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A CESÀRO AVERAGE FOR AN ADDITIVE PROBLEM WITH PRIME POWERS

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Abstract. In this paper we extend and improve our results on weighted averages for the number of representations of an integer as a sum of two powers of primes (the paper of the authors in Forum Math. 27 (2015), see also the paper of A.L., Riv. Mat. Univ. di Parma 7 (2016), Theorem 2.2). Let $1 \le \ell_1 \le \ell_2$ be two integers, Λ be the von Mangoldt function and $r_{\ell_1,\ell_2}(n) = \sum_{m_1^{\ell_1}+m_2^{\ell_2}=n} \Lambda(m_1)\Lambda(m_2)$ be the weighted counting function for the number of representation of an integer as a sum of two prime powers. Let $N \ge 2$ be an integer. We prove that the Cesàro average of weight $k > 1$ of r_{ℓ_1,ℓ_2} over the interval $[1, N]$ has a development as a sum of terms depending explicitly on the zeros of the Riemann zeta-function.

1. Introduction. We continue our recent work on the number of representations of an integer as a sum of primes. In [7] we studied the *average* number of representations of an integer as a sum of two primes, whereas in [8] we considered individual integers. In [10], see also Theorem 2.2 of [6], we studied a Cesàro weighted partial *explicit* formula for Goldbach numbers. Here we generalise and improve this last result by working on the Cesàro weighted counting function for the number of representation of an integer as a sum

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of two prime powers. We let $1 \le \ell_1 \le \ell_2$ be two integers and set

$$r_{\ell_1,\ell_2}(n) = \sum_{m_1^{\ell_1}+m_2^{\ell_2}=n} \Lambda(m_1)\Lambda(m_2).$$

We also use the following convenient abbreviations for the various terms of the development:

$$\mathcal{M}_{1,k,\ell_1,\ell_2}(N) = \frac{N^{1/\ell_1+1/\ell_2}}{\Gamma(k+1+1/\ell_1+1/\ell_2)} \frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1\ell_2},$$

$$\mathcal{M}_{2,k}(N) = \frac{\log^2(2\pi)}{\Gamma(k+1)},$$

$$\mathcal{M}_{3,k,\ell}(N) = -\log(2\pi)\frac{N^{1/\ell}}{\Gamma(k+1+1/\ell)}\frac{\Gamma(1/\ell)}{\ell} + \log(2\pi)\sum_\rho \Gamma\Bigl(\frac{\rho}{\ell}\Bigr)\frac{N^{\rho/\ell}}{\Gamma(k+1+\rho/\ell)}, \tag{1}$$

$$\mathcal{M}_{4,k,\ell_1,\ell_2}(N) = -N^{1/\ell_2}\frac{\Gamma(1/\ell_2)}{\ell_2}\sum_\rho \Gamma\Bigl(\frac{\rho}{\ell_1}\Bigr)\frac{N^{\rho/\ell_1}}{\Gamma(k+1+1/\ell_2+\rho/\ell_1)}, \tag{2}$$

$$\mathcal{M}_{5,k,\ell_1,\ell_2}(N) = \sum_{\rho_1}\sum_{\rho_2}\frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(k+1+\rho_1/\ell_1+\rho_2/\ell_2)} N^{\rho_1/\ell_1+\rho_2/\ell_2}. \tag{3}$$

Here ρ, with or without subscripts, runs over the non-trivial zeros of the Riemann zeta-function ζ and Γ is Euler's function. The main result of the paper is the following theorem.

THEOREM 1. *Let $1 \le \ell_1 \le \ell_2$ be two integers, and N be a positive integer. For $k > 1$ we have*

$$\sum_{n\le N} r_{\ell_1,\ell_2}(n)\frac{(1-n/N)^k}{\Gamma(k+1)} = \mathcal{M}_{1,k,\ell_1,\ell_2}(N) + \mathcal{M}_{2,k}(N) + \mathcal{M}_{3,k,\ell_1}(N) + \mathcal{M}_{3,k,\ell_2}(N) + \mathcal{M}_{4,k,\ell_1,\ell_2}(N) + \mathcal{M}_{4,k,\ell_2,\ell_1}(N) + \mathcal{M}_{5,k,\ell_1,\ell_2}(N) + \mathcal{O}_{k,\ell_1,\ell_2}(N^{-1/2+1/\ell_1}).$$

Clearly, depending on the size of ℓ_1, ℓ_2, some of the previous listed terms should be included in the error term. We remark that the double series over zeros in (3) converges absolutely for $k > 1/2$, and it seems reasonable to believe that the stated equality holds for the same values of k, possibly with a weaker error term, although the bound $k > 1$ appears in several places of the proof and it seems to be the limit of the method.

Theorem 1 generalises and improves our Theorem in [10], see also Theorem 2.2 of [6], which corresponds to the case $\ell_1 = \ell_2 = 1$. In fact in this case Theorem 1 leads to

$$\begin{aligned}\sum_{n\le N} r_G(n)\frac{(1-n/N)^k}{\Gamma(k+1)} &= \frac{N^2}{\Gamma(k+3)} - 2\sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{\rho+1} \\ &\quad - 2\log(2\pi)\frac{N}{\Gamma(k+2)} + 2\log(2\pi)\sum_\rho \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^\rho \\ &\quad + \sum_{\rho_1}\sum_{\rho_2}\frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(k+1+\rho_1+\rho_2)} N^{\rho_1+\rho_2} + \mathcal{O}_k(N^{1/2}),\end{aligned} \tag{4}$$

where $r_G(n) = r_{1,1}(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$, that is, we are now able to detect the term $\mathcal{M}_{3,k,1}$. Very recently Brüdern, Kaczorowski and Perelli [2] proved that (4) holds for every $k > 0$. We point out that Theorem 1 covers other interesting and classical cases like the sum of two prime squares ($\ell_1 = \ell_2 = 2$) or a prime and a prime square ($\ell_1 = 1$, $\ell_2 = 2$).

We recall that our method is based on a formula due to Laplace [12], namely

$$\frac{1}{2\pi i}\int_{(a)} v^{-s} e^{v}\,\mathrm{d}v = \frac{1}{\Gamma(s)}\,, \tag{5}$$

where $\Re(s) > 0$ and $a > 0$, see, e.g., formula 5.4(1) on page 238 of [3]. We will need the general case of (5), which can be found in de Azevedo Pribitkin [1], formulae (8) and (9):

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{iDu}}{(a+iu)^{s}}\,\mathrm{d}u = \begin{cases} \dfrac{D^{s-1}e^{-aD}}{\Gamma(s)} & \text{if } D > 0,\\ 0 & \text{if } D < 0, \end{cases} \tag{6}$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$, and

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{(a+iu)^{s}}\,\mathrm{d}u = \begin{cases} 0 & \text{if } \Re(s) > 1,\\ 1/2 & \text{if } s = 1, \end{cases} \tag{7}$$

for $a \in \mathbb{C}$ with $\Re(a) > 0$. Formulae (6)–(7) enable us to write averages of arithmetical functions by means of line integrals as we will see in §2 below.

The improvement we get in Theorem 1 follows using Lemma 1 below, which is a generalised and refined version of Lemma 4.1 of [10], see also Lemma 5.1 of [6]. In fact Lemma 1 can be also used to generalise and improve our result in [9] about the Hardy–Littlewood numbers to the $p^{\ell} + m^2$, $\ell \geq 1$, problem; we will discuss this case in [11].

