THE MODULAR DISTRIBUTION OF STERN'S SEQUENCE

JEAN-MARC DESHOUILLERS

Institut Mathématique de Bordeaux, Université de Bordeaux
Bordeaux INP and CNRS, 33405 Talence, France
E-mail: jean-marc.deshouillers@math.u-bordeaux.fr

ANDRZEJ SCHINZEL

Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-656 Warszawa, Poland
E-mail: schinzel@impan.pl

To our Friend and Colleague Jerzy Kaczorowski,
for his 60th anniversary

Abstract. Let $(s_n)_n$ be Stern's sequence, $a, b, m > 0$ integers. The natural density of indices n such that $(s_n, s_{n+1}) \equiv (a, b) \bmod m$ exists and is determined. The main tools in the proof are the properties of the relevant automata.

1. Introduction. Stern [7] defined in 1858 the following sequence

$$s_0 = 0, \quad s_1 = 1 \quad \text{and, for } n \geq 1: \quad s_{2n} = s_n, \ s_{2n+1} = s_n + s_{n+1}. \tag{1}$$

The paper [5] presents the main properties of this sequence.

Klavžar, Milutinović and Petr [4] defined similarly the sequence of Stern polynomials:

$$B_0(x) = 0, \qquad B_1(x) = 1,$$
$$\text{and, for } n \geq 1: B_{2n}(x) = xB_n(x), \ B_{2n+1}(x) = B_n(x) + B_{n+1}(x).$$

Ulas [8] conjectured that the only rational zeroes of $B_n(x)$ are $0, -1, -1/2, -1/3$. This conjecture was proved by Gawron [3] who showed that the lower density of the sequences of indices n such that $B_n(-1/2) = 0$ or such that $B_n(-1/3) = 0$ is 0, and conjectured that

---

[37]

the density actually exists, and is equal to 0. This conjecture was proved by the second named author [6] who showed that for $\nu \in \{2, 3\}$ there is an infinite sequence of primes $p_n^{(\nu)}$ such that the upper density of the sequence of indices n such that $B_n(-1/\nu) \equiv 0 \bmod p_n^{(\nu)}$ exists and tends to 0 when the prime $p_n^{(\nu)}$ tends to infinity. He also observed that for any prime p the upper density of the sequence of indices n such that $s_n \equiv 0 \bmod p$ does not exceed $2/p$. This is improved in the following results.

THEOREM 1.1. *For any positive integer m and any pair (a, b) of rational integers, we have*

$$\lim_{x \to \infty} \frac{1}{x} \operatorname{Card}\{n \leq x \colon (s_n, s_{n+1}) \equiv (a, b) \bmod m\}$$

$$= \begin{cases} \frac{1}{m^2} \prod_{p \mid m} \left(1 - \frac{1}{p^2}\right)^{-1} & \text{if } \gcd(a, b, m) = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where the product is taken over the prime factors p of m.

The first corollary of this result was proposed by the second named author during the Number Theory Week held in Poznań in September 2017, and proved by him for $p < 10$.

COROLLARY 1.2. *For any prime p and any residue r modulo p, the natural density of the sequence of indices n such that*

$$s_n \equiv r s_{n+1} \bmod p \quad (3)$$

exists and is equal to $1/(p+1)$.

COROLLARY 1.3. *For any positive integer m and any rational integer a, the sequence of indices n such that $s(n) \equiv a \bmod m$ has a natural density which is equal to*

$$\frac{1}{m} \prod_{\substack{p \mid m \\ p \mid a}} \left(1 + \frac{1}{p}\right)^{-1} \prod_{\substack{p \mid m \\ p \nmid a}} \left(1 - \frac{1}{p^2}\right)^{-1}. \quad (4)$$

A 2-automatic structure associated to the sequence of the Stern polynomials has been noticed by Gawron [3]. It turns out that the relevant automata have two specific properties, which are easy to check and imply the existence and the value of the natural density of the sequences they generate, a point which seems to have been overlooked in the previous papers on the subject. We thus devote the next section to the study of such automata. We shall give in the last section the proof of the results which we have just stated.

