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ALGEBRA OF CONVOLUTION TYPE OPERATORS WITH CONTINUOUS DATA ON BANACH FUNCTION SPACES

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Abstract. We show that if the Hardy–Littlewood maximal operator is bounded on a reflexive Banach function space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$, then the space $X(\mathbb{R})$ has an unconditional wavelet basis. As a consequence of the existence of a Schauder basis in $X(\mathbb{R})$, we prove that the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ on the space $X(\mathbb{R})$ is contained in the Banach algebra generated by all operators of multiplication aI by functions $a \in C(\dot{\mathbb{R}})$, where $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and by all Fourier convolution operators $W^0(b)$ with symbols $b \in C_X(\dot{\mathbb{R}})$, the Fourier multiplier analogue of $C(\dot{\mathbb{R}})$.

1. Introduction. The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}$ is denoted by $\mathfrak{M}(\mathbb{R})$. Let $\mathfrak{M}^+(\mathbb{R})$ be the subset of functions in $\mathfrak{M}(\mathbb{R})$ whose values lie in $[0, \infty]$. For a measurable set $E \subset \mathbb{R}$, its Lebesgue measure and the characteristic function are denoted by $|E|$ and χ_E, respectively. Following [BS88, Chapter 1, Definition 1.1],

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a mapping $\rho : \mathfrak{M}^+(\mathbb{R}) \to [0,\infty]$ is called a *Banach function norm* if, for all functions f, g, f_n ($n \in \mathbb{N}$) in $\mathfrak{M}^+(\mathbb{R})$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{R}$, the following properties hold:

$$
\begin{aligned}
&\text{(A1)} \quad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \ \rho(af) = a\rho(f), \ \rho(f+g) \leq \rho(f) + \rho(g),\\
&\text{(A2)} \quad 0 \leq g \leq f \text{ a.e.} \ \Rightarrow \ \rho(g) \leq \rho(f) \quad \text{(the lattice property)},\\
&\text{(A3)} \quad 0 \leq f_n \uparrow f \text{ a.e.} \ \Rightarrow \ \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)},\\
&\text{(A4)} \quad |E| < \infty \Rightarrow \rho(\chi_E) < \infty,\\
&\text{(A5)} \quad |E| < \infty \Rightarrow \int_E f(x)\,dx \leq C_E \rho(f),
\end{aligned}
$$

where $C_E \in (0,\infty)$ may depend on E and ρ but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R})$ of functions $f \in \mathfrak{M}(\mathbb{R})$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X(\mathbb{R})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{R})} := \rho(|f|).$$

With this norm and under natural linear space operations, the set $X(\mathbb{R})$ becomes a Banach space (see [BS88, Chapter 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{R})$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{R}} f(x)g(x)\,dx : f \in \mathfrak{M}^+(\mathbb{R}), \ \rho(f) \leq 1 \right\}, \quad g \in \mathfrak{M}^+(\mathbb{R}).$$

By [BS88, Chapter 1, Theorem 2.2], ρ' is itself a Banach function norm. The Banach function space $X'(\mathbb{R})$ determined by the Banach function norm ρ' is called the *associate space* (Köthe dual) of $X(\mathbb{R})$. The associate space $X'(\mathbb{R})$ is a subspace of the (Banach) dual space $[X(\mathbb{R})]^*$.

Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform

$$(Ff)(x) := \widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{itx}\,dt, \quad x \in \mathbb{R},$$

and let $F^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of F,

$$(F^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}\,dx, \quad t \in \mathbb{R}.$$

It is well known that the Fourier convolution operator $W^0(a) := F^{-1}aF$ is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$. Let $X(\mathbb{R})$ be a separable Banach function space. Then $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see Lemma 2.1 below). A function $a \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $X(\mathbb{R})$ if the convolution operator $W^0(a) := F^{-1}aF$ maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function a is called the *symbol* of the Fourier convolution operator $W^0(a)$. The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))},$$

where $\mathcal{B}(X(\mathbb{R}))$ denotes the Banach algebra of all bounded linear operators on the space $X(\mathbb{R})$. Let $\mathcal{K}(X(\mathbb{R}))$ denote the ideal of all compact operators in the Banach algebra $\mathcal{B}(X(\mathbb{R}))$.

Recall that the (non-centered) Hardy–Littlewood maximal function Mf of a function $f \in L^1_{\rm loc}(\mathbb{R})$ is defined by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ of finite length containing x. The Hardy–Littlewood maximal operator M defined by the rule $f \mapsto Mf$ is a sublinear operator.

Suppose $X(\mathbb{R})$ is a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Let $C(\dot{\mathbb{R}})$ denote the C^*-algebra of continuous functions on the one-point compactification $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the real line. Further, let $C_X(\dot{\mathbb{R}})$ be the closure of $C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ in the norm of $\mathcal{M}_{X(\mathbb{R})}$, where $V(\mathbb{R})$ is the algebra of all functions of finite total variation on $\mathbb{R}$. Consider the smallest Banach subalgebra

$$\mathcal{A}_{X(\mathbb{R})} = \operatorname{alg}\{aI, W^0(b) : a \in C(\dot{\mathbb{R}}),\ b \in C_X(\dot{\mathbb{R}})\}$$

of the algebra $\mathcal{B}(X(\mathbb{R}))$ that contains all operators of multiplication aI by functions $a \in C(\dot{\mathbb{R}})$ and all Fourier convolution operators $W^0(b)$ with symbols $b \in C_X(\dot{\mathbb{R}})$.

The algebra $\mathcal{A}_{X(\mathbb{R})}$ is well understood in the case when $X(\mathbb{R}) = L^p(\mathbb{R}, w)$ is a Lebesgue space with $1 < p < \infty$ and a Muckenhoupt weight w (see, e.g., [BKS02, Chapter 17] and also [D79] for the non-weighted case). Surprisingly enough, the algebra $\mathcal{A}_{X(\mathbb{R})}$ has not been investigated for more general Banach function spaces $X(\mathbb{R})$. The aim of this paper is to start studying the algebra $\mathcal{A}_{X(\mathbb{R})}$ on reflexive Banach function spaces $X(\mathbb{R})$ under the assumption that the Hardy–Littlewood maximal operator M is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$.