2. Settings. Let $\ell \geq 1$, $1 \leq \ell_1 \leq \ell_2$ be integer numbers and

$$\widetilde{S}_{\ell}(z) = \sum_{m\geq 1} \Lambda(m) e^{-m^{\ell} z}, \tag{8}$$

where $z = a + iy$ with $y \in \mathbb{R}$ and real $a > 0$. Moreover let us define the density of the problem as

$$\lambda = 1/\ell_1 + 1/\ell_2. \tag{9}$$

We recall that the Prime Number Theorem (PNT) is equivalent to the statement

$$\widetilde{S}_{\ell}(a) \sim \frac{\Gamma(1/\ell)}{\ell a^{1/\ell}} \quad \text{for } a \to 0+. \tag{10}$$

By (8) we have

$$\widetilde{S}_{\ell_1}(z)\widetilde{S}_{\ell_2}(z) = \sum_{n\geq 1} r_{\ell_1,\ell_2}(n) e^{-nz}.$$

Hence, for $N \in \mathbb{N}$ with $N > 0$ and $a > 0$ we have

$$\frac{1}{2\pi i}\int_{(a)} e^{Nz} z^{-k-1} \widetilde{S}_{\ell_1}(z)\widetilde{S}_{\ell_2}(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{(a)} e^{Nz} z^{-k-1} \sum_{n\geq 1} r_{\ell_1,\ell_2}(n) e^{-nz}\,\mathrm{d}z. \tag{11}$$

Since

$$\sum_{n\geq 1} |r_{\ell_1,\ell_2}(n) e^{-nz}| = \widetilde{S}_{\ell_1}(a)\widetilde{S}_{\ell_2}(a) \asymp_{\ell_1,\ell_2} a^{-\lambda}$$

by (10), where $f \asymp g$ means $g \ll f \ll g$, we can exchange the series and the line integral in (11) provided that $k > 0$. In fact, if $z = a + iy$, taking into account the estimate

$$|z|^{-1} \asymp \begin{cases} a^{-1} & \text{if } |y| \le a, \\ |y|^{-1} & \text{if } |y| \ge a, \end{cases} \tag{12}$$

we have

$$|e^{Nz} z^{-k-1}| \asymp e^{Na} \begin{cases} a^{-k-1} & \text{if } |y| \le a, \\ |y|^{-k-1} & \text{if } |y| \ge a, \end{cases}$$

and hence, recalling (10), we obtain

$$\int_{(a)} |e^{Nz} z^{-k-1}| \Big| \sum_{n \ge 1} r_{\ell_1,\ell_2}(n) e^{-nz} \Big| \, |\mathrm{d}z| \ll a^{-\lambda} e^{Na} \Big[\int_0^a a^{-k-1} \, \mathrm{d}y + \int_a^{+\infty} y^{-k-1} \, \mathrm{d}y \Big],$$

which is $\ll_k a^{-\lambda-k} e^{Na}$, but the rightmost integral converges only for $k > 0$. Using (6) for $n \ne N$ and (7) for $n = N$, we see that for $k > 0$ the right-hand side of (11) is

$$= \sum_{n \ge 1} r_{\ell_1,\ell_2}(n) \Big(\frac{1}{2\pi i} \int_{(a)} e^{(N-n)z} z^{-k-1} \, \mathrm{d}z \Big) = \sum_{n \le N} r_{\ell_1,\ell_2}(n) \, \frac{(N-n)^k}{\Gamma(k+1)} \, .$$

Remark. As in [10] the previous computation reveals that we cannot get rid of the Cesàro weight in our method since, for $k = 0$, it is not clear whether the integral on the right-hand side of (11) converges absolutely or not.

Summing up, for $a > 0$ and $k > 0$ we have

$$\sum_{n \le N} r_{\ell_1,\ell_2}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \widetilde{S}_{\ell_1}(z) \widetilde{S}_{\ell_2}(z) \, \mathrm{d}z,$$

where $N \in \mathbb{N}$ with $N > 0$. This is the fundamental relation for the method.

3. Inserting zeros. In this section we need $k > 1$. By Lemma 1 below we have

$$\widetilde{S}_\ell(z) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_\rho z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) - \log(2\pi) + E(a, y, \ell) = M(\ell, z) + E(a, y, \ell),$$

say, where $E(a, y, \ell)$ satisfies (16). Hence

$$\begin{aligned} \widetilde{S}_{\ell_1}(z) \widetilde{S}_{\ell_2}(z) &= M(\ell_1, z) M(\ell_2, z) + E(a, y, \ell_1) E(a, y, \ell_2) \\ &\quad + E(a, y, \ell_2) M(\ell_1, z) + E(a, y, \ell_1) M(\ell_2, z). \end{aligned}$$

We have $|M(\ell, z)| = \big|\widetilde{S}_\ell(z) - E(a, y, \ell)\big| \le \widetilde{S}_\ell(a) + \big|E(a, y, \ell)\big| \ll_\ell a^{-1/\ell} + \big|E(a, y, \ell)\big|$ by (10) again, so that

$$\begin{aligned} \widetilde{S}_{\ell_1}(z) \widetilde{S}_{\ell_2}(z) &= M(\ell_1, z) M(\ell_2, z) + \mathcal{O}_{\ell_1,\ell_2}\big(\big|E(a, y, \ell_1) E(a, y, \ell_2)\big|\big) \\ &\quad + \mathcal{O}_{\ell_1,\ell_2}\big(\big|E(a, y, \ell_2)\big| a^{-1/\ell_1} + \big|E(a, y, \ell_1)\big| a^{-1/\ell_2}\big), \end{aligned} \tag{13}$$

choosing $0 < a \le 1$, since $1 \le \ell_1 \le \ell_2$. Recalling (12) and taking into account (16), for $k > 1$ we have

$$\int_{(a)} \bigl|E(a,y,\ell_1)E(a,y,\ell_2)\bigr|\,|e^{Nz}|\,|z|^{-k-1}\,|\mathrm{d}z|$$
$$\ll_{\ell_1,\ell_2} e^{Na}\int_0^a a^{-k}\,\mathrm{d}y + e^{Na}\int_a^{+\infty} y^{-k}(1+\log^2(y/a))^2\,\mathrm{d}y$$
$$\ll_{k,\ell_1,\ell_2} e^{Na}a^{-k+1} + e^{Na}a^{-k+1}\int_1^{+\infty} v^{-k}(1+\log^2 v)^2\,\mathrm{d}v \ll_{k,\ell_1,\ell_2} e^{Na}a^{-k+1}.$$

If we choose $a = 1/N$, the error term is $\ll_{k,\ell_1,\ell_2} N^{k-1}$ for $k > 1$. For $a = 1/N$, by (12) and (16), the second remainder term in (13) for $k > 1/2$ is

$$\ll_{\ell_1,\ell_2} N^{1/\ell_1}\int_{(1/N)} |E(y,1/N,\ell_2)|\,|e^{Nz}|\,|z|^{-k-1}\,|\mathrm{d}z|$$
$$\ll_{\ell_1,\ell_2} N^{1/\ell_1}\int_0^{1/N} N^{k+1/2}\,\mathrm{d}y + N^{1/\ell_1}\int_{1/N}^{+\infty} y^{-k-1/2}\log^2(Ny)\,\mathrm{d}y$$
$$\ll_{k,\ell_1,\ell_2} N^{k-1/2+1/\ell_1} + N^{k-1/2+1/\ell_1}\int_1^{+\infty} v^{-k-1/2}\log^2 v\,\mathrm{d}v \ll_{k,\ell_1,\ell_2} N^{k-1/2+1/\ell_1}.$$

Analogously, it is easy to see that the remaining term is $\ll_{k,\ell_1,\ell_2} N^{k-1/2+1/\ell_2}$.