2. The b-I-automata. We describe here the deterministic finite automata which are of interest for this paper and give the properties we use; we shall call b-I-automata (or simply I-automata) those automata. As much as we can, we refer to the Allouche–Shallit monograph [1]. We first give their formal definition, which follows pp. 128–129 of [1], then explain how they are used for generating sequences of integers, as in Chapter 5 of [1]; we then give the main property of sequences generated by I-automata and finally prove them, using the consequence of the Perron–Frobenius theorem given in Chapter 8 of [1].

The main result of this section thus combines well-known facts, but it seems convenient to state it independently of the precise context in which we are using it.

DEFINITION 2.1. A *b-I-automaton* is defined by a 5-tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, where

- Q is a non-empty set (the set of the *states*),
- $\Sigma = \{0, 1, \ldots, b-1\}$, where $b \geq 2$ (the *alphabet*),
- $\delta : Q \times \Sigma \to Q$ (the *transition function*),
- $q_0 \in Q$ (the *initial state*),
- $F \subset Q$ (the *accepting states*),

satisfying the following two properties

(i) $\delta(q_0, 0) = q_0$,
(ii) for any $e \in \Sigma$, the map $\delta(\cdot, e)$ from Q to Q is invertible.

Condition (ii) was not considered in [1]. M. Drmota and J. Morgenbesser [2] call *invertible* an automaton satisfying Definition 2.1 with a stronger condition than (i), namely: for any q in Q, one has $\delta(q, 0) = q$.

As in [1], p. 129, we extend the application δ to an application, still denoted as δ, from the set Σ^* of the words over Σ to Q satisfying $\delta(q, \emptyset) = q$ for any $q \in Q$ and $\delta(q, xa) = \delta(\delta(q, x), a)$ for any $q \in Q$, $x \in \Sigma^*$ and $a \in \Sigma$.

In our case, the set Σ can be understood as the digits used for representing an integer in the base b: with a *word* $x = e_k e_{k-1} \cdots e_0 \in \Sigma^*$ we can associate the number $n(w) = \sum_{\ell=0}^{k} e_\ell b^\ell$. In the other way round, there are several ways to associate to a non-negative integer n a word in Σ^*. However, one of the merits of condition (i) is that if x_1 and x_2 are two words associated to n, we have $\delta(q_0, x_1) = \delta(q_0, x_2)$, since x_1 and x_2 may at most differ by the number of 0's on their left. Thus, the representation used for n is not relevant; for convenience we shall choose $[n]$ to represent the *shortest* representation of n in base b.

Let $\mathcal{A}$ as above be an *I*-automaton. Following Chapter 5 of [1], we say that the set $\mathcal{T} \subset \mathbb{N}$ is *generated* by $\mathcal{A}$ if

$$n \in \mathcal{T} \Leftrightarrow \delta(q_0, [n]) \in F. \tag{5}$$

We say that a set $\mathcal{T} \subset \mathbb{N}$ has an *natural density* or simply a *density* (in [1] the expression *frequency* is used), if the limit

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{Card}\{n \leq N : n \in \mathcal{T}\}$$

exists. When it does, this limit is called the natural density (or density) of $\mathcal{T}$. Finally, following [1], p. 129, we say that a state $q \in Q$ is *reachable* if there exists an $x \in \Sigma^*$ such that $q = \delta(q_0, x)$, and *unreachable* otherwise. We are now in a position to state the main result of this section.

THEOREM 2.2. *Let* $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ *be an I-automaton. The natural density of the set of integers generated by* $\mathcal{A}$ *exists and is equal to the quotient of the number of the reachable states in* F *by the number of the reachable states in* Q.

Proof. If F is the disjoint union of F_1 and F_2, the sequence generated by $(Q, \Sigma, \delta, q_0, F)$ is the disjoint union of the sequences generated by $(Q, \Sigma, \delta, q_0, F_1)$ and $(Q, \Sigma, \delta, q_0, F_2)$ and

the number of reachable elements of F is the sum of the number of reachable elements of F_1 and F_2; thus if the theorem is valid for F_1 and F_2, it is valid for F. It is thus enough to prove it when F has only one element. Moreover, the result is trivially true when $F = \{q\}$, where q is unreachable. It is thus enough to prove the theorem when $F = \{q\}$, where q is reachable. Moreover, without loss of generality, we may assume that all the elements of Q are reachable. That is what we assume in the sequel.

