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THE CENTRE SYMMETRY SET

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Abstract. A centrally symmetric plane curve has a point called it's *centre of symmetry*. We define (following Janeczko) a set which measures the central symmetry of an arbitrary strictly convex plane curve, or surface in $\mathbf{R}^3$. We investigate some of it's properties, and begin the study of non-convex cases.

1. Introduction. The concept of *Central Symmetry* and the notion of a *centre* of symmetry for a closed plane curve is very familiar: a plane curve Γ is said to have a centre c if, for all points x on Γ, the point $2c - x$ is also on Γ, and we say that Γ is *centrally symmetric* about c. This concept is extremely restrictive in that most closed curves in the plane do not have a centre of symmetry, and this leads us to attempt to generalize this concept.

The idea of generalizing classical concepts of plane symmetry began with the study of the *Symmetry Set* (SS), a generalization of an *axis of reflexional symmetry*. The differential structure of the SS has been studied extensively in [4], [2], [5]: the structures of the SS of a generic plane curve have been classified, as have the possible transitions on the SS of a plane curve as it is deformed through a 1-parameter family. Higher dimensional analogues of the SS have also been studied. Current research focuses on attempts to introduce *affine invariant* analogues of the SS for plane curves (see [6]).

Now central symmetry is a global property of a curve; however, we may consider a closed curve to have some degree of *local* central symmetry between segments of itself, and hence we must attempt to broaden our idea of central symmetry in order to study the large class of curves which do not conform to the strict limitations of the classical definition of central symmetry.

One attempt at a generalization begins with the observation that, for a centrally symmetric curve Γ with centre c, the tangents at each pair of points x and $2c - x$ are

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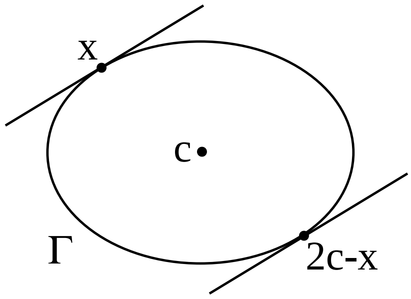
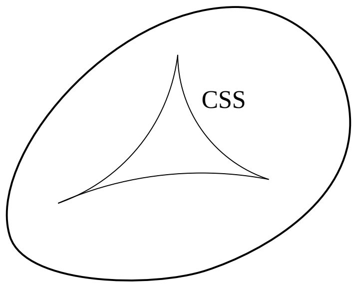
parallel (see the figure below, where Γ is an ellipse). Now if we were to construct the chord joining each of these pairs, then we notice that the envelope of these chords is the centre c (since the lines are concurrent). For a general strictly convex closed curve, consider the



following construction: we first of all find the *parallel tangent pairs* (the pairs of points on the curve at which the tangents are parallel), join them with a line, and find the envelope of these lines. In general we should expect, instead of a single point as a centre, a set of points which we can think of as capturing some aspect of the local symmetry of the curve. This leads us to the definition of the simplest version of the *Centre Symmetry Set*:

DEFINITION 1. The *Centre Symmetry Set* (CSS) of a strictly convex plane curve is the envelope of lines joining points of contact of parallel tangent pairs.

In fact, we shall be at pains to remove the condition of convexity in this definition by studying the effects of non-convex situations. Note that this definition depends only upon concepts which are affine invariant, and hence the construction of the CSS is itself affine invariant. Below is an example of the CSS of a strictly convex plane curve (which we will often refer to as an *oval*). Note that the CSS is a closed curve containing 3 cusps: in Section 4 we study this example in greater detail, and find that we are able to derive conditions on the number of cusps that the CSS of an oval may exhibit. Another inter-



esting example is that of a closed curve with constant width ('*constant width*' means that the chord between parallel tangent pairs is of some constant length; as such, the term '*constant width*' can only be applied to ovals). It is not hard to show that, for a curve of constant width, the line joining a pair of points of contact of parallel tangents is along the common normals to the curve at these points. Hence the CSS is set-wise equivalent to the envelope of normals to the curve, commonly known as the *evolute* (more precisely, the evolute is the double cover of the CSS).

The first attempt to create a generalization of central symmetry is made by S. Janeczko in [9]: here, the centre symmetry set of an oval is defined to be the *bifurcation set of a family of 'ratio-of-distances' functions on the curve parametrized by points in the plane*, and it is shown that the centre symmetry set of a generic oval has only fold and cusp singularities. In Sections 2 and 3 of this paper we show that the 'envelope' definition of the CSS above is entirely consistent with Janeczko's 'ratio-of-distances' definition of the CSS, when we restrict our study to that of strictly convex plane curves.