With a little effort we can give an explicit dependence on k for the implicit constants in the last three estimates.

Hence, by (9) and (11) we have

$$\sum_{n\le N} r_{\ell_1,\ell_2}(n)\frac{(N-n)^k}{\Gamma(k+1)}$$
$$= \frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}M(\ell_1,z)M(\ell_2,z)\,\mathrm{d}z + \mathcal{O}_{k,\ell_1,\ell_2}(N^{k-1/2+1/\ell_1})$$
$$= I_1(N;\ell_1,\ell_2,k) + I_2(N;k) + I_3(N;\ell_1,k) + I_3(N;\ell_2,k)$$
$$+ I_4(N;\ell_1,\ell_2,k) + I_4(N;\ell_2,\ell_1,k) + I_5(N;\ell_1,\ell_2,k) + \mathcal{O}_{k,\ell_1,\ell_2}(N^{k-1/2+1/\ell_1}),$$

say, where

$$I_1(N;\ell_1,\ell_2,k) = \frac{1}{2\pi i}\frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1\ell_2}\int_{(1/N)} e^{Nz}z^{-k-1-\lambda}\,\mathrm{d}z,$$
$$I_2(N;k) = \frac{\log^2(2\pi)}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\,\mathrm{d}z,$$
$$I_3(N;\ell,k) = \frac{\log(2\pi)}{2\pi i}\biggl\{-\frac{\Gamma(1/\ell)}{\ell}\int_{(1/N)} e^{Nz}z^{-k-1-1/\ell}\,\mathrm{d}z + \int_{(1/N)} e^{Nz}z^{-k-1}\sum_\rho z^{-\rho/\ell}\Gamma\Bigl(\frac{\rho}{\ell}\Bigr)\,\mathrm{d}z\biggr\},$$
$$I_4(N;\ell_1,\ell_2,k) = -\frac{1}{2\pi i}\frac{\Gamma(1/\ell_1)}{\ell_1}\int_{(1/N)} e^{Nz}z^{-k-1-1/\ell_1}\sum_\rho z^{-\rho/\ell_2}\Gamma\Bigl(\frac{\rho}{\ell_2}\Bigr)\,\mathrm{d}z,$$

$$I_5(N;\ell_1,\ell_2,k) = \frac{1}{2\pi i}\int_{(1/N)} e^{Nz} z^{-k-1} \sum_{\rho_1}\sum_{\rho_2} z^{-\rho/\ell_1-\rho/\ell_2}\Gamma\Big(\frac{\rho_1}{\ell_1}\Big)\Gamma\Big(\frac{\rho_2}{\ell_2}\Big)\,\mathrm{d}z.$$

The evaluation of the integrals I_j is a straightforward application of (5) with $s = Nz$, except that the interchange of the series with the integrals needs to be justified: see §5–7 for a proof that this is in fact permitted when $k > 1$. The proof that the double sum over zeros converges absolutely for $k > 1/2$ is given in §8 below. Combining the resulting expressions and dividing through by N^k we get Theorem 1.

4. Lemmas. We recall some basic facts in complex analysis. First, if $z = a + iy$ with $a > 0$, we see that for complex w we have

$$\begin{aligned} z^{-w} &= |z|^{-w}\exp(-iw\arctan(y/a)) \\ &= |z|^{-\Re(w)-i\Im(w)}\exp\big((-i\Re(w)+\Im(w))\arctan(y/a)\big), \end{aligned}$$

so that

$$|z^{-w}| = |z|^{-\Re(w)}\exp(\Im(w)\arctan(y/a)). \tag{14}$$

We also recall that, uniformly for $x \in [x_1, x_2]$, with x_1 and x_2 fixed, and for $|y| \to +\infty$, by the Stirling formula (see, e.g., Titchmarsh [14, §4.42]) we have

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}e^{-\pi|y|/2}|y|^{x-1/2}. \tag{15}$$

The following lemma generalises and improves Lemma 4.1 of [10], see also Lemma 5.1 of [6]. The improvement depends on the fact that the constant term $\log(2\pi)$ is now explicit since we realised that, in the application, this term leads, in some cases, to a non-trivial contribution in the final result. We follow the line of the proof in [10], but, in some cases, the integration path has to be changed; for clarity we repeat the whole argument.

LEMMA 1. *Let $\ell \geq 1$ be an integer, $z = a + iy$, where $a > 0$ and $y \in \mathbb{R}$. Then*

$$\widetilde{S}_\ell(z) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell}\sum_\rho z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big) - \log(2\pi) + E(a,y,\ell),$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a,y,\ell) \ll_\ell |z|^{1/2}\begin{cases} 1 & \text{if } |y| \leq a \\ 1+\log^2(|y|/a) & \text{if } |y| > a. \end{cases} \tag{16}$$

Proof. Following the line of Hardy and Littlewood, see [4, §2.2], [5, Lemma 4] and of §4 in Linnik [13], we have

$$\begin{aligned} \widetilde{S}_\ell(z) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} &- \frac{1}{\ell}\sum_\rho z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big) - \frac{\zeta'}{\zeta}(0) - \sum_{m=1}^{\ell/4}\Gamma\Big(-\frac{2m}{\ell}\Big)z^{2m/\ell} \\ &- \frac{1}{2\pi i}\int_{\mathcal{L}_\ell}\frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w}\,\mathrm{d}w, \end{aligned} \tag{17}$$

where $\mathcal{L}_\ell$ is the vertical line $\Re(w) = -1/2$ if $4 \nmid \ell$ and it is $\{-1/2+it\colon |t| > C\} \cup \{-1/2+it\colon 1/\ell \leq |t| \leq C\} \cup \gamma_\ell$ otherwise, $C > 1/\ell$ is an absolute constant to be chosen later and γ_ℓ is the right half-circle centred in $-1/2$ of radius $1/\ell$.

Now we estimate the integral in (17). Assume $4 \nmid \ell$. Writing $w = -1/2 + it$, we have $|(\zeta'/\zeta)(\ell w)| \ll_\ell \log(|t| + 2)$, $|z^{-w}| = |z|^{1/2} \exp(t \arctan(y/a))$ by (14) and, for $|t| > C$, $\Gamma(w) \ll |t|^{-1} \exp(-\frac{\pi}{2}|t|)$ by (15). Letting $L_C = \{-1/2 + it\colon |t| > C\}$ we have

$$\int_{L_C} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w}\,\mathrm{d}w \ll_\ell |z|^{1/2} \int_{L_C} \frac{\log|t|}{|t|} \exp\Bigl(-\frac{\pi}{2}|t| + t\arctan(y/a)\Bigr)\,\mathrm{d}t.$$

If $ty \le 0$ we call η the quantity $\frac{\pi}{2} + |\arctan(y/a)| \in [\pi/2, \pi)$. If $|y| \le a$ we define η as $\frac{\pi}{2} - \arctan(y/a) > \frac{\pi}{2} - \arctan(1) = \frac{\pi}{4}$. In the remaining case ($|y| > a$ and $ty > 0$) we set $\eta = \arctan(a/|y|) \gg a/|y|$. Now fix C such that $C\eta < 1$ (e.g., $C = 1/\pi$ is allowed). Letting $u = \eta t$, we get

$$\begin{aligned}\int_{L_C} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w}\,\mathrm{d}w &\ll_\ell |z|^{1/2} \int_C^{+\infty} e^{-\eta t}\,\frac{\log t}{t}\,\mathrm{d}t = |z|^{1/2} \int_{C\eta}^{+\infty} e^{-u}\,\frac{\log(u/\eta)}{u}\,\mathrm{d}u \\ &= |z|^{1/2} \int_{C\eta}^{+\infty} e^{-u}\,\frac{\log u}{u}\,\mathrm{d}u + |z|^{1/2}\log(1/\eta) \int_{C\eta}^{+\infty} e^{-u}\,\frac{\mathrm{d}u}{u} = J_1 + J_2. \qquad (18)\end{aligned}$$