We index the elements of Q (the indexation has no importance except that we keep the notation q_0 for the initial state) and write $Q = \{q_0, q_1, \ldots, q_{K-1}\}$, where $K = \mathrm{Card}(Q)$. For each digit $\ell \in \Sigma$, we define the matrix M_ℓ such that its entry at the intersection of the i-th row and the j-th column is 1 if $\delta(q_j, \ell) = q_i$ and 0 otherwise. Condition (ii) is equivalent to the fact that for each ℓ the matrix M_ℓ is a permutation matrix. The matrix $M = \sum_{\ell=0}^{b-1} M_\ell$ is called the *incidence matrix* (cf. [1], p. 248).

An important property of the incidence matrices associated to I-automata is the following lemma.

LEMMA 2.3. *The incidence matrix M of any I-automaton is* primitive, *which means that there exists an integer k such that M^k has only positive entries.*

Proof. We borrow the terminology *primitive* from [1] p. 250; notice that this definition differs from that used in several standard books. The lemma is equivalent to saying that there exists an integer k such that for any i and j in $\{0, 1, \ldots, K-1\}$, there exists a word $x \in \Sigma^*$ of length k such that $\delta(q_j, x) = q_i$. This is a simple consequence of the following three points.

(A) For any $i \in \{0, 1, \ldots, K-1\}$, there exists a word x_i such that $\delta(q_0, x_i) = q_i$. This is just a way of saying that q_i is reachable, which we assumed.

(B) For any non-negative m, we have $\delta(q_0, 0^{(m)}) = q_0$, where $0^{(m)}$ denotes the word $000 \cdots 000$ consisting of m consecutive zeroes. This is a trivial consequence of the property (i) stating that $\delta(q_0, 0) = q_0$.

(C) For any $j \in \{0, 1, \ldots, K-1\}$, there exists a word y_j such that $\delta(q_j, y_j) = q_0$. By (A), there exists a word $x_j = e_r e_{r-1} \cdots e_2 e_1$ such that $q_j = \delta(q_0, x_j)$; we claim that the word

$$z_j = e_1^{(K!-1)} e_2^{(K!-1)} \cdots e_{r-1}^{(K!-1)} e_r^{(K!-1)}$$

is an admissible word for y_j. Let us compute $\delta(q_j, z_j)$; we have

$$\delta(q_j, z_j) = \delta(\delta(q_0, x_j), z_j) = \delta(q_0, x_j z_j)$$

$$= \delta\Big(q_0, (e_r e_{r-1} \cdots e_2 e_1)\big(e_1^{(K!-1)} e_2^{(K!-1)} \cdots e_{r-1}^{(K!-1)} e_r^{(K!-1)}\big)\Big)$$

$$= \delta\Big(q_0, (e_r e_{r-1} \cdots e_2)\big(e_1 e_1^{(K!-1)}\big)\big(e_2^{(K!-1)} \cdots e_{r-1}^{(K!-1)} e_r^{(K!-1)}\big)\Big)$$

$$= \delta\Big(\delta\big(\delta(q_0, e_r \cdots e_2), e_1^{(K!)}\big), e_2^{(K!-1)} \cdots e_r^{(K!-1)}\Big).$$

The key point to notice here is that condition (ii) which says that for any e in Σ, the map $\delta(\cdot, e)$ is invertible, implies that its order is a divisor of that of the permutation group $\mathfrak{S}_K$ and thus $\delta(q, e^{(K!)})$ is equal to q for any state q and any digit e. We thus have by induction

$$\delta(q_j, z_j) = \delta\big(\delta(q_0, e_r \cdots e_2), e_2^{(K!-1)} \cdots e_r^{(K!-1)}\big) = \ldots = \delta(q_0, \emptyset) = q_0.$$

Equipped with (A), (B) and (C), let us finish the proof of Lemma 2.3. By Property (A), we can go from q_0 to any q_i in a finite number of steps (i.e. using words of finite lengths); this finite set of finite numbers is bounded, say by K_1 (it is indeed easy to see that K is a suitable bound, but we do not need this precision); the same applies to the number of steps to go from any q_i to q_0, which is bounded by some K_2 (again, K is admissible). Now, we see that we can always go from any q_j to any q_i in exactly $K_1 + K_2$ steps by going from q_j to q_0, stepping on q_0 a suitable number of times, and going from q_0 to q_i. Thus $K_1 + K_2$ (which can be taken to be equal to $2K$) is a suitable value for the k announced in the statement of Lemma 2.3. ∎