Sections 2–5 are concerned with discovering the local structure of the CSS using the envelope definition. Now since we are thinking of the CSS as an envelope, we expect it to be *smooth* in general, and exhibit *cusps* at isolated points, the points of regression of the envelope. For a generic plane curve, we expect to find *double tangents* — lines which are tangent at two distinct points — and *inflexions*: in Section 4 we consider in detail the effect that these cases have on the local structure of the CSS. In Section 5, we begin to study 1-parameter transitions on the CSS as the curve is deformed through a 1-parameter family: we expect to find some higher singularities of the CSS at isolated points. In Section 6, we begin to set out some ideas concerning the formulation of an analogous concept of a centre symmetry set in 3 dimensions. Finally, in Section 7, we consider some non-convex situations.

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2. The Centre Symmetry Set as an envelope of lines. Our first task is to find the envelope point of this family of lines joining points of contact of parallel tangent pairs, which we will refer to as the *CSS point*. We will set out in detail the method of coordinatewise calculation. First of all we set up our local coordinate system in the following way, as illustrated in Figure 1: consider two segments of a smooth plane curve γ — the first through $(0,0)$ and given by $\gamma_1(t) = (t, f(t))$, where $f(0) = f'(0) = 0$ (so this segment is tangent to the x-axis at $t = 0$), and the second through the point (c, d), given by $\gamma_2(u) = (c - u, d + g(u))$, with $g(0) = g'(0) = 0$ (so the tangent at $u = 0$ is parallel to the x-axis). The condition on t and u for the tangents at $\gamma_1(t)$ and $\gamma_2(u)$ to be parallel is

$$f'(t) = -g'(u), \tag{1}$$

where $'$ signifies the derivative with respect to the corresponding parameter t or u. Assuming that we have no inflexion on the upper curve segment, we solve this *parallel tangents condition* for $u = U(t)$. This gives us a family of lines parametrized by t:

$$F(t, x, y) \equiv (x - t)\big(d + g(U(t)) - f(t)\big) - \big(y - f(t)\big)(c - U(t) - t). \tag{2}$$

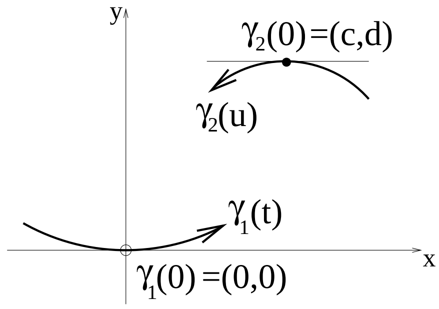

Figure 1

The envelope of this family of lines is given by solving $F = \partial F/\partial t = 0$ for x and y. At $t = U(t) = 0$ we find that the CSS point along the line joining $(0, 0)$ to (c, d) is given by the simultaneous equations

$$\begin{aligned} xd - yc &= 0 \\ -d + y(U'(0) + 1) &= 0. \end{aligned}$$

Note that if $d = 0$ then these conditions are satisfied for arbitrary x, and hence the entire x-axis $(x, 0)$ is a solution to this system. This tells us that, in the case of a double tangent, the tangent itself is part of the CSS. For $d \neq 0$, if we denote the curvature of the lower and upper segments by κ_1 and κ_2 respectively, then we have solution

$$(x, y) = \left(\frac{c\kappa_2}{\kappa_1 + \kappa_2}, \frac{d\kappa_2}{\kappa_1 + \kappa_2} \right),$$

with the assumption that $\kappa_1 + \kappa_2 \neq 0$ (here κ_1, κ_2 are evaluated at $t = U = 0$). We note that if $\kappa_1 + \kappa_2 = 0$, then the CSS point is *at infinity*. Thus we have:

THEOREM 1. *The CSS point* (x, y) *is the point on the chord joining pairs of points of contact of parallel oriented tangents* (*with opposite orientation*) *for which the ratio of distances from the two points is the same as the reciprocal ratio of the two curvatures*, *i.e. the CSS point divides the segment joining points with parallel tangents in the ratio*

$$v_1 : v_2 = \kappa_2 : \kappa_1,$$

where v_i *is the oriented distance from the envelope point to the curve segment* γ_i.

This result is expressed implicitly in [9], and it follows that we have done enough to show that the envelope definition and the ratio-of-distances definition of the CSS are identical in the case of strictly convex plane curves.

In general, taking (x_1, y_1) and (x_2, y_2) to be the points on the respective curve segments for which the oriented tangents are parallel but with opposite direction, we find that the CSS point is given by

$$(x, y) = \left(\frac{x_1\kappa_1 + x_2\kappa_2}{\kappa_1 + \kappa_2}, \frac{y_1\kappa_1 + y_2\kappa_2}{\kappa_1 + \kappa_2} \right). \tag{3}$$

If the orientations of the curve segments are the same, then the sign of one of the curvatures κ_i is reversed. There is an interesting corollary to Theorem 1: consider Figure 2, where e_1, e_2 denote the centres of curvature of curve segments γ_1 and γ_2 evaluated at $t = u = 0$. Line l_1 joins the points of contact of parallel tangent pairs at parameter values $t = u = 0$, and line l_2 joins the centres of curvature e_1, e_2. By considering similar triangles, it is not hard to see that the intersection of l_1 and l_2 is the corresponding CSS point p.

