(\mathcal{U}^1\cos 2\alpha_1 - \mathcal{U}^2\cos 2\alpha_2)\\
&= \frac{(3-\cos 2\phi)}{2}\Big[(\mathcal{H}^1 - \mathcal{H}^2)\frac{(1+\cos 2\phi)}{(3-\cos 2\phi)} + (\mathcal{U}^1\cos 2\alpha_1 - \mathcal{U}^2\cos 2\alpha_2)\Big].
\end{aligned}$$

2) *The principal directions at a non-umbilic point* p *in* $C\backslash M$ *are characterized by the condition of making an angle* θ *with* T_1, *given by*

$$\tan 2\theta = \frac{2[\tau_g^2(r) - \tau_g^1(r)]\sin\phi}{\sin^2\phi[k_n^2(r) - k_n^1(r)] - [k_{n\perp}^2(r) - k_{n\perp}^1(r)]} = \frac{2\Upsilon\sin\phi}{\Psi}.$$

Proof. Immediate from Proposition 1. ■

PROPOSITION 4. *Assume that* $p = p_m \in M$ *and that* $p_i = \Pi_i(p)$ *are non-umbilic points in* B_i. *The normal curvature* $k_n = k_n(\theta)$ *of* C *in the direction of an angle* θ_i *with the first vector of the parallel basis* $\{E_1^i, E_2^i, N_i\}$ *is given by*

$$k_n(\theta) = -\frac{1}{2}\Big[\frac{k_n^2(\theta_2)}{1 + r k_n^2(\theta_2)} - \frac{k_n^1(\theta_1)}{1 + r k_n^1(\theta_1)}\Big].$$

Proof. Similar to that of Proposition 1, replacing the frame $\{T_1, T_2, N\}$ by any tangent frame to C at p. ■

Remark 2. Taking into account that the two tangent frames differ by an angle α, and therefore $\theta_2 = \theta_1 + \alpha$ the above equation can be simplified to:

$$k_n(\theta) = -\frac{1}{2}\Big[\frac{k_n^2(\theta_1+\alpha)}{1 + r k_n^2(\theta_1+\alpha)} - \frac{k_n^1(\theta_1)}{1 + r k_n^1(\theta_1)}\Big].$$

Remark 3. If both p_i are umbilic then the point $p = p_m$ is umbilic, the other case leading to umbilic points on M is when the principal frames at p_i are parallel and the principal curvatures verify $k_2^2 - k_2^1 = k_1^2 - k_1^1 = -k$. The case $k = 0$ is studied in partial detail in next section. Pertinent changes for $k \neq 0$ are made in Remark 4.

3. An introduction to umbilics on conflict surfaces. In this section it will be shown how to determine the *Darbouxian type* of an umbilic point $p_m \in M$. Recall from [1], [4], that this *type*, (D_1, D_2, D_3), depends on the 3-jet of the surface at the point and it determines the behavior of lines of curvature near the umbilic point. To this end in what follows, the 3-jet of the conflict surface at p, in Monge form will be calculated in terms of the corresponding 3-jets of the surfaces B_1 and B_2 at p_1 and p_2.

Let the convex surfaces B_1 and B_2 be locally given in Monge charts (x, y) and (u, v) by the graphs of

$$\begin{aligned}
f_1(x,y) &= r + \frac{a}{2}x^2 + \frac{b}{2}y^2 + \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{30}y^3 + \ldots),\\
f_2(u,v) &= -r - \frac{a}{2}u^2 - \frac{b}{2}v^2 - \frac{1}{6}(b_{30}u^3 + 3b_{21}u^2v + 3b_{12}uv^2 + b_{30}v^3 + \ldots).
\end{aligned}$$

Let $P = (X, Y, Z)$ be a point of R^3. By the considerations above it follows that $p = (0, 0, 0) \in C$.

The coordinate expressions of the second order jets at p of the projections $\Pi_1(P)$ and $\Pi_2(P)$, with targets expressed in the Monge coordinates (x, y) and (u, v) will be computed now.

The projection $\Pi_1(X, Y, Z) = (x, y)$, in coordinate expression, is defined implicitly by the equations

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0,$$

where $F(x, y, X, Y, Z) = d(P, B_1)^2 = (X - x)^2 + (Y - y)^2 + (Z - f_1(x, y))^2$.