We come back to the proof of Theorem 2.2. Once we know that the incidence matrix M is primitive, Theorem 8.4.7 of [1] tells us that the natural density of the sequence generated by our I-automaton $(Q, \Sigma, \delta, q_0, \{q\})$ exists. Moreover, the vector of densities of the sequences generated by the automaton $(Q, \Sigma, \delta, q_0, \{q_i\})$ is proportional to the Perron–Frobenius vector of the incidence matrix. Because of (ii), the sum of all the entries of any row from M are equal to b. This implies that the vector ${}^t(1, 1, \ldots, 1, 1)$ is a Perron–Frobenius vector of M, and thus, the natural densities of all the sequences generated by the automaton $(Q, \Sigma, \delta, q_0, \{q_i\})$ are equal. This implies that their common natural density is $1/\operatorname{Card}(Q)$, which proves Theorem 2.2. ∎

3. Proof of the main results. In this section, we consider a positive integer m and for a rational integer x, we shall denote by $\overline{x}$ the class of x modulo m. For proving Theorem 1.1, we shall describe a 2-I-automaton which generates the sequence $(\overline{s_n}, \overline{s_{n+1}})_n$ and study the set of its reachable states.

We introduce two maps U_0 and U_1 from $(\mathbb{Z}/m\mathbb{Z})^2$ to $(\mathbb{Z}/m\mathbb{Z})^2$ defined by

$$U_0(\overline{x}, \overline{y}) = (\overline{x}, \overline{x} + \overline{y}), \qquad U_1(\overline{x}, \overline{y}) = (\overline{x} + \overline{y}, \overline{y}). \tag{6}$$

Relation (1), read modulo m, leads to

$$\overline{s_0} = \overline{0}, \quad \overline{s_1} = \overline{1} \quad \text{and, for } n \geq 1 : \ \overline{s_{2n}} = \overline{s_n}, \ \overline{s_{2n+1}} = \overline{s_n} + \overline{s_{n+1}}.$$

and a short computation leads to

$$(\overline{s_{2n}}, \overline{s_{2n+1}}) = (\overline{s_n}, \overline{s_n} + \overline{s_{n+1}}) = U_0(\overline{s_n}, \overline{s_{n+1}}) \tag{7}$$

and

$$(\overline{s_{2n+1}}, \overline{s_{2n+2}}) = (\overline{s_n} + \overline{s_{n+1}}, \overline{s_{n+1}}) = U_1(\overline{s_n}, \overline{s_{n+1}}). \tag{8}$$

PROPOSITION 3.1. *Let a and b be two rational integers. The sequence of the integers n such that $(\overline{s_n}, \overline{s_{n+1}}) = (\overline{a}, \overline{b})$ is generated by the 2-I-automaton*

$$\mathcal{A}_m = \left((\mathbb{Z}/m\mathbb{Z})^2, \{0, 1\}, \delta_m, (\overline{0}, \overline{1}), \{(\overline{a}, \overline{b})\}\right),$$

where

$$\forall q \in (\mathbb{Z}/m\mathbb{Z})^2 : \delta_m(q, 0) = U_0(q) \ and \ \delta_m(q, 1) = U_1(q).$$

Proof. All the pairs $\xi_n = (\overline{s_n}, \overline{s_{n+1}})$ belong to $(\mathbb{Z}/m\mathbb{Z})^2$, which we take as the set of states. Relations (7) and (8) tell us that ξ_{2n} and ξ_{2n+1} depend on ξ_n, regardless of n, which justifies that we have a 2-automaton (i.e. $\Sigma = \{0, 1\}$) and the same relations tell us how they depend on ξ_n, which justifies the definition of δ. We further have $\xi_0 = (\overline{s_0}, \overline{s_1}) = (\overline{0}, \overline{1})$, whence the choice of q_0. Finally, the choice of F to be $\{(\overline{a}, \overline{b})\}$ corresponds to the

fact that we wish to detect those integers n for which $(\overline{s_n}, \overline{s_{n+1}}) = (\overline{a}, \overline{b})$. It remains to show that the automaton we have introduced is a 2-I-automaton.