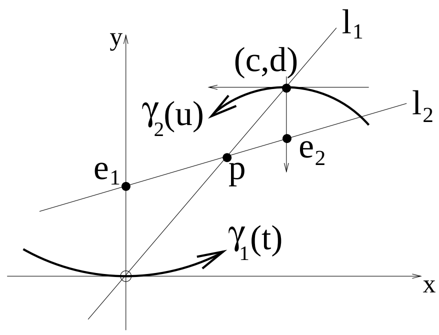

Figure 2

3. The local structure of the CSS. We now begin to study the local structure of the CSS of a generic plane curve γ. Note we are concerned with the *local* properties of the CSS, and will not assume that γ is convex. Now we are thinking of F (see (2)) as a function of t with parameters x and y, and in this setting the CSS is the bifurcation diagram of zeroes of F, i.e. points (x, y) for which $F = \partial F/\partial t = 0$ for some t. We may deduce the local structure of the CSS by examining the multiplicity of these zeroes: this is equivalent to finding conditions under which F has an $A_{\geq k}$ singularity on its zero level for $k \geq 2$. We begin by deriving conditions on the curvature of γ for F to have an $A_{\geq 2}$ point on its zero level, i.e. for the CSS to have a cusp, a situation which we expect to observe on a generic envelope. A short calculation gives us:

THEOREM 2. *The CSS is singular if and only if the same line is tangent at both points* $(d = 0)$, *or*

$$\kappa_1' \kappa_2^2 - \kappa_1^2 \kappa_2' = 0. \tag{4}$$

The double tangent situation $(d = 0)$ is considered in detail in Section 4. For now we will concentrate on the second condition above, which we will refer to as the '*cusp condition*'. Now we can write this cusp condition in a more succinct form: writing $\rho_i = 1/\kappa_i$ as the radius of curvature of curve segment γ_i, we see that the condition for a non-smooth CSS (away from $d = 0$) becomes

$$\rho_1' = \rho_2'.$$

Remember that $'$ is used to denote the derivative with respect to the parameter along the corresponding segment. To get a better geometric sense of this cusp condition, we parametrize both curve segments by the same parameter t, and use the fact that $dU/dt = \kappa_1/\kappa_2$ (recall that $u = U(t)$ comes from 1, the *parallel tangents condition*).

A brief calculation shows that

$$\kappa_1'\kappa_2^2 - \kappa_1^2\kappa_2' = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{\kappa_2}{\kappa_1}\right) = 0. \tag{5}$$

Now we recall that the envelope point (x, y) divides the chord joining points of contact of parallel tangents pairs in the ratio $v_1/v_2 = \kappa_2/\kappa_1$. Thus a cusp appears on the CSS if the ratio-of-distances function v_1/v_2 has a critical point. This leads us back once again to the original definition of the CSS as the bifurcation set of the family of ratio-of-distances function (see [9]), and verifies that these two definitions are identical for strictly convex curves.

R e m a r k 1. We can write this cusp condition in an affine invariant way. A short calculation shows that the *affine normal* (see Buchin Su [11], p. 9, for a definition of an affine normal, and a general introduction to affine differential geometry) to curve segment γ_1 at $t = 0$ has direction $(-f''', 3(f'')^2)$, and the affine normal to the curve segment γ_2 at $u = 0$ has direction $(g''', 3(g'')^2)$, where γ_1, γ_2 are as given in Section 2 (see Figure 1). Now (4) holds if and only if

$$\frac{f'''}{(f'')^2} = \frac{-g'''}{(g'')^2}$$

which in turn implies that the affine normals at $(0, 0)$ and (c, d) are parallel. In fact, we have more information than this: by comparing the ratio of these two affine normals, we see that the necessary and sufficient affine condition for the CSS to exhibit a cusp is that

$$\ddot{\gamma}_1(0) + \left(\frac{\kappa_1}{\kappa_2}\right)^{1/3} \ddot{\gamma}_2(0) = \mathbf{0},$$

where $\dot{}$ denotes the derivative with respect to the affine arclength parameter.