Our main result is the following.

THEOREM 1.1. *Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy–Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the ideal of compact operators $\mathcal{K}(X(\mathbb{R}))$ is contained in the Banach algebra $\mathcal{A}_{X(\mathbb{R})}$.*

Theorem 1.1 implies that the quotient Banach algebra

$$\mathcal{A}^\pi_{X(\mathbb{R})} := \mathcal{A}_{X(\mathbb{R})}/\mathcal{K}(X(\mathbb{R}))$$

is well-defined. It follows from [K15b, Theorem 2.9] that if $X(\mathbb{R})$ is either a reflexive rearrangement-invariant Banach function space with nontrivial Boyd indices or a reflexive variable Lebesgue space such that the Hardy–Littlewood maximal operator is bounded on $X(\mathbb{R})$, then $\mathcal{A}^\pi_{X(\mathbb{R})}$ is commutative.

QUESTION 1.2. Is it true that the quotient algebra $\mathcal{A}^\pi_{X(\mathbb{R})}$ is commutative under the assumptions of Theorem 1.1?

In order to prove Theorem 1.1, we have to insure that the space $X(\mathbb{R})$ has a Schauder basis. In Section 2, we prove a stronger result that might be of independent interest. It says that, under the assumptions of Theorem 1.1, the space $X(\mathbb{R})$ has an unconditional wavelet basis. Similar questions were considered earlier in [NPR14] and [INS15] under hypotheses on the space $X(\mathbb{R})$, which are different from ours (see also [Ho11a, Ho11b, So97, W12]).

In Section 3, we observe that the multiplication operators aI with $a \in L^\infty$ and the Fourier convolution operators $W^0(b)$ with $b \in \mathcal{M}_{X(\mathbb{R})}$ cannot be compact on the space $X(\mathbb{R})$ unless they are trivial. Thus, the nontrivial generators of the algebra $\mathcal{A}_{X(\mathbb{R})}$ are noncompact.

In Section 4, we state that a rank one operator T_1 defined by

$$(T_1 f)(x) = a(x) \int_{\mathbb{R}} b(y) f(y)\, dy,$$

where a and b are continuous and compactly supported functions, can be written as a product of generators of the algebra $\mathcal{A}_{X(\mathbb{R})}$. We prove Theorem 1.1 by showing that each compact operator can be approximated in the operator norm by finite rank operators and, further, by a finite sum of operators of the form T_1.

2. Wavelet bases in Banach function spaces

2.1. Density of nice functions in separable Banach function spaces. Let $C_0(\mathbb{R})$ and $C_0^\infty(\mathbb{R})$ denote the sets of continuous compactly supported functions on $\mathbb{R}$ and infinitely differentiable compactly supported functions on $\mathbb{R}$, respectively.

LEMMA 2.1. *Let $X(\mathbb{R})$ be a separable Banach function space. Then the sets $C_0(\mathbb{R})$, $C_0^\infty(\mathbb{R})$ and $L^2(\mathbb{R}) \cap X(\mathbb{R})$ are dense in the space $X(\mathbb{R})$.*

The density of $C_0(\mathbb{R})$ and $C_0^\infty(\mathbb{R})$ in $X(\mathbb{R})$ is shown in [KS14, Lemma 2.12]. Since $C_0(\mathbb{R}) \subset L^2(\mathbb{R}) \cap X(\mathbb{R}) \subset X(\mathbb{R})$, we conclude that $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$.

2.2. Uniform boundedness of families of operators satisfying local sharp maximal operator estimates uniformly. For $s > 0$ and $f \in L^s_{\rm loc}(\mathbb{R})$, consider the local s-sharp maximal function of f defined by

$$f_s^\#(x) := \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \left(\frac{1}{|Q|} \int_Q |f(y) - c|^s\, dy \right)^{1/s},$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ of finite length containing x.

The theorem below follows from [KS14, Theorem 3.6].

THEOREM 2.2. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Assume that $0 < s < 1$ and Ω is an index set. Let $\{T_\omega\}_{\omega \in \Omega}$ be a family of linear operators such that*

(a) *for each $\omega \in \Omega$, the operator T_ω is bounded on the space $L^2(\mathbb{R})$;*
(b) *there exists a constant $c_s \in (0, \infty)$ depending only on s and such that for every $\omega \in \Omega$, every $f \in C_0^\infty(\mathbb{R})$ and every $x_0 \in \mathbb{R}$, one has*

$$(T_\omega f)_s^\#(x_0) \le c_s (Mf)(x_0).$$

Then each operator T_ω, $\omega \in \Omega$, is bounded on $X(\mathbb{R})$ and

$$\sup_{\omega \in \Omega} \|T_\omega\|_{\mathcal{B}(X(\mathbb{R}))} < \infty.$$

2.3. Estimates for local sharp maximal operators of families of operators associated with kernels. Let $\mathcal{D}'(\mathbb{R})$ be the space of distributions, that is, the dual space of $C_0^\infty(\mathbb{R})$. The action of a distribution $a \in \mathcal{D}'(\mathbb{R})$ on a function $f \in C_0^\infty(\mathbb{R})$ is denoted by $a(f) = \langle a, f\rangle$. A locally integrable function $K : \mathbb{R}^2 \setminus \{(x,x) : x \in \mathbb{R}\} \to \mathbb{C}$ is said to be a kernel. One says that a linear and continuous operator $T_K : C_0^\infty(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ *is associated* with a kernel K if

$$\langle T_K f, g\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)g(x)f(y)\,dx\,dy$$

whenever $f, g \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$.

