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AN ALTERNATIVE TO PLANCHEREL'S CRITERION FOR BILINEAR OPERATORS

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Abstract. We prove that bilinear operators associated with L^q multipliers with sufficiently many derivatives in L^∞ are bounded from $L^2 \times L^2$ to L^1 when $q < 4$. In the absence of Plancherel's identity on L^1, the range $q < 4$ in the bilinear case should be compared to $q = \infty$ in the classical $L^2 \to L^2$ boundedness for linear multiplier operators.

1. Introduction. Function spaces provide quantitative ways to measure integrability, smoothness, and to certain extent, cancellation properties of functions. A space of central importance is $L^2(\mathbb{R}^n)$ which appears at the crossroads of many echelons of function spaces. An important feature of $L^2(\mathbb{R}^n)$ is *Plancherel's identity*, which says that the Fourier transform

$$\widehat{f}(\xi) = \lim_{N\to\infty} \int_{|x|\le N} f(x) e^{-2\pi i x\cdot\xi}\, dx \quad (\text{limit in } L^2)$$

of a square-integrable function f satisfies

$$\big\|f\big\|_{L^2} = \big\|\widehat{f}\,\big\|_{L^2} \tag{1}$$

(here $x \cdot y$ is the dot product on $\mathbb{R}^n$). This simply identity provides an alternative way to calculate L^2 norms. It also trivializes the characterization of the L^2-boundedness of convolution operators $\varphi \mapsto \varphi * K$, where K is a tempered distribution. Plancherel's identity yields that such a convolution operator is bounded on $L^2(\mathbb{R}^n)$ if and only if the distributional Fourier transform of K is a bounded function. Convolution operators can

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also be expressed as multiplier operators. A multiplier operator has the form

$$S_m(\varphi)(x) = \int_{\mathbb{R}^n} m(\xi)\widehat{\varphi}(\xi)e^{2\pi i x\cdot\xi}\, d\xi,$$

where m is a bounded function on $\mathbb{R}^n$ and is initially defined on Schwartz functions φ. We note that $S_m(\varphi) = \varphi * K$ whenever $\widehat{K} = m$. In view of Plancherel's identity we have

$$\big\|S_m(f)\big\|_{L^2} = \big\|\widehat{S_m(f)}\,\big\|_{L^2} = \big\|m\widehat{f}\,\big\|_{L^2}$$

and it follows from this that S_m is L^2 bounded if and only if m is an L^∞ function. Moreover, the norm of S_m from L^2 to itself is equal to $\|m\|_{L^\infty}$. This simple characterization of the $L^2 \to L^2$ boundedness of multiplier operators is a direct consequence of Plancherel's identity, and for this reason we simply refer to it as *Plancherel's criterion.*

In this note we ask whether there exist boundedness criteria for *bilinear translation-invariant operators* analogous to Plancherel's criterion. Bilinear translation-invariant operators have the form

$$T(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y, x-z)f(y)g(z)\, dy\, dz, \quad x \in \mathbb{R}^n,$$

where f, g are Schwartz functions and K is a distribution on $\mathbb{R}^{2n}$ that coincides with a suitable function on $\mathbb{R}^{2n} \setminus \{(0,0)\}$. These operators can also be expressed as *bilinear multiplier operators*, i.e., operators of the form

$$T_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta\,,$$

initially defined for Schwartz functions f, g where m is a bounded function on $\mathbb{R}^{2n}$. Note that m coincides with the distributional Fourier transform of K. We refer to [4, Section 6] for general material related to the bilinear translation-invariant operators. These operators may map the product $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p_1 + 1/p_2 = 1/p$ but in this note, we only focus on the $L^2 \times L^2 \to L^1$ boundedness of such operators. Such estimates are central and play the same role in bilinear theory as the L^2 boundedness plays in linear multiplier theory. As Plancherel's identity (1) does not hold on L^1, there does not seem to be a straightforward way to characterize the boundedness of bilinear multiplier operators from $L^2 \times L^2 \to L^1$. But for functions m with bounded derivatives up to a certain order, such a characterization is possible.