4. Inflexions and double tangents

4.1. Inflexion on one curve segment. Suppose that there is an inflexion on the lower curve segment of Figure 1, i.e. that $\kappa_1(0) = 0$, and suppose also that $d \neq 0$ (so we don't have a double tangent). Theorem 1 tells us that the CSS passes through the upper curve segment at the point where we have a parallel tangent to the inflexional tangent to the lower curve segment, namely the point (c, d). Furthermore, Theorem 2 tells us that the CSS is smooth at this point if and only if $\kappa_1'(0)\kappa_2(0)^2 \neq 0$, i.e. if and only if the lower curve segment has an ordinary inflexion and the upper curve segment has no inflexion. This gives us:

THEOREM 3. *Suppose* (i) *there is an ordinary inflexion at one point* $\gamma_1(t)$, (ii) *there is no inflexion at the corresponding point* $\gamma_2(U(t))$ *for which there is a parallel tangent*, *and* (iii) $\gamma_2(U(t))$ *does not lie on the inflexional tangent at* $\gamma_1(t)$ (*i.e.* $d \neq 0$). *Then the CSS is smooth and passes through* $\gamma_2(U(t))$.

Now it is trivial to see that, near to $\gamma_2(U(t))$, the CSS lies entirely to one side of the line joining $\gamma_1(t)$ to $\gamma_2(U(t))$. Setting up our coordinate system as in Figure 1, with the lower curve segment $\gamma_1(t) = (t, f(t))$ having an inflexion at $t = 0$, and the upper curve segment $\gamma_2(u) = (c - u, d + g(u))$ having no inflexion at $u = 0$, we use the following expansions of $f(t)$ and $g(u)$:

$$f(t) = a_3t^3 + a_4t^4 + \ldots$$
$$g(u) = b_2u^2 + b_3u^3 + \ldots$$

with $a_3b_2 \neq 0$. A short calculation shows that the CSS is quadratic near $\gamma_2(0) = (c, d)$, and by examining the ratio a_3/b_2 we can determine which side of the envelope line the CSS lies, local to (c, d). Figure 3 gives a schematic illustration of the results (the CSS is

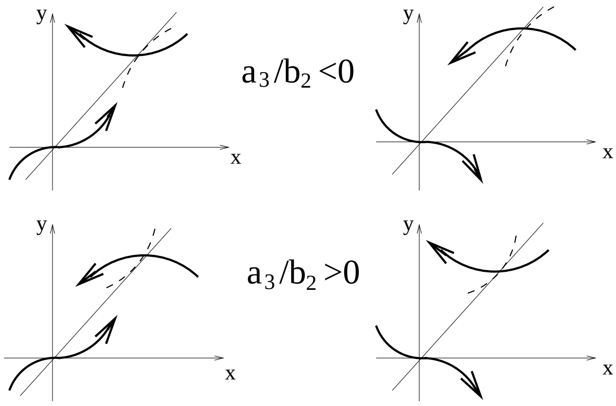

Figure 3

shown dashed). We now move on to consider how the existence of a double tangent to our plane curve effects the structure of the CSS. Consider the following set-up, as illustrated

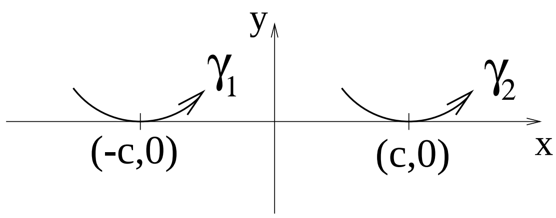

Figure 4

in Figure 4, where we have a double tangent to the curve segments γ_1 and γ_2 given by

$$\gamma_1(t) = (-c + t, a_2t^2 + a_3t^3 + \ldots),$$
$$\gamma_2(u) = (c + u, b_2u^2 + b_3u^3 + \ldots).$$

We assume that at least one of $a_2, b_2 \neq 0$, and that $c \neq 0$. In Theorem 2, we saw that the CSS is locally non-smooth at a double tangent, and in fact it can be shown using this set-up that the CSS always inflects the double tangent. The position of the inflexion along a double tangent can be easily determined using the corollary to Theorem 1 (see the end of Section 2), which tells us that the CSS point lies at the intersection of the line joining parallel tangent pairs (which in this case is the double tangent) with the line joining the corresponding centres of curvature: Figure 5 illustrates the result, where κ_i

denotes the curvature of the corresponding curve segment γ_i evaluated at $t = u = 0$ (the two points of contact of the curve segments and the double tangent). When $\kappa_1 = \kappa_2$,

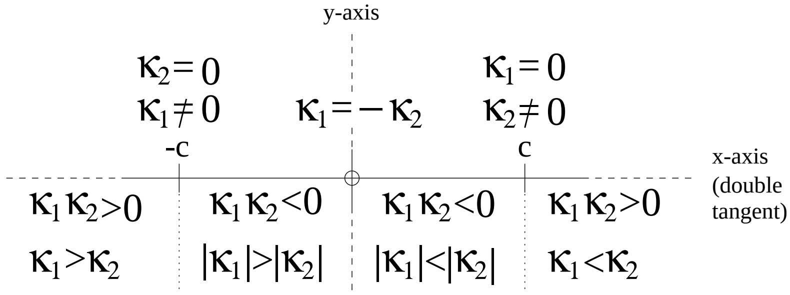

Figure 5. Position of inflexion along double tangent

the CSS point is at infinity along the double tangent: it can be shown that the CSS still inflects the double tangent at infinity. Our next theorem states a global result concerning the structure of the CSS for a plane curve. It states that the number of inflexions on the CSS of a curve γ is equal to the number of double tangents to γ.