For each point $x_0 \in \mathbb{R}$, each radius $r > 0$ and a kernel K, we consider the interval

$$I = I(x_0, r) = (x_0 - r, x_0 + r)$$

and the function

$$(D_I K)(y) := \frac{1}{|I|^2}\iint_{I\times I} |K(z,y) - K(x,y)|\,dx\,dz.$$

Let Ω be an index set. Following [AP94, Section 2.1], a family of kernels $\{K_\omega\}_{\omega\in\Omega}$ is said to *satisfy Condition* (D) *uniformly in* Ω if there are constants $C_D, N \in (0,\infty)$ such that for all $\omega \in \Omega$, all $f \in C_0^\infty(\mathbb{R})$ and all $x_0 \in \mathbb{R}$,

$$\sup_{r>0}\int_{|y-x_0|>Nr} (D_I K_\omega)(y)|f(y)|\,dy \le C_D (Mf)(x_0). \tag{2.1}$$

THEOREM 2.3. *Let Ω be an index set, $\{K_\omega\}_{\omega\in\Omega}$ be a family of kernels and let $\{T_{K_\omega}\}_{\omega\in\Omega}$ be the family of operators associated with the kernels in the family $\{K_\omega\}_{\omega\in\Omega}$. If*

(a) *the family $\{K_\omega\}_{\omega\in\Omega}$ satisfies Condition (D) uniformly in Ω with some constants $C_D, N \in (0,\infty)$;*
(b) *the operators T_{K_ω} extend to bounded operators from $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$ uniformly in Ω, that is, there exists a constant $C_{1,1} \in (0,\infty)$ such that for all $\omega \in \Omega$ and all $f \in C_0^\infty(\mathbb{R})$,*

$$\sup_{\lambda>0}\big(\lambda\big|\{x \in \mathbb{R} : |(T_{K_\omega} f)(x)| > \lambda\}\big|\big) \le C_{1,1}\|f\|_{L^1(\mathbb{R})},$$

then for all $\omega \in \Omega$, all $s \in (0,1)$, all $f \in C_0^\infty(\mathbb{R})$ and all $x_0 \in \mathbb{R}$,

$$\big(T_{K_\omega} f\big)_s^{\#}(x_0) \le C_s (Mf)(x_0), \tag{2.2}$$

where

$$C_s := 2^{2/s-1}\big(N(1-s)^{-1/s}C_{1,1} + C_D\big).$$

Proof. This theorem is proved by analogy with [AP94, Theorem 2.1] (see also [KJLH09, Theorem 2.6] for tracing the constant C_s). Since the definition of the local s-sharp maximal function adopted in this paper slightly differs from that of [AP94, KJLH09], we provide some details here.

Fix $\omega \in \Omega$, $f \in C_0^\infty(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Let Q be an interval of finite length containing x_0 and I be the smallest interval centered at x_0, which contains Q. Then $|Q| \le |I| \le 2|Q|$ and

$$\inf_{c\in\mathbb{C}}\left(\frac{1}{|Q|}\int_Q |(T_{K_\omega} f)(y) - c|^s\,dy\right)^{1/s} \le 2^{1/s}\inf_{c\in\mathbb{C}}\left(\frac{1}{|I|}\int_I |(T_{K_\omega} f)(y) - c|^s\,dy\right)^{1/s}. \tag{2.3}$$

Let $f = f_1 + f_2$ where $f_1 = f\chi_{I(x_0,Nr)}$, $I = I(x_0, r)$, and N is given by Condition (D). Set

$$(T_{K_\omega} f_2)_I = \frac{1}{|I|}\int_I (T_{K_\omega} f_2)(y)\,dy.$$

Since $||a|^s - |b|^s| \le |a-b|^s$ and $(|a|+|b|)^{1/s} \le 2^{1/s-1}(|a|^{1/s}+|b|^{1/s})$ for $a, b \in \mathbb{C}$ and $0 < s < 1$, we have

$$\begin{aligned}
&\inf_{c\in\mathbb{C}}\left(\frac{1}{|I|}\int_I |(T_{K_\omega} f)(y) - c|^s\,dy\right)^{1/s}\\
&\le \left(\frac{1}{|I|}\int_I \big|(T_{K_\omega} f_1)(y) + (T_{K_\omega} f_2)(y) - (T_{K_\omega} f_2)_I\big|^s\,dy\right)^{1/s}\\
&\le \left(\frac{1}{|I|}\int_I |(T_{K_\omega} f_1)(y)|^s\,dy + \frac{1}{|I|}\int_I |(T_{K_\omega} f)_2(y) - (T_{K_\omega} f_2)_I|^s\,dy\right)^{1/s}\\
&\le 2^{1/s-1}\left[\left(\frac{1}{|I|}\int_I |(T_{K_\omega} f_1)(y)|^s\,dy\right)^{1/s} + \left(\frac{1}{|I|}\int_I |(T_{K_\omega} f_2)(y) - (T_{K_\omega} f_2)_I|^s\,dy\right)^{1/s}\right]\\
&=: 2^{1/s-1}(J_1 + J_2).
\end{aligned}\tag{2.4}$$

By [KJLH09, formulas (2.48)–(2.49)],

$$J_1 \le N(1-s)^{-1/s} C_{1,1}(Mf)(x_0), \quad J_2 \le C_D (Mf)(x_0). \tag{2.5}$$

Combining (2.3)–(2.5), we arrive at (2.2). ∎

2.4. Families of standard kernels in the sense of Coifman and Meyer. Let Ω be an index set. We say that a family of kernels $\{K_\omega\}_{\omega\in\Omega}$ is a uniform in Ω family of standard kernels (in the sense of Coifman and Meyer, see, e.g., [MC97, p. 9]) if there exist constants $C_1, C_2, C_3 \in (0,\infty)$ such that for all $\omega \in \Omega$ and all pairs (x, y), (z, y), (x, w) in $\mathbb{R}^2 \setminus \{(x, x) : x \in \mathbb{R}\}$, one has

$$|K_\omega(x,y)| \le \frac{C_1}{|x-y|}, \tag{2.6}$$

$$|K_\omega(z,y) - K_\omega(x,y)| \le \frac{C_2|z-x|}{|x-y|^2} \quad \text{if} \quad |z-x| \le \frac{1}{2}|x-y|, \tag{2.7}$$

$$|K_\omega(x,w) - K_\omega(x,y)| \le \frac{C_3|w-y|}{|x-y|^2} \quad \text{if} \quad |w-y| \le \frac{1}{2}|x-y|. \tag{2.8}$$

