FUNCTION SPACES XII
BANACH CENTER PUBLICATIONS, VOLUME 119
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2019

ON DECOMPOSITION PROBLEM IN WEIGHTED HARDY SPACE

VOLODYMYR DILNYI

Institute of Mathematics, Cracow University of Technology
Warszawska str., 24, 31-155 Kraków, Poland

Department of Applied Mathematics, University of Agriculture
Balicka str., 253c, 30-198 Kraków, Poland

Department of Mathematics, Drohobych State Pedagogical University
Stryiska str., 3, 82100 Drohobych, Ukraine
E-mail: dilnyi@ukr.net

KHRYSTYNA HUK

Department of Mathematics, Drohobych State Pedagogical University
Stryiska str., 3, 82100 Drohobych, Ukraine
E-mail: khrystyna.huk2711@gmail.com

Abstract. We study in this paper the problem of decomposition of functions in the Paley–Wiener space into the sum of two functions, each being “large” only on some part of the complex plane.

1. Introduction. Representations of a function as a sum or a product of two functions of simpler nature is an important and useful method of studies in the analytic function theory. Some results in this direction were obtained by R. S. Yulmukhametov [13], [14], Yu. I. Lyubarskii [10], I. E. Chyzhykov [4] and others.

We denote by W^p_σ, $\sigma > 0$, the Paley–Wiener space, i.e., the space of entire functions f of exponential type $\leq \sigma$ belonging to $L^p(\mathbb{R})$. R. Paley and N. Wiener proved the following fundamental theorem.

2010 *Mathematics Subject Classification*: Primary 30H10; Secondary 30D10, 47B40.
Key words and phrases: weighted Hardy space, decomposition, Paley–Wiener space.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc119-7

THEOREM A. *The space W^2_σ coincides with the space of functions representable as*

$$f(z) = \int_{-\sigma}^{\sigma} \varphi(it)e^{itz}\, dt, \qquad \varphi \in L^2(-i\sigma; i\sigma). \tag{1}$$

The space W^p_σ can be defined (see [9]) as the space of entire functions satisfying the condition

$$\sup_{\varphi\in(0;2\pi)} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r|\sin\varphi|}\, dr \right\}^{1/p} < +\infty.$$

R. Boas [3] and G. Ber [2] obtained the representation theorems for the space W^1_σ.

THEOREM B ([6]). *The space W^1_σ coincides with the space of functions f represented by* (1), *where*

$$\varphi(t) = \frac{1}{2\sigma} \sum_{k=-\infty}^{+\infty} c_k e^{-ik\pi t/\sigma}, \qquad (c_k) \in l^1, \tag{2}$$

and

$$\sum_{m=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} (-1)^{k+m} c_{k+m} \frac{k}{1+k^2} \right| < +\infty.$$

Let $E^p[\mathbb{C}(\alpha;\beta)]$, $0 < \beta - \alpha < 2\pi$, $1 \le p < +\infty$, be the space of analytic functions f in $\mathbb{C}(\alpha;\beta) = \{z : \alpha < \arg z < \beta\}$, for which

$$\sup_{\alpha<\varphi<\beta} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|\, dr \right\} < +\infty.$$

The functions $f \in E^p[(\alpha;\beta)]$ have almost everywhere on $\partial\mathbb{C}(\alpha;\beta)$ the angular boundary values [8], which we denote by f, and $f \in L^p[\mathbb{C}(\alpha;\beta)]$.

2. The main result. B. Vynnytskyi and V. Dilnyi considered [12] the following problem of decomposition for functions in the Paley–Wiener space into the sum of two functions, each of which is "large" only in some domain. This problem has applications in the studies of the completeness [6] and is interesting in the theory of integral operators as well as in the invariant subspaces theory (see [1], [11]).

PROBLEM. *Which functions $f \in W^p_\sigma$, $1 \le p \le 2$, admit the decomposition $f = \chi - \mu$ with entire functions χ and μ, where $\chi \in E^p[\mathbb{C}(0;\pi/2)]$, $\mu \in E^p[\mathbb{C}(-\pi/2;0)]$?*

For the case $p = 2$ there exists the elementary solution of the Problem based on the Paley–Wiener theorem:

$$\chi(z) = \int_0^{\sigma} \varphi(it)e^{itz}\, dt, \quad \mu(z) = -\int_{-\sigma}^{0} \varphi(it)e^{itz}\, dt.$$

The case $p = 1$ is more difficult and important for applications (see [6], [12]). For some partial cases solutions of the Problem are known (see [5], [7]).

THEOREM 1 ([7]). *If $f \in W^1_\sigma$ and $c_k = 0$ for each odd $k \in \mathbb{N}$, then for f there exists a decomposition in the sense of above Problem.*

In [6] also the following simpler problem is studied:

Is it possible to decompose each $f \in W_\sigma^p$, $1 \le p \le 2$, as $f = \chi - \mu$, where χ, μ are entire functions and $\chi \in E^p[\mathbb{C}(0;\pi)]$, $\mu \in E^p[\mathbb{C}(-\pi;0)]$?