THEOREM 4. *The CSS has an inflexion only at a double tangent of the curve.*

Proof. Consider the dual of the CSS, which is defined to be the set of tangents to the CSS, that is the set of original chords joining points of contact of parallel tangents. We regard these chords as points in the dual plane, and the locus of these points is then the dual-CSS. Now we know that inflexions on the CSS correspond to cusps on the dual-CSS, and hence we may use the dual to find inflexions on the CSS by finding conditions under which the dual has a cusp. A few short calculations show that the dual-CSS is non-smooth if and only if $d = 0$, which corresponds to the case of a bitangent line. Thus inflexions occur on the CSS only at a double tangent, in which case we know that there is always one and only one inflexion. ∎

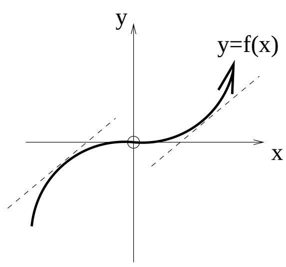

Figure 6

We now consider the local structure of the CSS at the other generic situation we expect to find on a plane curve, namely an *inflexion*. Consider Figure 6 above, where we have an ordinary inflexion at the origin. Now there will be pairs of points on opposite sides of the inflexion where the tangents are parallel, and therefore the inflexion contributes to the CSS. We would like to find the limiting point of the CSS as the pairs of points with parallel tangents approach the inflexion. We find:

Theorem 5. *The limiting point of the CSS for a curve segment having an ordinary inflexion is at the inflexion, and the CSS is tangent to the curve there.*

If we consider the expansion $f(x) = a_3x^3 + a_4x^4 + \ldots$, with $a_3 \neq 0$, then it can be shown that the direction in which the CSS approaches the inflexion depends upon the signs of a_3 and a_4. Figure 7 summarizes these results (the CSS is shown dashed). We briefly note

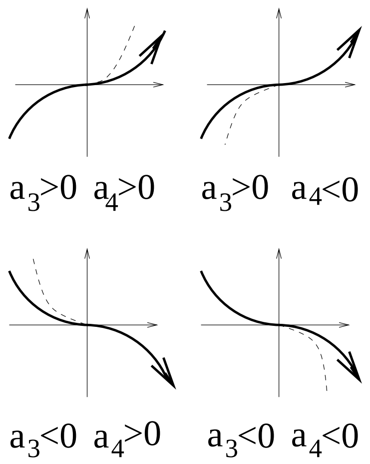

Figure 7

that there is a bifurcation of the CSS when $a_4 = 0$ (even when we have $a_3 \neq 0$): in this case, the CSS is still a semi-cubical parabola, with endpoint at the inflexion and tangent to the curve there. However, the direction of the CSS as it approaches the inflexion now depends upon the signs of a_3 and a_6 — the local diagrams are the same as those in Figure 7, with a_6 replacing a_4.

The next result is another example of using the envelope definition to deduce a *global* result concerning the structure of the CSS of an oval:

Theorem 6. *The number of cusps on the CSS of an oval is odd and* ≥ 3.

Proof. It can easily be shown that the CSS of an oval Γ is a continuous closed curve, and since an oval has no double tangents, Theorem 4 tells us that the CSS of Γ has no inflexions. By considering the chords joining pairs of points of Γ having parallel tangents (by definition, these chords are the tangents to the CSS), we can show that the CSS of an oval always has rotation number $1/2$: thus there is an odd number of cusps on the CSS of an oval. Furthermore, it is not hard to see that there exists no continuous closed curve of rotation number $1/2$ containing just a single cusp and no inflexions. Hence the odd number of cusps on the CSS of an oval must be at least 3. ■

We end this section with the interesting example illustrated in Figure 8 below, where the CSS is shown as a solid line and the original curve is shown dashed: we start with a circle, centre c; we then deform this circle into a non-centrally symmetric curve with no zeroes of curvature, an oval; we further deform this curve until it exhibits a zero of curvature, at which stage the CSS touches the curve at the corresponding parallel tangent point; further deformation results in the final curve, having 2 inflexions and a double tangent: note that the CSS inflects the double tangent, has end-points at the inflexions of the curve, and tends to infinity along the asymptotic line (shown dotted). This example sums up all the phenomena that we know about the CSS so far. The next section continues this experimental approach concerning transitions on the CSS.