An operator T_{K_ω} associated with a standard kernel K_ω is called a *Calderón–Zygmund operator.*

LEMMA 2.4. *Let Ω be an index set and let $\{K_\omega\}_{\omega\in\Omega}$ be a uniform in Ω family of standard kernels. Then $\{K_\omega\}_{\omega\in\Omega}$ satisfies Condition (D) uniformly in Ω with the constants $C_D = 8C_2$ and $N = 2$.*

Proof. Take $N = 2$. Fix $x_0 \in \mathbb{R}$ and $r > 0$. If $y \notin I = I(x_0, 2r)$, then for $x, z \in I(x_0, r)$,

$$|x - x_0| < r \le \frac{1}{2}|y - x_0|, \quad |z - x_0| < r \le \frac{1}{2}|y - x_0|.$$

Hence, taking into account (2.7), we obtain for all $\omega \in \Omega$ and $x, z \in I(x_0, r)$,

$$\begin{aligned}|K_\omega(z,y) - K_\omega(x,y)| &\le |K_\omega(z,y) - K_\omega(x_0,y)| + |K_\omega(x,y) - K_\omega(x_0,y)| \\ &\le \frac{C_2|z-x_0|}{|y-x_0|^2} + \frac{C_2|x-x_0|}{|y-x_0|^2} \le \frac{2C_2 r}{|y-x_0|^2}\,.\end{aligned}$$

Then

$$(D_I K_\omega)(y) = \frac{1}{|I(x_0,r)|^2}\iint_{I(x_0,r)\times I(x_0,r)} |K_\omega(z,y) - K_\omega(x,y)|\,dx\,dz \le \frac{2C_2 r}{|y-x_0|^2}$$

and

$$\begin{aligned}\int_{|y-x_0|>2r} (D_I K_\omega)(y)|f(y)|\,dy &\le 2C_2 r \int_{|y-x_0|>2r} \frac{|f(y)}{|y-x_0|^2}\,dy \\ &= 2C_2 r \sum_{n=0}^{\infty} \int_{2^{n+1}r<|y-x_0|\le 2^{n+2}r} \frac{|f(y)|}{|y-x_0|^2}\,dy \\ &\le 2C_2 r \sum_{n=0}^{\infty} \frac{1}{(2^{n+1}r)^2} \int_{2^{n+1}r<|y-x_0|\le 2^{n+2}r} |f(y)|\,dy \\ &\le 2C_2 \sum_{n=0}^{\infty} \frac{2^{-n}}{2^{n+2}r} \int_{I(x_0,2^{n+2}r)} |f(y)|\,dy \\ &= 4C_2 \sum_{n=0}^{\infty} \frac{2^{-n}}{|I(x_0,2^{n+2}r)|} \int_{I(x_0,2^{n+2}r)} |f(y)|\,dy \\ &\le 4C_2 \Big(\sum_{n=0}^{\infty} 2^{-n}\Big)(Mf)(x_0) = 8C_2 (Mf)(x_0),\end{aligned}$$

which implies (2.1) with $N = 2$ and $C_D = 8C_2$. ∎

2.5. Families of Calderón–Zygmund operators associated with kernels defined by orthonormal wavelets. As usual, let

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx. \tag{2.9}$$

be the standard inner product in $L^2(\mathbb{R})$.

Following [HW96, Section 5.3], a function

$$W : [0,\infty) \to (0,\infty)$$

is said to be a *radial decreasing L^1-majorant* of a function $g : \mathbb{R} \to \mathbb{C}$ if $|g(x)| \le W(|x|)$ for a.e. $x \in \mathbb{R}$, and $W \in L^1([0,\infty))$, W is decreasing, $W(0) < \infty$.

Recall that a function $\psi \in L^2(\mathbb{R})$ is called an *orthonormal wavelet* if the family

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R}, \quad j,k \in \mathbb{Z},$$

forms an orthonormal basis in $L^2(\mathbb{R})$.

Let $\mathcal{E}$ be the family of all sequences $\varepsilon = \{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}}$ with $\varepsilon_{j,k} \in \{-1,1\}$ for all $j,k \in \mathbb{Z}$. For an orthonormal wavelet ψ and a sequence $\varepsilon = \{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}} \in \mathcal{E}$, consider the kernel

$$K_\varepsilon(x,y) := \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \varepsilon_{j,k}\psi_{j,k}(x)\overline{\psi_{j,k}(y)}, \quad x,y \in \mathbb{R}. \tag{2.10}$$

If the wavelet ψ has a radial decreasing L^1-majorant W, it follows from [HW96, Section 5.3, Lemma 3.12] that

$$\begin{aligned}|K_\varepsilon(x,y)| &\le \sum_{j\in\mathbb{Z}} 2^j \sum_{k\in\mathbb{Z}} |\psi(2^j x-k)\psi(2^j y-k)| \le \sum_{j\in\mathbb{Z}} 2^j \sum_{k\in\mathbb{Z}} W(|2^j x-k|)W(|2^j y-k|)\\ &\le C(W)\sum_{j\in\mathbb{Z}} 2^j W(2^{j-1}|x-y|),\end{aligned}$$

where $C(W)$ depends only on W and, by the proof of [HW96, Section 5.6, Theorem 6.12],

$$\sum_{j\in\mathbb{Z}} 2^j W(2^{j-1}|x-y|) \le \frac{4}{|x-y|}\,\|W\|_{L^1([0,\infty))}.$$

Hence, $|K_\varepsilon(x,y)|<\infty$ for all $x,y\in\mathbb{R}$ such that $x\ne y$.