The answer to this question is in the negative. For example, for the function

$$f(z) = (1 - \cos \sigma z)/z^2$$

such a decomposition is impossible. T. I. Hishchak has obtained the following result.

THEOREM 2 ([7]). *Let $f \in W_\sigma^1$. The functions $\chi(z) = \chi_1(z) + i\chi_2(-iz)$ and $\mu = \chi - f$, where*

$$\chi_1(z) = \int_0^\sigma \varphi(it)e^{itz}\,dt, \quad \chi_2(z) = -\int_{-\sigma}^0 \varphi(it)e^{itz}\,dt$$

are a solution of the Problem if and only if both of the following conditions are fulfilled

$$\sum_{m=1}^{+\infty}\left|\sum_{k=-\infty}^{+\infty} c_k \frac{k}{(m-i/2-k)(m-i/2-ik)}\right| < +\infty, \tag{3}$$

$$\sum_{m=1}^{+\infty}\left|\sum_{k=-\infty}^{+\infty} c_k \frac{k}{(m+i/2+ik)(m+i/2-k)}\right| < +\infty. \tag{4}$$

As a consequence of equalities (1)–(2) we have the representation

$$f(z) = \sum_{k=-\infty}^{+\infty} c_k \frac{\sin \sigma z}{\sigma z - \pi k}\,. \tag{5}$$

We obtain the following result.

THEOREM 3. *If $c_k \in l^1$ and $c_{2k} = -c_{2k+1}$ for each $k \in \mathbb{Z}$, then for the function f defined by (1)–(2) there exists a solution of the Problem and functions χ, μ can be defined as in the previous theorem.*

We note that if a decomposition in the sense of the Problem exists, then it is not unique.

LEMMA 4. *If $(c_k) \in l^1$ and $c_{2k} = -c_{2k+1}$ for every $k \in \mathbb{Z}$, then the function f defined by (1) and (2) belongs to W_σ^1.*

Proof. Consider representation (5). Since $c_{2k} = -c_{2k+1}$, we obtain

$$f(z) = -\sum_{k=-\infty}^{+\infty} \sin \sigma z \,\frac{c_{2k}\pi}{(\sigma z - 2\pi k)(\sigma z - 2\pi k - \pi)}\,.$$

Hence

$$\int_{-\infty}^{+\infty} |f(x+i)|\,dx \le M_1 \sum_{k=-\infty}^{+\infty} |c_{2k}| \int_{-\infty}^{+\infty} \frac{\pi\,dx}{|\sigma x + \sigma i - 2\pi k|\,|\sigma x + \sigma i - 2\pi k - \pi|}\,.$$

Since the sum below converges absolutely and uniformly, we have

$$\int_{-\infty}^{+\infty} |f(x+i)|\,dx \le M_1 \sum_{k=-\infty}^{+\infty} |c_{2k}| \Bigg(\int_{-\infty}^{-2\pi k-2\pi} \frac{\pi\,dx}{|\sigma x+\sigma i-2\pi k|\,|\sigma x+\sigma i-2\pi k-\pi|}$$
$$+ \int_{-2\pi k-2\pi}^{-2\pi k+2\pi} \frac{\pi\,dx}{|\sigma x+\sigma i-2\pi k|\,|\sigma x+\sigma i-2\pi k-\pi|}$$
$$+ \int_{-2\pi k+2\pi}^{+\infty} \frac{\pi\,dx}{|\sigma x+\sigma i-2\pi k|\,|\sigma x+\sigma i-2\pi k-\pi|} \Bigg)$$
$$\le M_1 \sum_{k=-\infty}^{+\infty} |c_{2k}| \int_{-\infty}^{+\infty} \frac{\pi}{(\sigma x-2\pi k)^2+\sigma^2}\,dx = M_2 \sum_{k=-\infty}^{+\infty} |c_{2k}| < +\infty.$$

Therefore $f \in W^1_\sigma$ (see, for example, [9]). ∎

Proof of Theorem 3. Let us denote by L the left side of inequality (3). Since $c_{2k} = -c_{2k+1}$, from (2) we obtain

$$L = \sum_{m=1}^{+\infty} \Bigg| \sum_{k=-\infty}^{+\infty} \left(\frac{c_k}{m-i/2-k} - \frac{c_k}{m-i/2-ki} \right) \frac{1}{1-i} \Bigg|$$
$$= \frac{1}{\sqrt{2}} \sum_{m=1}^{+\infty} \Bigg| \sum_{n=-\infty}^{+\infty} \frac{c_{2n}}{m-i/2-2n} - \frac{c_{2n}}{m-i/2-2ni}$$
$$+ \sum_{n=-\infty}^{+\infty} \frac{c_{2n+1}}{m-i/2-2n-1} - \frac{c_{2n+1}}{m-i/2-2ni-i} \Bigg|$$
$$= \frac{1}{\sqrt{2}} \sum_{m=1}^{+\infty} \Bigg| \sum_{n=-\infty}^{+\infty} \frac{-c_{2n}}{(m-i/2-2n)(m-i/2-2n-1)}$$
$$+ \frac{c_{2n} i}{(m-i/2-2ni)(m-i/2-2ni-i)} \Bigg|$$
$$\le \frac{1}{\sqrt{2}} \sum_{m=1}^{+\infty} \Bigg(\sum_{n=-\infty}^{+\infty} \frac{|c_{2n}|}{|m-i/2-2n||m-i/2-2n-1|}$$
$$+ \sum_{n=-\infty}^{+\infty} \frac{|c_{2n}|}{(m-i/2-2ni)(m-i/2-2ni-i)} \Bigg).$$