Condition (i) from Definition 2.1 comes simply from the fact that $U_0(\overline{0}, \overline{1}) = (\overline{0}, \overline{1})$ and condition (ii) from the fact that U_0 and U_1 are invertible linear operators on $(\mathbb{Z}/m\mathbb{Z})^2$ (with respective inverses U_0^{-1} and U_1^{-1} defined by $U_0^{-1}(\overline{x}, \overline{y}) = (\overline{x}, -\overline{x} + \overline{y})$, $U_1^{-1}(\overline{x}, \overline{y}) = (\overline{x} - \overline{y}, \overline{y}))$. ∎

We now determine the set R_m of reachable states of the automaton $\mathcal{A}_m$.

PROPOSITION 3.2. *The reachable states of the automaton $\mathcal{A}_m$ introduced in Proposition* 3.1 *is the set*

$$R_m = \{(\overline{a}, \overline{b}) \colon \gcd(a, b, m) = 1\}. \tag{9}$$

Proof. The fact that all reachable states $(\overline{s_n}, \overline{s_{n+1}})$ are in R_m is a direct consequence of the fact that for any integer $n \geq 0$, the consecutive Stern numbers s_n and s_{n+1} are coprime ([7], p. 199).

The proof of the reverse assertion (Lemma 3.4 below) will use the following result.

LEMMA 3.3. *Let (a, b) be rational integers such that $\gcd(a, b, m) = 1$. There exists a positive integer k such that $\gcd(a - kb, m) = 1$.*

Proof. We let k to be the greatest factor of m which is coprime to a. Let p be a prime factor of m.

If p divides a, by assumption p does not divide b and by definition p does not divide k so that p does not divide $a - kb$, and thus does not divide $\gcd(a - kb, m)$.

If p does not divide a, by definition p divides k, thus p does not divide $a - kb$, and thus does not divide $\gcd(a - kb, m)$. ∎

Proposition 3.2 will follow from the next lemma.

LEMMA 3.4. *For any pair (a, b) of rational integers such that $\gcd(a, b, m) = 1$, the pair $(\overline{a}, \overline{b})$ is reachable.*

Proof. For any integer $k \geq 0$ we have $U_1^k(\overline{0}, \overline{1}) = (\overline{k}, \overline{1})$ and $U_0^k(\overline{1}, \overline{1}) = (\overline{1}, \overline{k+1})$, which gives us a first set of reachable elements.

Let x be an integer coprime to m; we can find a positive integer ℓ such that $\ell x \equiv -1 \bmod m$. If we apply ℓ times U_0 to $(\overline{x}, \overline{1})$, we obtain $(\overline{x}, \ell\overline{x} + \overline{1}) = (\overline{x}, \overline{0})$ which is thus reachable.

Let x be an integer coprime to m; then for any $y \in \mathbb{Z}/m\mathbb{Z}$, $(\overline{x}, y)$ is reachable: indeed, since $(\overline{x}, \overline{0})$ is reachable, for any h, the element $U_0^h(x, \overline{0}) = (\overline{x}, h\overline{x})$ is reachable, and since $\overline{x}$ is invertible in $\mathbb{Z}/m\mathbb{Z}$, the family $h\overline{x}$ covers $\mathbb{Z}/m\mathbb{Z}$.

Let finally a and b be two rational integers such that $\gcd(a, b, m) = 1$. By Lemma 3.3, there exists a positive integer k such that $a - kb$ is coprime to m; we apply the previous claim with $x = a - kb$ and $y = \overline{b}$: the state $(\overline{a} - k\overline{b}, \overline{b})$ is reachable, and so is $U_1^k(\overline{a} - k\overline{b}, \overline{b})$, which is equal to $(\overline{a}, \overline{b})$.

This ends the proof of Lemma 3.4, as well as that of Proposition 3.2. ∎

We now compute the cardinality of the set R_m of reachable states of the automaton $\mathcal{A}_m$.

PROPOSITION 3.5. *The number of reachable states of the automaton $\mathcal{A}_m$ introduced in Proposition* 3.1 *is*

$$\text{Card}(R_m) = m^2 \prod_{p \mid m} \left(1 - \frac{1}{p^2}\right). \tag{10}$$

Proof. We use the characterization of the reachable states given by Proposition 3.2. We have

$$\text{Card}(R_m) = \sum_{\substack{0 \le a < m \\ 0 \le b < m \\ \gcd(a,b,m)=1}} 1 = \sum_{d \mid m} \sum_{\substack{0 \le a < m \\ \gcd(a,m)=d}} \sum_{\substack{0 \le b < m \\ \gcd(b,d)=1}} 1$$

$$= m \sum_{d \mid m} \varphi\left(\frac{m}{d}\right) \frac{\varphi(d)}{d} = m\psi(m),$$

where $\psi(m)$ is the Dirichlet convolution of two multiplicative functions. Thus $\psi(m)$ is a multiplicative function and it is sufficient to evaluate it on prime powers.