Theorem 2.5. *Suppose that ψ is an orthonormal and differentiable wavelet such that ψ and its derivative ψ' have a common radial decreasing L^1-majorant W satisfying*

$$\int_0^\infty sW(s)\,ds<\infty. \tag{2.11}$$

Then the family $\{K_\varepsilon\}_{\varepsilon\in\mathcal{E}}$ given by (2.10) *is a uniform in $\mathcal{E}$ family of standard kernels with the constants C_1, C_2, C_3 in* (2.6)–(2.8) *depending only on W.*

The proof of this theorem is analogous to the proof of [HW96, Section 5.6, Theorem 6.12] and therefore is omitted.

Let us consider the operator T_{K_ε} associated with K_ε, which is given for $f\in L^2(\mathbb{R})$ by

$$(T_{K_\varepsilon}f)(x)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\varepsilon_{j,k}\langle f,\psi_{j,k}\rangle\psi_{j,k}(x),\quad x\in\mathbb{R}. \tag{2.12}$$

For each $\varepsilon=\{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}}\in\mathcal{E}$, the operator T_{K_ε} is an isometry on $L^2(\mathbb{R})$.

Theorem 2.6. *Suppose that ψ is an orthonormal wavelet having a radial decreasing L^1-majorant W satisfying* (2.11). *There exists a constant $C_{1,1}(W)$ depending only on W such that for every sequence $\varepsilon=\{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}}\in\mathcal{E}$ and every function $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$,*

$$\sup_{\lambda>0}\big(\lambda\big|\{x\in\mathbb{R}:|(T_{K_\varepsilon}f)(x)|>\lambda\}\big|\big)\le C_{1,1}(W)\|f\|_{L^1(\mathbb{R})},$$

where the family of operators $\{T_{K_\varepsilon}\}_{\varepsilon\in\mathcal{E}}$ is defined by (2.10) *and* (2.12) *on $L^2(\mathbb{R})$.*

Since the function $s\mapsto W(|s|)$ belongs to $L^\infty(\mathbb{R})$ and, by (2.11),

$$\int_0^\infty W(s)\ln(1+s)\,ds\le\int_0^\infty sW(s)\,ds<\infty,$$

Theorem 2.6 follows from [KaSa99, Chapter 7, Theorem 9].

Combining Theorems 2.3, 2.5, 2.6 with Lemma 2.4, we arrive at the following.

Theorem 2.7. *Suppose that ψ is an orthonormal and differentiable wavelet such that ψ and its derivative have a common radial decreasing L^1-majorant W satisfying* (2.11). *Then there exist constants $C_{1,1}(W), C_D(W)\in(0,\infty)$ depending only on W such that for every $\varepsilon=\{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}}\in\mathcal{E}$, every $s\in(0,1)$, every $f\in C_0^\infty(\mathbb{R})$ and every $x_0\in\mathbb{R}$, one has*

$$(T_{K_\varepsilon}f)_s^{\#}(x_0)\le C_s(W)(Mf)(x_0),$$

where

$$C_s(W) := 2^{2/s-1}\big(2(1-s)^{-1/s}C_{1,1}(W) + C_D(W)\big),$$

and the family of operators $\{T_{K_\varepsilon}\}_{\varepsilon\in\mathcal{E}}$ *is defined by* (2.10) *and* (2.12).

2.6. Boundedness of square functions associated with orthonormal wavelets. Let ψ be an orthonormal wavelet. Consider the family of kernels $\{K_\varepsilon\}_{\varepsilon\in\mathcal{E}}$ defined by (2.10) and the family of operators $\{T_{K_\varepsilon}\}_{\varepsilon\in\mathcal{E}}$ associated with these kernels, which are given by (2.12) for $f \in C_0^\infty(\mathbb{R})$.

THEOREM 2.8. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that ψ is an orthonormal and differentiable wavelet such that ψ and its derivative have a common radial decreasing L^1-majorant W satisfying* (2.11). *Then for all $\varepsilon = \{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}} \in \mathcal{E}$, the operators T_{K_ε}, defined initially on $C_0^\infty(\mathbb{R})$, extend to bounded linear operators on $X(\mathbb{R})$ and*

$$N := \sup_{\varepsilon\in\mathcal{E}} \|T_{K_\varepsilon}\|_{\mathcal{B}(X(\mathbb{R}))} < \infty.$$

This theorem follows from Theorems 2.2 and 2.7.

Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space. It follows from the Hölder inequality for Banach function spaces (see [BS88, Chapter 1, Theorem 2.4]) that for $f \in X(\mathbb{R})$ and $g \in X'(\mathbb{R})$, the pairing $\langle f, g\rangle$ is correctly defined by (2.9).

One of the main ingredients of the proof of the existence of unconditional wavelet bases in Banach function spaces is the following theorem. Its proof is inspired by Meyer's approach (see [M95, Section 6.2] for the case of Lebesgue spaces and [INS15, Theorem 4.2] for weighted variable Lebesgue spaces).

THEOREM 2.9. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that ψ is an orthonormal and differentiable wavelet such that ψ and its derivative have a common radial decreasing L^1-majorant W satisfying* (2.11) *and such that $\psi_{j,k} \in X'(\mathbb{R})$ for all $j, k \in \mathbb{Z}$. Then the sublinear operator V defined by*

$$(Vf)(x) := \Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle \psi_{j,k}(x)|^2\Big)^{1/2}, \quad x \in \mathbb{R}, \tag{2.13}$$

is bounded on the space $X(\mathbb{R})$.