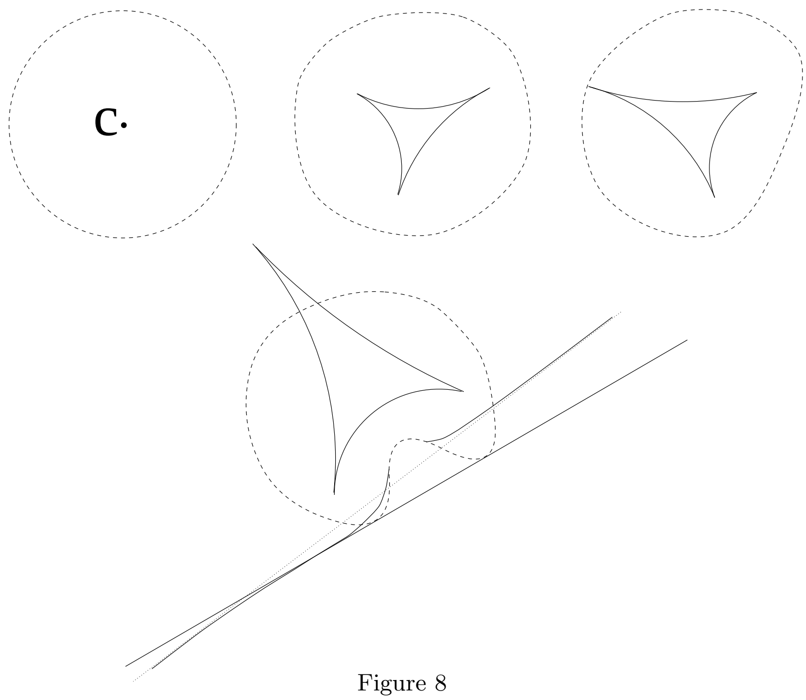

Figure 8

5. 1-parameter families of plane curves. We would like to further our investigation of the CSS by analysing the possible transitions that may occur on the CSS of a plane curve as it is deformed through a 1-parameter family. Using the same coordinate system as illustrated in Figure 1, and with reference to Section 3, we are now looking for conditions for F (see (2)) to have an $A_{\geq 3}$ singularity on its zero-level. Some simple analysis leads us to:

THEOREM 7. *F has an $A_{\geq 3}$ point on its zero-level in the following situations*:

- $(c, d) = (0, 0)$: *both curve segments pass through the origin and are tangent there.*
- $\kappa_1' \kappa_2^2 - \kappa_1^2 \kappa_2' = d = 0$: *the cusp condition is satisfied along a double tangent.*
- $\kappa_1 = d = 0$: *we have a double tangent where there is an inflexion on one of the curve segments.*
- $d \neq 0$ *and both*
  (i) $\kappa_1' \kappa_2^2 = \kappa_1^2 \kappa_2'$
  (ii) $\kappa_1'' \kappa_2^3 = \kappa_1^3 \kappa_2''$

occur simultaneously.

The first three cases are simply situations in which we find some higher degree of degeneracy of F. The last situation is the one which interests us most: this is the condition for a '*swallowtail point*' to appear on the CSS (generically $\partial^4 F/\partial t^4 \neq 0$). As noted before, we expect to observe swallowtail transitions on 1-parameter families of the CSS. We can rewrite this *swallowtail condition* in terms of radii of curvature of the curve segments: the condition for a swallowtail point becomes

$$\begin{aligned} &\text{(i)} \quad \rho_1' = \rho_2', \\ \text{and } &\text{(ii)} \quad \rho_1 \rho_1'' = \rho_2 \rho_2''. \end{aligned}$$

where $\rho_i = 1/\kappa_i$ is the radius of curvature of curve segment γ_i. As in the case of the cusp condition, we can parametrize both curves by the same parameter t, and a simple calculation then shows that

$$\kappa_1' \kappa_2^2 - \kappa_1^2 \kappa_2' = \kappa_1'' \kappa_2^3 - \kappa_1^3 \kappa_2'' = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{\kappa_2}{\kappa_1}\right) = \frac{d^2}{dt^2}\left(\frac{\kappa_2}{\kappa_1}\right) = 0.$$

Hence we expect a swallowtail point on the CSS when we have a degenerate critical point of the ratio of curvatures function κ_2/κ_1.

The experimental theme of the previous section is continued with the help of LSMP [10], a software package implemented on an SGI machine, which allows us to plot the CSS for specific plane curve segments. In the schematic example below, an inflexion on our curve segment is deformed through a family, and we can clearly see a (semi-)swallowtail transition occuring. Experiments such as this one make it clear that we are in fact dealing with *boundary singularities* (see [1], [3], p. 409, [7]).