We remark that $0 \le u^{-1}\log u \le e^{-1}$ for $u \ge 1$, since the maximum of $u^{-1}\log u$ is attained at $u = e$. Since

$$0 \le \int_1^{+\infty} e^{-u}\,\frac{\log u}{u}\,\mathrm{d}u \le e^{-1} \int_1^{+\infty} e^{-u}\,\mathrm{d}u \ll 1$$

and

$$\Bigl|\int_{C\eta}^1 e^{-u}\,\frac{\log u}{u}\,\mathrm{d}u\Bigr| \le \int_{C\eta}^1 \frac{-\log u}{u}\,\mathrm{d}u = \Bigl[-\frac{1}{2}\log^2 u\Bigr]_{C\eta}^1 \ll \log^2(1/\eta)$$

we have $J_1 \ll |z|^{1/2}\log^2(1/\eta)$. For J_2 it is sufficient to remark that

$$0 \le J_2 \le |z|^{1/2}\log(1/\eta)\Bigl(\int_{C\eta}^1 \frac{\mathrm{d}u}{u} + \int_1^{+\infty} e^{-u}\mathrm{d}u\Bigr) \ll |z|^{1/2}\log^2(1/\eta).$$

Inserting the last two estimates in (18), recalling the definition of η, remarking that the integration over $|t| \le C$ gives immediately a contribution $\ll_\ell |z|^{1/2}$, we get

$$\int_{\mathcal{L}_\ell} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w}\,\mathrm{d}w \ll_\ell |z|^{1/2} \begin{cases} 1 & \text{if } |y| \le a \\ 1 + \log^2(|y|/a) & \text{if } |y| > a. \end{cases}$$

provided that $4 \nmid \ell$. Recalling $(\zeta'/\zeta)(0) = \log(2\pi)$ and remarking that

$$\sum_{m=1}^{\ell/4} \Gamma\Bigl(-\frac{2m}{\ell}\Bigr) z^{2m/\ell} \ll_\ell |z|^{1/2}, \qquad (19)$$

we see that the case $4 \nmid \ell$ of the lemma is proved.

Assume now that $4 \mid \ell$. The computation over L_C can be performed as in the previous case; we can also choose $C = 1/\pi$ as we did before. On the vertical segments $\mathcal{S}$ given by $\Re(w) = -1/2$, $|\Im(w)| \in [1/\ell, C]$, we exploit the boundedness of the Γ-function and the estimate $|z^{-w}| \ll |z|^{1/2}$ which holds on $\mathcal{S}$ since the argument of z is bounded there. This gives

$$\frac{1}{2\pi i} \int_{\mathcal{S}} \frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w}\,\mathrm{d}w \ll_\ell |z|^{1/2}.$$

It remains to consider the contribution over γ_ℓ; on this path we can again make use of the boundedness of the Γ-function and that $|z^{-w}| \ll |z|^{1/2}$ since the argument of z is bounded on γ_ℓ. This leads to

$$\frac{1}{2\pi i}\int_{\gamma_\ell}\frac{\zeta'}{\zeta}(\ell w)\Gamma(w)z^{-w}\,\mathrm{d}w \ll_\ell |z|^{1/2}.$$

Summing up, for $4 \mid \ell$ we see that the integral in (17) is dominated by the right hand side of (16) and this, together with (19) and $(\zeta'/\zeta)(0) = \log(2\pi)$, proves this case of the lemma. ■

We remark that, at the cost of some other complications in the details, Lemma 1 can be extended to the case $\ell \in \mathbb{R}$, $\ell > 0$.

In the next sections we will need to perform several times a set of similar computations; we collected them in the following two lemmas, which extend Lemmas 4.2 and 4.3 in [10].

LEMMA 2. *Let $\ell \geq 1$ be an integer, let $\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta-function and $\alpha > 1$ be a parameter. For any fixed $c \geq 0$ the series*

$$\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\int_1^{+\infty}(\log u)^c\exp\Big(-\frac{\gamma}{\ell}\arctan\frac{1}{u}\Big)\frac{\mathrm{d}u}{u^{\alpha+\beta/\ell}}$$

converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge.

Proof. Setting $y = \arctan(1/u)$, for any real $\gamma > 0$ we have

$$\begin{aligned}\int_1^{+\infty}\exp\Big(-\frac{\gamma}{\ell}\arctan\frac{1}{u}\Big)\frac{\mathrm{d}u}{u^{\alpha+\beta/\ell}} &= \int_0^{\pi/4}\exp\Big(-\frac{\gamma y}{\ell}\Big)\frac{(\sin y)^{\alpha+\beta/\ell-2}}{(\cos y)^{\alpha+\beta/\ell}}\,\mathrm{d}y\\ \ll_\alpha \int_0^{\pi/4}\exp\Big(-\frac{\gamma y}{\ell}\Big)y^{\alpha+\beta/\ell-2}\,\mathrm{d}y &= \Big(\frac{\gamma}{\ell}\Big)^{1-\alpha-\beta/\ell}\int_0^{\pi\gamma/(4\ell)}\exp(-w)\,w^{\alpha+\beta/\ell-2}\,\mathrm{d}w\\ &\ll_{\alpha,\ell}\Big(\frac{\gamma}{\ell}\Big)^{1-\alpha-\beta/\ell}\big(\Gamma(\alpha-1)+\Gamma(\alpha+1/\ell-1)\big),\end{aligned}$$

since $0 < \beta < 1$. This shows that the series over γ converges for $\alpha > 3/2$. For $\alpha = 3/2$ essentially the same computation shows that the integral is $\gg \gamma^{-1/2-\beta/\ell}$ and it is well known that in this case the series over zeros diverges. ■

LEMMA 3. *Let $\ell \geq 1$ be an integer, $\alpha > 1$, $z = a+iy$, $a \in (0,1)$ and $y \in \mathbb{R}$. Let further $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function. We have*

$$\sum_{\rho}\Big|\frac{\gamma}{\ell}\Big|^{\beta/\ell-1/2}\int_{\mathbb{Y}_1\cup\mathbb{Y}_2}\exp\Big(\frac{\gamma}{\ell}\arctan\frac{y}{a}-\frac{\pi}{2}\Big|\frac{\gamma}{\ell}\Big|\Big)\frac{\mathrm{d}y}{|z|^{\alpha+\beta/\ell}} \ll_{\alpha,\ell} a^{1-\alpha-1/\ell},$$

where $\mathbb{Y}_1 = \{y \in \mathbb{R}\colon y\gamma \leq 0\}$ and $\mathbb{Y}_2 = \{y \in [-a,a]\colon y\gamma > 0\}$. The result remains true if we insert in the integral a factor $(\log(|y|/a))^c$, for any fixed $c \geq 0$.