Proof. Fix $f \in X(\mathbb{R})$ and $g \in X'(\mathbb{R})$ such that $\|g\|_{X'(\mathbb{R})} \le 1$. Let the set $\mathcal{E} = \{-1, 1\}^{\mathbb{Z}\times\mathbb{Z}}$ be equipped with the Bernoulli probability measure μ obtained by taking the product of the measures on each factor $\{-1, 1\}$, which give a mass $1/2$ to each of the points -1 and 1. By Khintchine's inequality (see, e.g., [Gut05, Corollary 8.1] and also [M95, Section 6.2, Lemma 2]), there exists a constant $L \in (0, \infty)$ such that

$$(Vf)(x) \le L \int_{\mathcal{E}} |(T_{K_\varepsilon} f)(x)\| \, d\mu(\varepsilon), \quad x \in \mathbb{R}, \tag{2.14}$$

where the family of operators $\{T_{K_\varepsilon}\}_{\varepsilon\in\mathcal{E}}$ is defined by (2.10) and (2.12). Then, by inequality (2.14), Fubini's theorem and Hölder's inequality for Banach function spaces (see

[BS88, Chapter 1, Theorem 2.4]), we obtain

$$\begin{aligned}\int_{\mathbb{R}} |(Vf)(x)g(x)|\,dx &\le L\int_{\mathbb{R}}\Big(\int_{\mathcal{E}} |(T_{K_\varepsilon}f)(x)|\,d\mu(\varepsilon)\Big)|g(x)|\,dx\\ &= L\int_{\mathcal{E}}\Big(\int_{\mathbb{R}} |(T_{K_\varepsilon}f)(x)g(x)|\,dx\Big)\,d\mu(\varepsilon)\\ &\le L\int_{\mathcal{E}} \|T_{K_\varepsilon}f\|_{X(\mathbb{R})}\|g\|_{X'(\mathbb{R})}\,d\mu(\varepsilon) \le L\int_{\mathcal{E}} \|T_{K_\varepsilon}f\|_{X(\mathbb{R})}\,d\mu(\varepsilon).\end{aligned} \tag{2.15}$$

By Theorem 2.8, for all $\varepsilon = \{\varepsilon_{j,k}\}_{j,k\in\mathbb{Z}} \in \mathcal{E}$, we have

$$\|T_{K_\varepsilon}f\|_{X(\mathbb{R})} \le N\|f\|_{X(\mathbb{R})}. \tag{2.16}$$

Combining (2.15) and (2.16), we see that for all $f \in X(\mathbb{R})$ and all $g \in X'(\mathbb{R})$ satisfying $\|g\|_{X'(\mathbb{R})} \le 1$,

$$\int_{\mathbb{R}} |(Vf)(x)g(x)|\,dx \le LN\int_{\mathcal{E}} \|f\|_{X(\mathbb{R})}\,d\mu(\varepsilon) = LN\|f\|_{X(\mathbb{R})}.$$

It follows from the above inequality and the Lorentz–Luxemburg theorem (see [BS88, Chapter 1, Theorem 2.7]) that for all $f \in X(\mathbb{R})$,

$$\begin{aligned}\|Vf\|_{X(\mathbb{R})} &= \|Vf\|_{X''(\mathbb{R})}\\ &= \sup\Big\{\int_{\mathbb{R}} |(Vf)(x)g(x)|\,dx : g \in X'(\mathbb{R}),\ \|g\|_{X'(\mathbb{R})} \le 1\Big\} \le LN\|f\|_{X(\mathbb{R})},\end{aligned}$$

which completes the proof. ∎

2.7. Existence of a wavelet basis in a Banach function space. By [HW96, Section 2.3, Theorem 3.29], for every $r \in \{0, 1, 2, \dots\}$, there exists an orthonormal wavelet ψ with compact support such that ψ has bounded derivatives up to order r.

Recall that a function f in a Banach function space $X(\mathbb{R})$ is said to have absolutely continuous norm in $X(\mathbb{R})$ if $\|f\chi_{E_n}\|_{X(\mathbb{R})} \to 0$ for every sequence $\{E_n\}_{n=1}^\infty$ of measurable sets on $\mathbb{R}$ such that $\chi_{E_n} \to 0$ a.e. on $\mathbb{R}$ as $n \to \infty$. If all functions $f \in X(\mathbb{R})$ have this property, then the space $X(\mathbb{R})$ itself is said to have absolutely continuous norm (see [BS88, Chapter 1, Definition 3.1]).

Now we are in a position to prove the main result of this section.

Theorem 2.10. *Let $X(\mathbb{R})$ be a reflexive Banach function space such that the Hardy–Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that ψ is an orthonormal C^1-wavelet with compact support. Then the system*

$$\{\psi_{j,k} : j, k \in \mathbb{Z}\}$$

is an unconditional basis in $X(\mathbb{R})$ and the wavelet expansion

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\psi_{j,k}$$

holds for every $f \in X(\mathbb{R})$, where the convergence is unconditional in $X(\mathbb{R})$.

Proof. If a Banach function space $X(\mathbb{R})$ is reflexive, then it follows from [BS88, Chapter 1, Corollary 4.4] that the spaces $X(\mathbb{R})$ and $X'(\mathbb{R})$ have absolutely continuous norms. Hence, by [BS88, Chapter 1, Corollary 5.6], the spaces $X(\mathbb{R})$ and $X'(\mathbb{R})$ are separable. Then, in view of Lemma 2.1, $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ and $L^2(\mathbb{R}) \cap X'(\mathbb{R})$ is dense in

$X'(\mathbb{R})$. Since $\psi_{j,k} \in C_0(\mathbb{R})$, we have $\psi_{j,k} \in X(\mathbb{R})$ and $\psi_{j,k} \in X'(\mathbb{R})$ for all $j,k \in \mathbb{Z}$. Moreover, there exist constants $C, \delta > 0$ such that $W(s) = Ce^{-\delta s}$, $s \in [0,\infty)$, is a common L^1-majorant for ψ and ψ' that satisfies (2.11). By Theorem 2.9, the operator V given by (2.13) is bounded on the spaces $X(\mathbb{R})$ and $X'(\mathbb{R})$. Then the desired result follows from [INS15, Theorem 4.1]. ∎

3. Noncompactness of multiplication and Fourier convolution operators

3.1. Noncompactness of nontrivial multiplication operators. The following theorem can be extracted from [HKK06, Theorem 2.4].