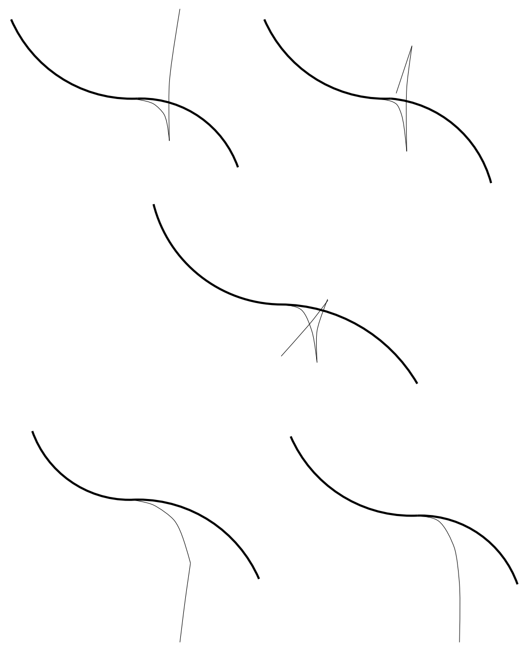

Figure 9. A schematic representation of a semi-swallowtail transition

6. The 3-dimensional surface case. In this section we shall show briefly how to apply the idea of 'ratio-of-distances' to the case of a surface. Specifically, we consider a smooth closed surface M and pairs of points of M at which the tangent planes are parallel. For now, let us assume that M is strictly convex, so that all points are elliptic. We shall indicate below some of the interesting complications that arise when this assumption is dropped, but will leave a detailed exposition of the general structure of the CSS for surfaces to another article. In fact, it may be better to use a different definition of the CSS to overcome these complications — see Section 7 for details.

We can set up local coordinates so that we are examining two local pieces of surface, one of which, M say, is given by a parametrization $(x, y, g(x, y))$ where $g = g_x = g_y = 0$ at $x = y = 0$ (see Figure 10). The base point $(0, 0, 0)$ will be referred to as P. The other surface piece, N say, will have a horizontal tangent plane at some point Q. All our constructions are affinely invariant, so for purposes of calculation we can first perform an affine transformation which moves Q on to the z-axis. Better still, we can always do this by means of a transformation of the form

$$x = x' + \alpha z', \quad y = y' + \beta z', \quad z = z',$$

and a short calculation shows that such a transformation *leaves the second degree terms of the surfaces M and N unchanged*. We suppose this is done; N is then given by a parametrization of the form $(u, v, k + h(u, v))$, where $Q = (0, 0, k)$ and $h = h_u = h_v = 0$ at $u = v = 0$. The parallel tangent plane condition is now $g_x = h_u, g_y = h_v$. Using the

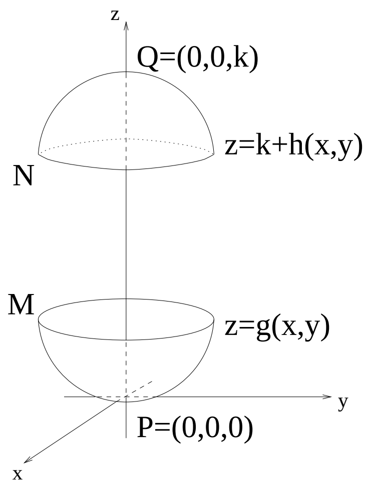

Figure 10

implicit function theorem, we can express u, v locally as functions of x, y provided Q is not parabolic, which we are assuming here. We can now write down the function $R(x, y)$ which represents the ratio of the distances of a fixed point (p, q, r) from the parallel tangent planes at two points near P and Q. Thus

$$R = \frac{(x-p)g_x + (y-q)g_y + r - g}{\sqrt{g_x^2 + g_y^2 + 1}} \times \frac{\sqrt{h_u^2 + h_v^2 + 1}}{(p-u)h_u + (q-v)h_v + h + k - r},$$

where u, v are functions of x, y as above. We seek the conditions on (p, q, r) for R to be (i) singular, (ii) degenerate, at $x = y = 0$. Naturally this is just a matter of calculation! The results are:

(i) R is singular at $x = y = 0$ (i.e., $R_x = R_y = 0$) if and only if

$$\begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 0, \tag{6}$$

the derivatives being evaluated at $x = y = 0$. We shall use G to denote this matrix of the second derivatives of G. Note that G was not affected by the initial affine transformation which placed Q on the z-axis. The letter H will denote the corresponding matrix from h.

Granted that P is not parabolic, this shows that points (p, q, r) for which R is singular at $x = y = 0$ are precisely those with $p = q = 0$, i.e., *points on the line joining P and Q*.

(ii) R is degenerate at $x = y = 0$ (i.e., also $R_{xx}R_{yy} = R_{xy}^2$) if and only if $r = \lambda k$ where λ satisfies

$$\det(\lambda G + (1 - \lambda)H) = 0.$$

When, as here, G is nonsingular, this amounts to saying that $(1 - \lambda)/\lambda$ is an eigenvalue of $G^{-1}H$.