Changing the order of summation and employing the notation $l = m-2n$, we obtain

$$L \le \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{m=-\infty}^{+\infty} \frac{1}{\sqrt{(m-2n)^2+1/4}\,\sqrt{(m-2n-1)^2+1/4}}$$
$$+ \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{m=-\infty}^{+\infty} \frac{1}{\sqrt{m^2+(1/2+2n)^2}\,\sqrt{m^2+(1/2+2n+1)^2}}$$
$$\le \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{l=-\infty}^{+\infty} \frac{1}{\sqrt{l^2+1/4}\,\sqrt{(l-1)^2+1/4}} + \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{m=-\infty}^{+\infty} \frac{1}{m^2}.$$

The second term converges absolutely, so we need to show that the first series converges

$$\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{+\infty}|c_{2n}|\sum_{l=-\infty}^{+\infty}\frac{1}{\sqrt{l^2+1/4}\sqrt{(l-1)^2+1/4}}$$

$$=\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{+\infty}|c_{2n}|\Big(\sum_{l=-\infty}^{-1}\frac{1}{\sqrt{l^2+\frac{1}{4}}\sqrt{(l-1)^2+\frac{1}{4}}}+\frac{4}{\sqrt{5}}+\sum_{l=1}^{+\infty}\frac{1}{\sqrt{l^2+\frac{1}{4}}\sqrt{(l-1)^2+\frac{1}{4}}}\Big)$$

$$\le M_2\Big(\sum_{l=-\infty}^{-1}\frac{1}{l^2+1/4}+\frac{4}{\sqrt{5}}+\sum_{l=1}^{+\infty}\frac{1}{(l-1)^2+1/4}\Big)<+\infty.$$

Therefore $L<+\infty$, and condition (3) is proved. The proof of (4) is analogous. It remains to apply Theorem 2. ∎

References

[1] E. Albrecht, M. Ptak, *Invariant subspaces for doubly commuting contractions with rich Taylor spectrum*, J. Operator Theory 40 (1998), 373–384.

[2] G. Z. Ber, *The phenomenon of interference in an integral metric, and approximation by entire functions of exponential type* (Russian), Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 34 (1980), 11–24.

[3] R. P. Boas, Jr., *Entire Functions*, Academic Press, New York, 1954.

[4] I. E. Chyzhykov, *Growth of p-th means of analytic and subharmonic functions in the unit disc and angular distribution of zeros*, arXiv:1509.02141.

[5] V. M. Dilnyi, *Splitting of some spaces of analytic functions*, Ufa Math. J. 6 (2014), no. 2, 25–34.

[6] V. M. Dilnyi, *On cyclic functions in weighted Hardy spaces*, Zh. Mat. Fiz. Anal. Geom. 7 (2011), 19–33.

[7] V. M. Dilnyi, T. I. Hishchak, *On splitting functions in Paley–Wiener space*, Mat. Stud. 45 (2016), 137–148.

[8] M. M. Djrbashian, *Integral transforms and representations of functions in the complex domain* (Russian), Nauka, Moscow, 1966.

[9] B. Ya. Levin, Yu. I. Lyubarskiĭ, *Interpolation by means of special classes of entire functions and related expansions in series of exponentials*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 675–702; English transl.: Math. USSR-Izv. 9 (1975), 621–662.

[10] Yu. I. Lyubarskiĭ, *Representation of functions from $\mathcal{H}^p$ in a half plane and some of its applications* (Russian), Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 38 (1982), 76–84.

[11] P. Mleczko, *On Beurling type theorem for abstract Hardy spaces*, J. Math. Anal. Appl. 403 (2013), 1–4.

[12] B. V. Vynnytskyi, V. M. Dilnyi, *On an analogue of the Paley–Wiener theorem for weighted Hardy spaces* (Ukrainian), Mat. Stud. 14 (2000), 35–40.

[13] R. S. Yulmukhametov, *Splitting of entire functions with zeros in a strip*, Mat. Sb. 186 (1995), no. 7, 147–160; English transl.: Sb. Math. 186 (1995), 1071–1084.

[14] R. S. Yulmukhametov, *Solution of the L. Ehrenpreis problem on factorization*, Mat. Sb. 190 (1999), no. 4, 123–157; English transl.: Sb. Math. 190 (1999), 597–629.