Proof. We first work on $\mathbb{Y}_1$. By symmetry, we may assume that $\gamma > 0$. For $y \in (-\infty, 0]$ we have $(\gamma/\ell)\arctan(y/a) - \frac{\pi}{2}|\gamma/\ell| \leq -\frac{\pi}{2}|\gamma/\ell|$ and hence the quantity we are estimating becomes

$$\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{2}\,\frac{\gamma}{\ell}\Big)\int_{-\infty}^{0}\frac{\mathrm{d}y}{|z|^{\alpha+\beta/\ell}}$$
$$\ll_{\alpha,\ell}\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{2}\,\frac{\gamma}{\ell}\Big)a^{1-\alpha-\beta/\ell}\ll_{\alpha,\ell}a^{1-\alpha-1/\ell},$$

using $0<\beta<1$, standard zero-density estimates and (12). We consider now the integral over $\mathbb{Y}_2$. Again by symmetry we can assume that $\gamma>0$ and so we get

$$\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\int_{0}^{a}\exp\Big(\frac{\gamma}{\ell}\Big(\arctan\frac{y}{a}-\frac{\pi}{2}\Big)\Big)\frac{\mathrm{d}y}{|z|^{\alpha+\beta/\ell}}$$
$$\ll\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{4}\frac{\gamma}{\ell}\Big)\int_{0}^{a}\frac{\mathrm{d}y}{|z|^{\alpha+\beta/\ell}}$$
$$\ll_{\alpha,\ell}\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{4}\frac{\gamma}{\ell}\Big)a^{1-\alpha-\beta/\ell}\ll_{\alpha,\ell}a^{1-\alpha-1/\ell}$$

arguing as above. The other assertions are proved in the same way. ∎

5. Interchange of summation over zeros with the line integral in I_3. We need $k>1/2$ in this section. We need to establish the convergence of

$$\sum_{\rho}\Big|\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|\int_{(1/N)}|e^{Nz}|\,|z|^{-k-1}\,|z^{-\rho/\ell}|\,|\mathrm{d}z|. \tag{20}$$

By (14) and the Stirling formula (15), we are left with estimating

$$\sum_{\rho}\Big|\frac{\gamma}{\ell}\Big|^{\beta/\ell_j-1/2}\int_{\mathbb{R}}\exp\Big(\frac{\gamma}{\ell}\arctan(Ny)-\frac{\pi}{2}\Big|\frac{\gamma}{\ell}\Big|\Big)\frac{\mathrm{d}y}{|z|^{k+1+\beta/\ell}}\,. \tag{21}$$

We have just to consider the case $\gamma y>0$, $|y|>1/N$ since in the other cases the total contribution is $\ll_{k,\ell}N^{k+1/\ell}$ by Lemma 3 with $\alpha=k+1$ and $a=1/N$. By symmetry, we may assume that $\gamma>0$. We see that the integral in (21) is

$$\ll_{\ell}\sum_{\rho\colon\,\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\int_{1/N}^{+\infty}\exp\Big(-\frac{\gamma}{\ell}\arctan\frac{1}{Ny}\Big)\frac{\mathrm{d}y}{y^{k+1+\beta/\ell}}$$
$$=N^{k}\sum_{\rho\colon\,\gamma>0}N^{\beta/\ell}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\int_{1}^{+\infty}\exp\Big(-\frac{\gamma}{\ell}\arctan\frac{1}{u}\Big)\frac{\mathrm{d}u}{u^{k+1+\beta/\ell}}\,.$$

For $k>1/2$ this is $\ll_{k,\ell}N^{k+1/\ell}$ by Lemma 2. This implies that the integrals in (21) and in (20) are both $\ll_{k,\ell}N^{k+1/\ell}$ and hence the exchange steps for I_3 are fully justified.

6. Interchange of summation over zeros with the line integral in I_4. We need $k>1/2-1/\ell_2$ in this section. We need to establish the convergence of

$$\sum_{\rho}\Big|\Gamma\Big(\frac{\rho}{\ell_1}\Big)\Big|\int_{(\frac{1}{N})}|e^{Nz}|\,|z|^{-k-1-1/\ell_2}\,|z^{-\rho/\ell_1}|\,|\mathrm{d}z| \tag{22}$$

and of the case in which ℓ_1 and ℓ_2 are interchanged. By (14) and the Stirling formula (15), we are left with estimating

$$\sum_{\rho}\Big|\frac{\gamma}{\ell_1}\Big|^{\beta/\ell_1-1/2}\int_{\mathbb{R}}\exp\Big(\frac{\gamma}{\ell_1}\arctan(Ny)-\frac{\pi}{2}\Big|\frac{\gamma}{\ell_1}\Big|\Big)\frac{\mathrm{d}y}{|z|^{k+1+1/\ell_2+\beta/\ell_1}}\,. \tag{23}$$

We have just to consider the case $\gamma y>0$, $|y|>1/N$ since in the other cases the total contribution is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ by Lemma 3 with $\alpha=k+1+1/\ell_2$ and $a=1/N$. By symmetry, we may assume that $\gamma>0$. We have that the integral in (23) is

$$\begin{aligned}
&\ll_{\ell_1}\sum_{\rho:\ \gamma>0}\Big(\frac{\gamma}{\ell_1}\Big)^{\beta/\ell_1-1/2}\int_{1/N}^{+\infty}\exp\Big(-\frac{\gamma}{\ell_1}\arctan\frac{1}{Ny}\Big)\frac{\mathrm{d}y}{y^{k+1+1/\ell_2+\beta/\ell_1}}\\
&=N^{k+1/\ell_2}\sum_{\rho:\ \gamma>0}N^{\beta/\ell_1}\Big(\frac{\gamma}{\ell_1}\Big)^{\beta/\ell_1-1/2}\int_{1}^{+\infty}\exp\Big(-\frac{\gamma}{\ell_1}\arctan\frac{1}{u}\Big)\frac{\mathrm{d}u}{u^{k+1+1/\ell_2+\beta/\ell_1}}\,.
\end{aligned}$$

For $k>1/2-1/\ell_2$ this is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ by Lemma 2. This implies that the integrals in (23) and in (22) are both $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ and hence the exchange step for I_4 is fully justified.

7. Interchange of the double summation over zeros with the line integral in I_5. We need $k>1$ in this section. Arguing as in Sections 5–6, we first need to establish the convergence of

$$\sum_{\rho_1}\Big|\Gamma\Big(\frac{\rho_1}{\ell_1}\Big)\Big|\int_{(1/N)}\Big|\sum_{\rho_2}\Gamma\Big(\frac{\rho_2}{\ell_2}\Big)z^{-\rho_2/\ell_2}\Big||e^{Nz}|\,|z|^{-k-1}\,|z^{-\rho_1/\ell_1}|\,|\mathrm{d}z|. \tag{24}$$

Using the Prime Number Theorem and (16), we first remark that

$$\Big|\sum_{\rho_2}\Gamma\Big(\frac{\rho_2}{\ell_2}\Big)z^{-\rho_2/\ell_2}\Big|\ll_{\ell_2} N^{1/\ell_2}+|z|^{1/2}\log^2(2N|y|). \tag{25}$$