By using the Implicit Function Theorem, after extensive calculation, it is obtained that the solution of the system of equations above is given by

$$\begin{aligned} x = {} & \frac{1}{1+ra}X - \frac{ra_{30}}{2(1+ra)^3}X^2 - \frac{ra_{21}}{(1+rb)(1+ra)^2}XY \\ & - \frac{ra_{12}}{2(1+ra)(1+rb)^2}Y^2 + \frac{a}{(1+ra)^2}XZ + \dots, \\ y = {} & \frac{1}{1+rb}Y - \frac{ra_{21}}{2(1+rb)(1+ra)^2}X^2 - \frac{ra_{12}}{(1+ra)(1+rb)^2}XY \\ & - \frac{ra_{03}}{2(1+rb)^3}Y^2 + \frac{b}{(1+rb)^2}YZ + \dots . \end{aligned}$$

Similarly, considering the function

$$G(x, y, X, Y, Z) = d(p, B_2)^2 = (X - u)^2 + (Y - v)^2 + (Z - f_2)^2,$$

we obtain that the projection $\Pi_2(X, Y, Z) = (u, v)$, in coordinate expression, is given by

$$\begin{aligned} u = {} & \frac{1}{1+ra}X + \frac{rb_{30}}{2(1+ra)^3}X^2 + \frac{rb_{21}}{(1+rb)(1+ra)^2}XY \\ & + \frac{rb_{12}}{2(1+ra)(1+rb)^2}Y^2 - \frac{a}{(1+ra)^2}XZ + \dots, \\ v = {} & \frac{1}{1+rb}Y + \frac{rb_{21}}{2(1+rb)(1+ra)^2}X^2 + \frac{rb_{12}}{(1+ra)(1+rb)^2}XY \\ & + \frac{rb_{03}}{2(1+rb)^3}Y^2 - \frac{b}{(1+rb)^2}YZ + \dots . \end{aligned}$$

Recall from the Introduction that the conflict surface C is defined by the equation

$$c(X, Y, Z) = d\big((X, Y, Z), (u, v, f_2(u, v))\big) - d\big((X, Y, Z), (x, y, f_1(x, y))\big) = 0,$$

with (x, y) and (u, v) representing the projections $\Pi_1(X, Y, Z)$ and $\Pi_2(X, Y, Z)$, given by the expressions above.

An extensive calculation leads to the following expression for the 3-jet of the Monge representation of C in a neighborhood of $p = (0, 0, 0)$:

$$\begin{aligned} Z = {} & \frac{(a_{30} - b_{30})}{2(1+ar)^3}\frac{X^3}{6} + \frac{(a_{21} - b_{21})}{2(1+br)(1+ar)^2}\frac{X^2Y}{2} \\ & + \frac{(a_{12} - b_{12})}{2(1+ar)(1+br)^2}\frac{XY^2}{2} + \frac{(a_{03} - b_{03})}{2(1+br)^3}\frac{Y^3}{6} + \dots . \end{aligned}$$

This follows by using the Implicit Function Theorem applied to the function $c(X, Y, Z)$, and observing that $c(0,0,0) = 0$ and $\frac{\partial c}{\partial Z}(0,0,0) = 4r \neq 0$.

Since the Darbouxian type of the umbilic is defined by semialgebraic conditions on the coefficients of $j^3 Z(0,0)$, the behavior of lines of curvature near p can be expressed in terms of the coefficients of the 3-jets of the surfaces B_i at p_i.

Remark 4. Under generic conditions on the coefficients, the cubic form above determines the index ($\pm\frac{1}{2}$) as well as the singularity (hyperbolic, elliptic) and Darbouxian (D_1, D_2, D_3) types of the umbilic of the conflict surface C.

In the example discussed above the caustic umbilic (on the focal surface) is located at infinity. By taking $a+k$ and $b+k$, $k \neq 0$, instead of a and b in f_1 and keeping f_2 unchanged, the caustic umbilic of the conflict surface moves to $(0, 0, \frac{1}{k})$. The index of the umbilic, as well as the Darbouxian and singularity types of the umbilic will be also determined by the coefficients of the cubic form, changed accordingly to the presence of constant k. The generic conditions referred invoked above are expressed by the non-vanishing of a quadratic form, for the index, to which the non-vanishing of pertinent cubic forms should be added for the singularity and Darbouxian classifications [2].

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