We have

$$\psi(p) = (p-1) \cdot 1 + 1 \cdot \left(1 - \frac{1}{p}\right) = p - \frac{1}{p} = p\left(1 - \frac{1}{p^2}\right)$$

and, for $k \ge 2$,

$$\psi(p^k) = (p^{k-1} - p^{k-2} + \cdots + p - 1)(1 - 1/p) + (p^k - p^{k-1}) = p^k(1 - 1/p^2).$$

The multiplicative function $\psi(m)$ coincide on all prime powers with the multiplicative function $m \prod_{p \mid m}(1 - 1/p^2)$. Those two functions are thus equal and we have proved (10). ∎

Proof of Theorem 1.1. Theorem 1.1 is a straightforward application of Theorem 2.2 which explains how to compute the density of a sequence generated by a *b-I*-automaton in terms of the reachable states, Proposition 3.1, which states that the sequences under consideration in Theorem 1.1 are generated by a 2-*I*-automaton, Proposition 3.2, which describes the reachable states and Proposition 3.5, which counts those reachable states. ∎

Proof of Corollary 1.2. In the case when $m = p$ is a prime number, all the states are reachable, except $(\overline{0}, \overline{0})$, which leads to a density of $1/(p^2-1)$ for each reachable pair. Thus, for any r, the number of reachable pairs $(\overline{a}, \overline{b})$ such that $\overline{a} = r\overline{b}$ is $p-1$ and the density of the sequence of indices n such that $s_n \equiv rs_{n+1} \bmod p$ is $(p-1)/(p^2-1) = 1/(p+1)$. ∎

Proof of Corollary 1.3. We can either deduce Corollary 1.3 from our general result, or notice that the sequence of the integers n such that $s(n) \equiv a \bmod m$ is a union of sequences, the natural densities of which exist and are given by Theorem 1.1. In either case, the result easily follows from the following lemma. ∎

LEMMA 3.6. *For any rational integer a, we have*

$$\text{Card}\{b \in [0, m-1] \colon \gcd(a, b, m) = 1\} = m \prod_{\substack{p \mid m \\ p \mid a}} \left(1 - \frac{1}{p}\right). \tag{11}$$

Proof. We have

$$\mathrm{Card}\{b \in [0, m-1]\colon \gcd(a, b, m) = 1\} = \sum_{\substack{0 \le b < m \\ \gcd(b, \gcd(a,m))=1}} 1$$

$$= \frac{m}{\gcd(a,m)}\, \varphi(\gcd(a,m)) = m\, \frac{\varphi(\gcd(a,m))}{\gcd(a,m))}$$

$$= m \prod_{\substack{p \,|\, a \\ p \,|\, m}} \left(1 - \frac{1}{p}\right). \quad \blacksquare$$

References

[1] J-P. Allouche, J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge Univ. Press, Cambridge, 2003.

[2] M. Drmota, J. Morgenbesser, *Generalized Thue–Morse sequences of squares*, Israel J. Math. 190 (2012), 157–193.

[3] M. Gawron, *A note on the arithmetic properties of the Stern polynomials*, Publ. Math. Debrecen 85 (2014), 453–465.

[4] S. Klavžar, U. Milutinović, C. Petr, *Stern polynomials*, Adv. in Appl. Math. 39 (2007), 86–95.

[5] S. Northshield, *Stern's diatomic sequence* 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, . . . , Amer. Math. Monthly 117 (2010), 581–598.

[6] A. Schinzel, *On the factors of Stern polynomials II. Proof of a conjecture of M. Gawron*, Publ. Math. Debrecen 91 (2017), 515–524.

[7] M. A. Stern, *Über eine zahlentheoretische Funktion*, J. Reine Angew. Math. 55 (1858), 193–220.

[8] M. Ulas, *On certain arithmetic properties of Stern polynomials*, Publ. Math. Debrecen 79 (2011), 55–81.