By symmetry, we may assume that $\gamma_1>0$. By (25), (12), (14) and (9), for $y\in(-\infty,0]$ we are first led to estimate

$$\begin{aligned}
&\sum_{\rho_1:\ \gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\Big)\Big(\int_{-1/N}^{0}N^{k+1+1/\ell_2+\beta_1/\ell_1}\,\mathrm{d}y\\
&+N^{1/\ell_2}\int_{-\infty}^{-1/N}\frac{\mathrm{d}y}{|y|^{k+1+\beta_1/\ell_1}}+\int_{-\infty}^{-1/N}\log^2(2N|y|)\frac{\mathrm{d}y}{|y|^{k+1/2+\beta_1/\ell_1}}\Big)\ll_{k,\ell_1,\ell_2} N^{k+\lambda}
\end{aligned}$$

by the same argument used in the proof of Lemma 3 with $\alpha=k+1/2$ and $a=1/N$. On the other hand, for $y>0$ we split the range of integration into $(0,1/N]\cup(1/N,+\infty)$. By (25), (12) and Lemma 3 with $\alpha=k+1$ and $a=1/N$, on $[0,1/N]$ we have

$$N^{1/\ell_2}\sum_{\rho_1:\ \gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\int_0^{1/N}\exp\Big(\frac{\gamma_1}{\ell_1}\Big(\arctan(Ny)-\frac{\pi}{2}\Big)\Big)\frac{\mathrm{d}y}{|z|^{k+1+\beta_1/\ell_1}}\ll_{k,\ell_1,\ell_2} N^{k+\lambda}.$$

On the other interval, again by (12), we have to estimate

$$\sum_{\rho_1\,:\,\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\int_{1/N}^{+\infty}\exp\Big(-\frac{\gamma_1}{\ell_1}\arctan\frac{1}{Ny}\Big)\frac{N^{1/\ell_2}+y^{1/2}\log^2(2Ny)}{y^{k+1+\beta_1/\ell_1}}\,\mathrm{d}y$$
$$= N^k\sum_{\rho_1\,:\,\gamma_1>0}N^{\beta_1/\ell_1}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}$$
$$\times\int_1^{+\infty}\exp\Big(-\frac{\gamma_1}{\ell_1}\arctan\frac{1}{u}\Big)\frac{N^{1/\ell_2}+u^{1/2}N^{-1/2}\log^2(2u)}{u^{k+1+\beta_1/\ell_1}}\,\mathrm{d}u.$$

Recalling (9), Lemma 2 with $\alpha = k + 1/2$ shows that the last term is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$. This implies that the integral in (24) is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ provided that $k > 1$ and hence we can exchange the first summation with the integral in this case.

To exchange the second summation we have to consider

$$\sum_{\rho_1}\Big|\Gamma\Big(\frac{\rho_1}{\ell_1}\Big)\Big|\sum_{\rho_2}\Big|\Gamma\Big(\frac{\rho_2}{\ell_2}\Big)\Big|\int_{(1/N)}|e^{Nz}|\,|z|^{-k-1}|z^{-\rho_1/\ell_1}|\,|z^{-\rho_2/\ell_1}|\,|\mathrm{d}z|. \tag{26}$$

By symmetry, we can consider $\gamma_1,\gamma_2 > 0$ or $\gamma_1 > 0$, $\gamma_2 < 0$.

Assuming $\gamma_1,\gamma_2 > 0$, for $y \le 0$ we have $(\gamma_j/\ell_j)\arctan(Ny) - \frac{\pi}{2}(\gamma_j/\ell_j) \le -\frac{\pi}{2}(\gamma_j/\ell_j)$, $j = 1, 2$, and, by (14), the corresponding contribution to (26) is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ since

$$\sum_{\rho_1\,:\,\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\Big)$$
$$\times\sum_{\rho_2\,:\,\gamma_2>0}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_2}{\ell_2}\Big)\Big(\int_{-\infty}^{0}\frac{\mathrm{d}y}{|z|^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}\Big)$$
$$\ll_k N^{k+\lambda}\sum_{\rho_1\,:\,\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\Big)\sum_{\rho_2\,:\,\gamma_2>0}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_2}{\ell_2}\Big),$$

using standard zero-density estimates, (12) and (9). On the other hand, for $y > 0$ we split the range of integration into $(0, 1/N] \cup (1/N, +\infty)$. On the first interval we have

$$\sum_{\rho_1\,:\,\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\sum_{\rho_2\,:\,\gamma_2>0}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}$$
$$\times\int_0^{1/N}\exp\Big(\Big(\frac{\gamma_1}{\ell_1}+\frac{\gamma_2}{\ell_2}\Big)\Big(\arctan(Ny)-\frac{\pi}{2}\Big)\Big)\frac{\mathrm{d}y}{|z|^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}$$
$$\ll\sum_{\rho_1\,:\,\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\sum_{\rho_2\,:\,\gamma_2>0}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}$$
$$\times\exp\Big(-\frac{\pi}{4}\Big(\frac{\gamma_1}{\ell_1}+\frac{\gamma_2}{\ell_2}\Big)\Big)\int_0^{1/N}N^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}\,\mathrm{d}y$$
$$\ll_{k,\ell_1,\ell_2} N^{k+\lambda}\sum_{\rho_1\,:\,\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{4}\frac{\gamma_1}{\ell_1}\Big)\sum_{\rho_2\,:\,\gamma_2>0}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}\exp\Big(-\frac{\pi}{4}\frac{\gamma_2}{\ell_2}\Big),$$

which is also $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$, by the same argument as above. With similar computations, on the other interval we have

$$\sum_{\rho_1\colon\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\sum_{\rho_2\colon\gamma_2>0}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}$$
$$\times\int_{1/N}^{+\infty}\exp\Big(\Big(\frac{\gamma_1}{\ell_1}+\frac{\gamma_2}{\ell_2}\Big)\Big(\arctan(Ny)-\frac{\pi}{2}\Big)\Big)\frac{\mathrm{d}y}{y^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}$$
$$=N^k\sum_{\rho_1\colon\gamma_1>0}N^{\beta_1/\ell_1}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\sum_{\rho_2\colon\gamma_2>0}N^{\beta_2/\ell_2}\Big(\frac{\gamma_2}{\ell_2}\Big)^{\beta_2/\ell_2-1/2}$$
$$\times\int_{1}^{+\infty}\exp\Big(-\Big(\frac{\gamma_1}{\ell_1}+\frac{\gamma_2}{\ell_2}\Big)\arctan\frac{1}{u}\Big)\frac{\mathrm{d}u}{u^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}\,.$$

Arguing as in the proof of Lemma 2, we prove that the integral on the right is $\asymp_{k,\ell_1,\ell_2}$ $(\gamma_1+\gamma_2)^{-k-\beta_1/\ell_1-\beta_2/\ell_2}$. The inequality

$$\frac{\gamma_1^{\beta_1/\ell_1-1/2}\gamma_2^{\beta_2/\ell_2-1/2}}{(\gamma_1+\gamma_2)^{\beta_1/\ell_1+\beta_2/\ell_2}}\le\frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}}\tag{27}$$

shows, by using (9), that it is sufficient to consider

$$N^k\sum_{\rho_1\colon\gamma_1>0}\sum_{\rho_2\colon\gamma_2>0}N^{\beta_1/\ell_1+\beta_2/\ell_2}\frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}(\gamma_1+\gamma_2)^k}$$
$$\ll_{k,\ell_1,\ell_2}N^{k+\lambda}\sum_{\rho_1\colon\gamma_1>0}\frac{1}{\gamma_1^{k+1/2}}\sum_{\rho_2\colon 0<\gamma_2\le\gamma_1}\frac{1}{\gamma_2^{1/2}}\ll_{k,\ell_1,\ell_2}N^{k+\lambda}\sum_{\rho_1\colon\gamma_1>0}\frac{\log\gamma_1}{\gamma_1^{k}}$$

and the last series over zeros converges for $k>1$.