Note that by our assumption that G is a *definite* matrix (since M has only elliptic points), the eigenvalues will always be real. It is, however, possible for them to coincide. In fact, rotating axes so that G has the form $\frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)+$ h.o.t., and writing

$$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

the condition for equal eigenvalues comes to

$$(\kappa_2 a - \kappa_1 c)^2 + 4\kappa_1\kappa_2 b^2 = 0, \quad \text{i.e.} \quad \kappa_2 a = \kappa_1 c, \ b = 0,$$

since $\kappa_1\kappa_2 \neq 0$. Since $b = 0$, the principal directions at P and Q are parallel. In addition, the ratios of principal curvatures are equal: $\kappa_1/\kappa_2 = a/c$. We can expect this to occur at isolated points (if at all), so the two real sheets of the CSS will come together at isolated points, in rather the same way that the two sheets of the focal surface of a given smooth surface come together over the umbilic points.

PROPOSITION 8. *For a smooth strictly convex surface the CSS is a real 2-sheeted surface with the two sheets coming together at isolated points. Furthermore, it can be shown that the chords joining pairs of points of contact of parallel tangent planes are all tangent to the CSS.*

As a simple example, consider a surface M of '*constant width*': the chords joining pairs of points with parallel tangent planes are the (common) normals to the surface, and these chords are of some constant length. It follows that the distances d, e from a fixed point (p, q, r) to two parallel tangent planes have constant sum, and hence the ratio-of-distances function R has the same singularities as d. However, it is easy to see that d has the same singularities as the standard distance-squared function from (p, q, r)

to M, and hence the bifurcation set of the ratio R coincides with the bifurcation set of standard distance-squared function on the surface, namely the *focal set*. Thus it follows that the CSS of a surface of constant width is the focal set of the surface. (Compare this with the analogous example for a curve of constant width, outlined in Section 1.) It is interesting to note that, in this example, the set of chords joining parallel tangent planes are normals to a surface (which is in fact the original surface itself): this is not true in general.

When we move on to consider non-convex objects, the situation becomes rather more complicated. For example, equation (6) holds when P is parabolic but Q is not, and then we can take g of the form $\frac{1}{2}\kappa_1 x^2+$ h.o.t. The condition is then just $p = 0$, which means that there is a whole *plane* of points (p, q, r) which make R singular at $x = y = 0$, rather than just a line of such points, as we should expect. Thus for a surface with parabolic points, the ratio-of-distances definition turns out to be unsatisfactory.

7. Conclusions & further investigations. So far, we have proposed two definitions of the CSS — one via a ratio-of-distances function, and the other, in the plane curve case, in terms of an envelope. We found that both definitions are entirely successful when applied to strictly convex curves — we are able to analyse the structure of the CSS in a variety of convex situations, and the two definitions lead to identical sets.

Now it is quite natural to ask whether we can extend these ideas to non-convex situations, both for plane curves *and* surfaces: however, as we have seen in Section 6, this may lead to some unsatisfactory results. One of our objectives is that each pair of parallel tangents (or parallel tangent planes) to our curve (or surface) should contribute a single point to the CSS. In the last section, we saw that the ratio-of-distances approach was unsatisfactory when we considered certain non-convex surface situations, namely the case where one of the surface segments is parabolic. In fact, analogous problems occur when we consider both the ratio-of-distances definition *and* the envelope definition for the CSS of a plane curve. We will illustrate how these definitions may 'fail' in two specific non-convex plane curve situations:

(i) We have an inflexion on one curve segment, and another non-inflecting curve segment (see Figure 3);

(ii) We have a single inflecting curve segment — here, pairs of parallel tangents straddle the inflexion (see Figure 6).

For each definition, we will calculate the CSS 'point' corresponding to the inflexion (the 'point' may in fact turn out to be some larger set, as we shall see). First of all we consider the ratio-of-distances definition. A short calculation shows that this definition is unsuitable in both situations: in case (i), the ratio-of-distances function is singular for an arbitrary base-point in the plane, and degenerate for all points on the line joining the inflexion and its corresponding parallel tangent point — thus the whole line is part of the CSS in this definition, when we should expect a single point on the line only; in case (ii), the ratio-of-distances function is in fact degenerate for any base-point — thus the whole plane is part of the CSS using the ratio-of-distances definition, a result which we consider to be unacceptable since we should expect the inflexion to contribute a single point to the CSS, namely the inflexion itself.

Consider now the envelope definition in the same two situations: in case (i), we find that the envelope definition gives a single point, namely the point of contact of the corresponding parallel tangent on the upper curve segment — thus the envelope definition in this situation gives an entirely acceptable result. However, in case (ii), we find that the envelope definition gives the inflexional tangent as part of the CSS, which is a situation we would rather avoid. Thus the envelope definition, although superior to the ratio-of-distances definition in some non-convex situations, fails in case (ii).

It is clear that further work is needed to cover the non-convex situation. In discussions with Volody Zakalyukin and Victor Goryunov, a new approach has been developed, and this will be the subject of a future article.

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