THEOREM 3.1. *Let $X(\mathbb{R})$ be a separable Banach function space and $a \in L^\infty(\mathbb{R})$. Then the multiplication operator aI is compact on the space $X(\mathbb{R})$ if and only if $a = 0$ almost everywhere on $\mathbb{R}$.*

We give another proof of this result based on the following lemma, which is of independent interest.

For a sequence of operators $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(X(\mathbb{R}))$, let

$$\operatorname*{s-lim}_{n\to\infty} A_n$$

denote the strong limit of the sequence, if it exists. For $\lambda, x \in \mathbb{R}$, consider the function

$$e_\lambda(x) := e^{i\lambda x}.$$

LEMMA 3.2. *Let $X(\mathbb{R})$ be a separable Banach function space and K be a compact operator on $X(\mathbb{R})$. Then*

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} K e_{h_n}^{-1} I = 0$$

on the space $X(\mathbb{R})$ for every sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} h_n = \pm\infty.$$

Proof. The idea of the proof is borrowed from the proof of [BKS02, Lemma 10.1] and [KJLH08, Lemma 3.8]. Let $f \in X(\mathbb{R})$ and $g \in X'(\mathbb{R})$. By Hölder's inequality for Banach function spaces (see [BS88, Chapter 1, Theorem 2.4]), $f\overline{g} \in L^1(\mathbb{R})$. Hence, by the Riemann–Lebesgue lemma (see, e.g., [Kat76, Chapter VI, Theorem 1.7]),

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{-ixh_n} f(x)\overline{g(x)}\, dx = \lim_{n\to\infty} (f\overline{g})\hat{}\,(-h_n) = 0 \tag{3.1}$$

whenever $h_n \to \pm\infty$ as $n \to \infty$. Since the space $X(\mathbb{R})$ is separable, it follows from [BS88, Chapter 1, Corollaries 4.3 and 5.6] that the associate space $X'(\mathbb{R})$ is canonically isometrically isomorphic to the Banach dual space $[X(\mathbb{R})]^*$ of $X(\mathbb{R})$. Hence equality (3.1) implies that the sequence of multiplication operators $\{e_{h_n}^{-1} I\}_{n\in\mathbb{N}}$ converges weakly to the zero operator on the space $X(\mathbb{R})$ as $n \to \infty$. It is clear that $\|e_{h_n} I\|_{\mathcal{B}(X(\mathbb{R}))} \le 1$ for all $n \in \mathbb{N}$. Since the sequence $\{e_{h_n} I\}_{n\in\mathbb{N}}$ is uniformly bounded, the operator K is compact, and the sequence $\{e_{h_n}^{-1} I\}_{n\in\mathbb{N}}$ converges weakly to the zero operator as $n \to \infty$, we conclude that in view of [RSS11, Lemmas 1.4.4 and 1.4.6], the sequence $\{e_{h_n} K e_{h_n}^{-1} I\}_{n\in\mathbb{N}}$ converges strongly to the zero operator on the space $X(\mathbb{R})$ as $n \to \infty$. ∎

Proof of Theorem 3.1. It is clear that if $a = 0$ a.e. on $\mathbb{R}$, then aI is the zero operator, whence it is compact. Assume that aI is compact and consider a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$

such that $h_n \to +\infty$ as $n \to \infty$. It is clear that $e_{h_n}(aI)e_{h_n}^{-1}I = aI$ for $n \in \mathbb{N}$. Then, by Lemma 3.2,

$$aI = \operatorname*{s-lim}_{n\to\infty} e_{h_n}(aI)e_{h_n}^{-1}I = 0,$$

which implies that $a = 0$ a.e. on $\mathbb{R}$. ∎

3.2. Noncompactness of nontrivial Fourier convolution operators. The following result was recently obtained by the authors.

Theorem 3.3 ([FKK19, Theorem 1.1]). *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that $b \in \mathcal{M}_{X(\mathbb{R})}$. Then the Fourier convolution operator $W^0(a)$ is compact on the space $X(\mathbb{R})$ if and only if $b = 0$ almost everywhere on $\mathbb{R}$.*

In the next section we will show that, along with the fact that each individual nontrivial multiplication operator aI with $a \in L^\infty(\mathbb{R})$ and each nontrivial Fourier convolution operator $W^0(b)$ with $b \in \mathcal{M}_{X(\mathbb{R})}$ is never compact on the space $X(\mathbb{R})$, the algebra generated by the operators aI and $W^0(b)$ contains all compact operators, similarly to Banach algebras of Toeplitz operators with continuous symbols on Hardy spaces.

4. Algebra of convolution type operators with continuous data

4.1. Fourier convolution operators with symbols in the algebra $V(\mathbb{R})$. Suppose that $a : \mathbb{R} \to \mathbb{C}$ is a function of finite total variation $V(a)$ given by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all partitions of $\mathbb{R}$ of the form

$$-\infty < x_0 < x_1 < \ldots < x_n < +\infty$$

with $n \in \mathbb{N}$. The set $V(\mathbb{R})$ of all functions of finite total variation on $\mathbb{R}$ with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a)$$

is a unital non-separable Banach algebra.

Theorem 4.1. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator M is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If a function $a : \mathbb{R} \to \mathbb{C}$ has a finite total variation $V(a)$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$ and*

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le c_X \|a\|_V \tag{4.1}$$

where c_X is a positive constant depending only on $X(\mathbb{R})$.

This result follows from [K15a, Theorem 4.3].

For Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, inequality (4.1) is usually called *Stechkin's inequality*, and the constant c_{L^p} is calculated explicitly:

$$c_{L^p} = \|S\|_{\mathcal{B}(L^p(\mathbb{R}))} = \begin{cases} \tan(\pi/(2p)) & \text{if } 1 < p \le 2, \\ \cot(\pi/(2p)) & \text{if } 2 < p < \infty, \end{cases} \tag{4.2}$$

where S is the Cauchy singular integral operator given by

$$(Sf)(x) := \frac{1}{\pi i} \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus(x-\varepsilon,x+\varepsilon)} \frac{f(t)}{t-x}\,dt. \tag{4.3}$$

We refer to [D79, Theorem 2.11] for the proof of (4.1) in the case of Lebesgue spaces $L^p(\mathbb{R})$ with $c_{L^p} = \|S\|_{\mathcal{B}(L^p(\mathbb{R}))}$ and to [GK92, Chapter 13, Theorem 1.3] for the calculation of the norm of S given in the second equality in (4.2).

4.2. One-dimensional operator with continuous compactly supported data. A proof of the next lemma can be extracted from the proof of [KILH13, Lemma 6.1].

LEMMA 4.2. *Suppose $X(\mathbb{R})$ is a separable Banach function space. Let $a, b \in C_0(\mathbb{R})$ and a one-dimensional operator T_1 be defined on the space $X(\mathbb{R})$ by*

$$(T_1 f)(x) = a(x)\int_{\mathbb{R}} b(y) f(y)\,dy. \tag{4.4}$$

Then there exists a function $c \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ such that $T_1 = aW^0(c)bI$.

4.3. Proof of Theorem 1.1. It follows from Theorem 2.10 that the space $X(\mathbb{R})$ has a Schauder basis. It is well known that every compact operator on a Banach space with a Schauder basis can be approximated in the operator norm by linear operators of finite rank (see, e.g., [S70, Chapter I, Corollary 17.7]). It follows from [BS88, Chapter 1, Corollaries 4.3 and 4.4] that the Banach space dual $[X(\mathbb{R})]^*$ of the space $X(\mathbb{R})$ is canonically isometrically isomorphic to the associate space $X'(\mathbb{R})$. Hence a finite rank operator on $X(\mathbb{R})$ is of the form

$$(T_m f)(x) = \sum_{j=1}^{m} a_j(x)\int_{\mathbb{R}} b_j(y) f(y)\,dy, \quad x \in \mathbb{R}, \tag{4.5}$$

where $a_j \in X(\mathbb{R})$ and $b_j \in X'(\mathbb{R})$ for $j \in \{1,\dots,m\}$ and some $m \in \mathbb{N}$. Since the set $C_0(\mathbb{R})$ is dense in $X(\mathbb{R})$ and in $X'(\mathbb{R})$ in view of Lemma 2.1, for every $\varepsilon \in (0,1)$ and every $j \in \{1,\dots,m\}$, there exist $a_{j,\varepsilon}, b_{j,\varepsilon} \in C_0(\mathbb{R})$ such that

$$\big|\|a_j\|_{X(\mathbb{R})} - \|a_{j,\varepsilon}\|_{X(\mathbb{R})}\big| < 1 \tag{4.6}$$

and

$$\|a_j - a_{j,\varepsilon}\|_{X(\mathbb{R})} < \frac{\varepsilon}{2m(\|b_j\|_{X'(\mathbb{R})}+1)}, \quad \|b_j - b_{j,\varepsilon}\|_{X'(\mathbb{R})} < \frac{\varepsilon}{2m(\|a_j\|_{X(\mathbb{R})}+1)}. \tag{4.7}$$

Let $T_{m,\varepsilon}$ denote the operator defined by (4.5) with $a_{j,\varepsilon}$ and $b_{j,\varepsilon}$ in place of a_j and b_j, respectively. It follows from Hölder's inequality for Banach function spaces (see [BS88, Chapter 1, Theorem 2.4]) and inequalities (4.6)–(4.7) that for $f \in X(\mathbb{R})$,

$$\begin{aligned}
&\|T_m f - T_{m,\varepsilon} f\|_{X(\mathbb{R})} \\
&\le \Big\|\sum_{j=1}^{m}(a_j - a_{j,\varepsilon})\int_{\mathbb{R}} b_j(y)f(y)\,dy\Big\|_{X(\mathbb{R})} + \Big\|\sum_{j=1}^{m} a_{j,\varepsilon}\int_{\mathbb{R}}(b_j(y) - b_{j,\varepsilon}(y))f(y)\,dy\Big\|_{X(\mathbb{R})} \\
&\le \sum_{j=1}^{m}\|a_j - a_{j,\varepsilon}\|_{X(\mathbb{R})}\|b_j\|_{X'(\mathbb{R})}\|f\|_{X(\mathbb{R})} + \sum_{j=1}^{m}\|a_{j,\varepsilon}\|_{X(\mathbb{R})}\|b_j - b_{j,\varepsilon}\|_{X'(\mathbb{R})}\|f\|_{X(\mathbb{R})} \\
&< \sum_{j=1}^{m}\frac{\varepsilon\|f\|_{X(\mathbb{R})}}{2m(\|b_j\|_{X'(\mathbb{R})}+1)}\|b_j\|_{X'(\mathbb{R})} + \sum_{j=1}^{m}(\|a_j\|_{X(\mathbb{R})}+1)\frac{\varepsilon\|f\|_{X(\mathbb{R})}}{2m(\|a_j\|_{X(\mathbb{R})}+1)} < \varepsilon\|f\|_{X(\mathbb{R})},
\end{aligned}$$

whence $\|T_m - T_{m,\varepsilon}\| \le \varepsilon$. Therefore, each compact operator on the space $X(\mathbb{R})$ can be approximated in the operator norm by a finite sum of rank one operators T_1 of the form (4.4) with $a, b \in C_0(\mathbb{R})$. By Lemma 4.2, each such operator can be written in the form $T_1 = aW^0(c)bI$ with $c \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$. It follows from Theorem 4.1 that $c \in C_X(\dot{\mathbb{R}})$. Hence $T_1 \in \mathcal{A}_{X(\mathbb{R})}$, which completes the proof. ∎

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