Assume now $\gamma_1>0$, $\gamma_2<0$. For $y\le0$ we have $\frac{\gamma_1}{\ell_1}\arctan(Ny)-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\le-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}$, by (12) and (9) the corresponding contribution to (26) is

$$\ll_{k,\ell_1,\ell_2}\sum_{\rho_1\colon\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\Big)$$
$$\times\Big\{\sum_{\rho_2\colon\gamma_2<0}\Big|\frac{\gamma_2}{\ell_2}\Big|^{\beta_2/\ell_2-1/2}\Big[\exp\Big(-\frac{\pi}{4}\Big|\frac{\gamma_2}{\ell_2}\Big|\Big)\int_{-1/N}^{0}N^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}\,\mathrm{d}y$$
$$+\int_{-\infty}^{-1/N}\exp\Big(-\Big|\frac{\gamma_2}{\ell_2}\Big|\Big(\arctan(Ny)+\frac{\pi}{2}\Big)\Big)\frac{\mathrm{d}y}{|y|^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}\Big]\Big\}$$
$$\ll_{k,\ell_1,\ell_2}N^{k+\lambda}\sum_{\rho_1\colon\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\Big)\sum_{\rho_2\colon\gamma_2<0}\Big|\frac{\gamma_2}{\ell_2}\Big|^{\beta_2/\ell_2-1/2}\exp\Big(-\frac{\pi}{4}\Big|\frac{\gamma_2}{\ell_2}\Big|\Big)$$
$$+N^{k+\lambda}\sum_{\rho_1\colon\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma_1}{\ell_1}\Big)\sum_{\rho_2\colon\gamma_2<0}\Big|\frac{\gamma_2}{\ell_2}\Big|^{\beta_2/\ell_2-1/2}$$
$$\times\int_{1}^{+\infty}\exp\Big(-\Big|\frac{\gamma_2}{\ell_2}\Big|\arctan\frac{1}{u}\Big)\frac{\mathrm{d}u}{u^{k+1+\beta_1/\ell_1+\beta_2/\ell_2}}$$
$$\ll_{k,\ell_1,\ell_2}N^{k+\lambda}+N^{k+\lambda}\sum_{\rho_1\colon\gamma_1>0}\Big(\frac{\gamma_1}{\ell_1}\Big)^{\beta_1/\ell_1-1/2}\exp\Big(-\frac{\pi}{2}\gamma_1\Big)\ll_{k,\ell_1,\ell_2}N^{k+\lambda}$$

for $k>1/2$, by Lemma 2 and standard zero-density estimates.

On the other hand, the case $\gamma_1 > 0$, $\gamma_2 < 0$ and $y > 0$ can be estimated in a similar way essentially exchanging the role of γ_1 and γ_2 in the previous argument.

This implies that the integral in (26) is $\ll_{k,\ell_1,\ell_2} N^{k+\lambda}$ provided that $k > 1$. Combining the convergence conditions for (24)–(26), we see that we can exchange both summations with the integral provided that $k > 1$.

8. Convergence of the double sum over zeros. In this section we prove that the double sum on the right of (3) converges absolutely for every $k > 1/2$; the other series in (1) and (2) clearly converge for $k > 0$ or better. We need (15) uniformly for $x \in [0, k+3]$ and $|y| \ge T$, where T is large but fixed: this provides both an upper and a lower bound for $|\Gamma(x+iy)|$. Let

$$\Sigma = \sum_{\rho_1}\sum_{\rho_2}\left|\frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(\rho_1/\ell_1+\rho_2/\ell_2+k+1)}\right|,$$

so that, by the symmetry of the zeros of the Riemann zeta-function, we have

$$\begin{aligned}\Sigma &= 2\sum_{\substack{\rho_1\colon \gamma_1>0\\ \rho_2\colon \gamma_2>0}}\sum\left|\frac{\Gamma(\rho_1/\ell_1)\Gamma(\rho_2/\ell_2)}{\Gamma(\rho_1/\ell_1+\rho_2/\ell_2+k+1)}\right| + 2\sum_{\substack{\rho_1\colon \gamma_1>0\\ \rho_2\colon \gamma_2>0}}\sum\left|\frac{\Gamma(\rho_1/\ell_1)\Gamma(\overline{\rho}_2/\ell_2)}{\Gamma(\rho_1/\ell_1+\overline{\rho}_2/\ell_2+k+1)}\right| \\ &= 2(\Sigma_1+\Sigma_2),\end{aligned}$$

say. It is clear that if both Σ_1 and Σ_2 converge, then the double sum on the right-hand side of (3) converges absolutely. In order to estimate Σ_1 we choose a large T and let

$$\begin{aligned}
D_0 &= \{(\rho_1,\rho_2)\colon (\gamma_1,\gamma_2)\in[0,2T]^2\}, & D_3 &= \{(\rho_1,\rho_2)\colon \gamma_2\ge T,\ T\le\gamma_1\le\gamma_2\},\\
D_1 &= \{(\rho_1,\rho_2)\colon \gamma_1\ge T,\ T\le\gamma_2\le\gamma_1\}, & D_4 &= \{(\rho_1,\rho_2)\colon \gamma_2\ge T,\ 0\le\gamma_1\le T\},\\
D_2 &= \{(\rho_1,\rho_2)\colon \gamma_1\ge T,\ 0\le\gamma_2\le T\},
\end{aligned}$$

so that $\Sigma_1 \le \Sigma_{1,0}+\Sigma_{1,1}+\Sigma_{1,2}+\Sigma_{1,3}+\Sigma_{1,4}$, say, where $\Sigma_{1,j}$ is the sum with $(\rho_1,\rho_2)\in D_j$. Now, D_0 contributes a bounded amount, that depends only on T, and, by symmetry again, $\Sigma_{1,1}=\Sigma_{1,3}$ and $\Sigma_{1,2}=\Sigma_{1,4}$. We also recall the inequality (27) which is valid for all couples of zeros considered in Σ_1. Hence

$$\begin{aligned}
\Sigma_{1,1} &\ll_{\ell_1,\ell_2} \sum_{\substack{\rho_1\colon \gamma_1\ge T\\ \rho_2\colon T\le\gamma_2\le\gamma_1}}\sum \frac{e^{-\pi(\gamma_1/\ell_1+\gamma_2/\ell_2)/2}(\gamma_1/\ell_1)^{\beta_1/\ell_1-1/2}(\gamma_2/\ell_2)^{\beta_2/\ell_2-1/2}}{e^{-\pi(\gamma_1/\ell_1+\gamma_2/\ell_2)/2}(\gamma_1/\ell_1+\gamma_2/\ell_2)^{\beta_1/\ell_1+\beta_2/\ell_2+k+1/2}}\\
&\ll_{\ell_1,\ell_2} \sum_{\substack{\rho_1\colon \gamma_1\ge T\\ \rho_2\colon T\le\gamma_2\le\gamma_1}}\sum \frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}(\gamma_1+\gamma_2)^{k+1/2}}\\
&\ll_{\ell_1,\ell_2} \sum_{\rho_1\colon \gamma_1\ge T}\frac{1}{\gamma_1^{k+1}}\sum_{\rho_2\colon T\le\gamma_2\le\gamma_1}\frac{1}{\gamma_2^{1/2}} \ll_{\ell_1,\ell_2} \sum_{\rho_1\colon \gamma_1\ge T}\frac{\log\gamma_1}{\gamma_1^{k+1/2}}.
\end{aligned}$$

A similar argument proves that

$$\Sigma_{1,2} \ll_{k,T,\ell_1,\ell_2} \sum_{\rho_1\colon \gamma_1\ge T}\frac{1}{\gamma_1^{k+1}},$$

since $\Gamma(\rho_2)$ is uniformly bounded, in terms of T, for $(\rho_1,\rho_2)\in D_2$. Summing up, we have

$$\Sigma_1 \ll_{k,T,\ell_1,\ell_2} 1+\sum_{\rho_1\colon\, \gamma_1\geq T}\frac{\log\gamma_1}{\gamma_1^{k+1/2}},$$

which is convergent provided that $k>1/2$. In order to estimate Σ_2 we use a similar argument. Choose a large T and for $\{i,j\}=\{1,2\}$ set

$$\begin{aligned}
E_0(i,j)&=\Big\{(\rho_1,\rho_2)\colon \Big(\frac{\gamma_i}{\ell_i},\frac{\gamma_j}{\ell_j}\Big)\in[0,2T]^2\Big\},\\
E_1(i,j)&=\Big\{(\rho_1,\rho_2)\colon \frac{\gamma_i}{\ell_i}\geq 2T,\ 0\leq\frac{\gamma_j}{\ell_j}\leq T\Big\},\\
E_2(i,j)&=\Big\{(\rho_1,\rho_2)\colon \frac{\gamma_i}{\ell_i}\geq 2T,\ T\leq\frac{\gamma_j}{\ell_j}\leq\frac{\gamma_i}{\ell_i}-T\Big\},\\
E_3(i,j)&=\Big\{(\rho_1,\rho_2)\colon \frac{\gamma_i}{\ell_i}\geq 2T,\ \frac{\gamma_i}{\ell_i}-T\leq\frac{\gamma_j}{\ell_j}\leq\frac{\gamma_i}{\ell_i}\Big\},
\end{aligned}$$

so that $\Sigma_2\leq\Sigma_0(1,2)+\Sigma_1(1,2)+\Sigma_2(1,2)+\Sigma_3(1,2)+\Sigma_3(2,1)+\Sigma_2(2,1)+\Sigma_1(2,1)$, say, where $\Sigma_r(i,j)$ is the sum with $(\rho_1,\rho_2)\in E_r(i,j)$. Now, E_0 contributes a bounded amount, that depends only on T, ℓ_1 and ℓ_2. We remark that similar arguments apply when dealing with $\Sigma_1(1,2)$ and $\Sigma_1(2,1)$; $\Sigma_2(1,2)$ and $\Sigma_2(2,1)$; $\Sigma_3(1,2)$ and $\Sigma_3(2,1)$ respectively. Again we use (15) as above; hence

$$\begin{aligned}
&\Sigma_2(1,2)\\
&\quad\ll_{\ell_1,\ell_2}\Big(\sum_{\substack{(\rho_1,\rho_2)\in E_2(1,2)\\ \gamma_2\leq\gamma_1^{1/2}}}\sum+\sum_{\substack{(\rho_1,\rho_2)\in E_2(1,2)\\ \gamma_2>\gamma_1^{1/2}}}\sum\Big)\frac{(\gamma_1/\ell_1)^{\beta_1/\ell_1-1/2}(\gamma_2/\ell_2)^{\beta_2/\ell_2-1/2}e^{-\pi\gamma_2/\ell_2}}{(\gamma_1/\ell_1-\gamma_2/\ell_2)^{\beta_1/\ell_1+\beta_2/\ell_2+k+1/2}}.
\end{aligned}$$

We bound the first sum by a further subdivision of the zeros ρ_2, treating differently those with $\beta_2<\ell_2/2$ and the other ones, if any. The first sum is

$$\begin{aligned}
&\ll_{\ell_1,\ell_2} e^{-\pi T}\sum_{\gamma_1\geq 2T\ell_1}\gamma_1^{\beta_1/\ell_1-1/2}\sum_{\gamma_2\in[T\ell_2,\gamma_1^{1/2}]}\frac{\gamma_2^{\beta_2/\ell_2-1/2}}{\gamma_1^{\beta_1/\ell_1+\beta_2/\ell_2+k+1/2}}\\
&\ll_{T,\ell_1,\ell_2}\sum_{\gamma_1\geq 2T\ell_1}\frac{1}{\gamma_1^{k+3/2}}\Big(\sum_{\substack{\beta_2<\ell_2/2\\ \gamma_2\in[T\ell_2,\gamma_1^{1/2}]}}+\sum_{\substack{\beta_2\geq\ell_2/2\\ \gamma_2\in[T\ell_2,\gamma_1^{1/2}]}}\Big)\Big(\frac{\gamma_2}{\gamma_1}\Big)^{\beta_2/\ell_2-1/2}\\
&\ll_{T,\ell_1,\ell_2}\sum_{\gamma_1\geq 2T\ell_1}\frac{1}{\gamma_1^{k+3/2}}\Big(\sum_{\substack{\beta_2<\ell_2/2\\ \gamma_2\in[T\ell_2,\gamma_1^{1/2}]}}\Big(\frac{\gamma_1}{T}\Big)^{1/2-\beta_2/\ell_2}+\gamma_1^{1/2}\log\gamma_1\Big)\\
&\ll_{T,\ell_1,\ell_2}\sum_{\gamma_1\geq 2T\ell_1}\frac{\log\gamma_1}{\gamma_1^{k+1/2}}.
\end{aligned}$$

The rightmost series over zeros plainly converges for $k > 1/2$. The second sum is

$$\begin{aligned}
&\ll_{T,\ell_1,\ell_2} \sum_{\gamma_1 \ge 2T\ell_1} \gamma_1^{\beta_1/\ell_1-1/2} e^{-\pi\gamma_1^{1/2}/\ell_2} \\
&\quad\times \sum_{\gamma_2 \in [\gamma_1^{1/2},(\gamma_1/\ell_1-T)\ell_2]} \frac{\gamma_2^{\beta_2/\ell_2-1/2}}{(\gamma_1/\ell_1-\gamma_2/\ell_2)^{\beta_1/\ell_1+\beta_2/\ell_2+k+1/2}} \\
&\ll_{T,\ell_1,\ell_2} \sum_{\gamma_1 \ge 2T\ell_1} \gamma_1^{\beta_1/\ell_1-1/2} e^{-\pi\gamma_1^{1/2}/\ell_2} (\gamma_1 \log \gamma_1) T^{-(\beta_1/\ell_1+k+1/2)} \gamma_1^{1/2},
\end{aligned}$$

which is very small. The contribution of zeros in $E_1(1,2)$ is treated in a similar fashion, using the uniform upper bound $\Gamma(\rho_2) \ll_T 1$, and is also small. We now deal with $\Sigma_3(1,2)$: we have

$$\begin{aligned}
\Sigma_3(1,2) &\ll_{\ell_1,\ell_2} \sum\sum_{(\rho_1,\rho_2)\in E_3} e^{-\pi\gamma_1/(2\ell_1)} \gamma_1^{\frac{\beta_1}{\ell_1}-\frac{1}{2}} e^{-\pi\gamma_2/(2\ell_2)} \gamma_2^{\frac{\beta_2}{\ell_2}-\frac{1}{2}} \Bigl(\min_{\substack{k+1\le x\le k+3 \\ 0\le t\le T}} |\Gamma(x+it)| \Bigr)^{-1} \\
&\ll_{k,T,\ell_1,\ell_2} \sum_{\rho_1 \colon \gamma_1 \ge 2T\ell_1} e^{-\pi\gamma_1/\ell_1} \gamma_1^{\beta_1/\ell_1+1/\ell_1} \log(\gamma_1+T),
\end{aligned}$$

provided that T is large enough. Here we are using Theorem 9.2 of Titchmarsh [15] with T large but fixed. The series at the extreme right is plainly convergent.

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