As we restrict attention to multipliers all of whose derivatives are bounded, we introduce the space

$$\mathcal{L}^\infty(\mathbb{R}^{2n}) = \big\{m : \mathbb{R}^{2n} \to \mathbb{C} :\ \partial^\alpha m \text{ exist for all } \alpha \text{ and } \|\partial^\alpha m\|_{L^\infty} < \infty\big\}.$$

In the linear setting we have $m \in L^\infty$ if and only if the corresponding linear operator is bounded on L^2. So one may guess that a bilinear operator T_m is bounded from $L^2 \times L^2$ to L^1 when m lies in $\mathcal{L}^\infty$. However Bényi and Torres [1] provided an example of a function $m \in \mathcal{L}^\infty$ for which the associated bilinear operator T_m is unbounded from $L^{p_1} \times L^{p_2}$ to L^p for any $1 \le p_1, p_2 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$. The counterexample of Bényi and Torres is also complemented by a subsequent positive result of He, Honzík, and the author [2, Corollary 8], who showed that the mere L^2 integrability of functions in $\mathcal{L}^\infty$ suffices to yield the $L^2 \times L^2 \to L^1$ boundedness of T_m.

It turns out that the magnitude of integrability of a function m in $\mathcal{L}^\infty$ characterizes the boundedness of the bilinear multiplier operator T_m from $L^2 \times L^2 \to L^1$. We provide a proof of the main direction of this equivalence, the one that yields the boundedness of the operator.

THEOREM 1.1 ([3]). *Let $1 \le q < 4$ and set $M_q = \lfloor \frac{2n}{4-q} \rfloor + 1$. Let m be a function in $L^q(\mathbb{R}^{2n}) \cap \mathcal{C}^{M_q}(\mathbb{R}^{2n})$ satisfying*

$$\|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty \quad \text{for all multiindices } \alpha \text{ with } |\alpha| \le M_q. \tag{2}$$

Then there is a constant C depending on n and q such that the bilinear operator T_m with multiplier m satisfies

$$\|T_m\|_{L^2\times L^2 \to L^1} \le C\, C_0^{1-q/4} \|m\|_{L^q}^{q/4}. \tag{3}$$

Additionally, we are aware of examples indicating that for any $q \ge 4$ there exist functions $m \in L^q(\mathbb{R}^{2n}) \cap \mathcal{L}^\infty(\mathbb{R}^{2n})$ such that the associated operator T_m does not map $L^2 \times L^2$ to L^1; see [3] for $q > 4$ and [5] for $q = 4$. These counterexamples complement Theorem 1.1 and indicate its sharpness; as this note is based on the lecture of the author at the *Function Spaces XII* conference, we do not describe these counterexamples here.

2. Product-type wavelets. We plan to outline the proof of Theorem 1.1. This is based on the product-type wavelet method initiated by He, Honzík and the author in [2]. Our approach here incorporates several crucial combinatorial improvements. For the sake of a simple and clear presentation, we prove Theorem 1.1 only in the case where $n = 1$.

We recall some facts related to product-type wavelets that will be crucial in our approach of proving Theorem 1.1. For a fixed $M \in \mathbb{N}$ there exist real-valued compactly supported functions ψ_F, ψ_M in $\mathcal{C}^k(\mathbb{R})$, called *father wavelet* and *mother wavelet*, respectively, that satisfy

$$\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$$

and

$$\int_{\mathbb{R}} x^k \psi_M(x)\, dx = 0 \qquad \text{for all } 0 \le k \le M.$$

Then the family of functions

$$\begin{aligned}
&\bigcup_{\mu_1,\mu_2\in\mathbb{Z}} \{\psi_F(x_1-\mu_1)\psi_F(x_2-\mu_2)\} \\
&\quad\cup \bigcup_{\mu_1,\mu_2\in\mathbb{Z}} \bigcup_{\lambda=0}^{\infty} \{2^{\lambda/2}\psi_F(2^\lambda x_1-\mu_1) 2^{\lambda/2}\psi_M(2^\lambda x_2-\mu_2)\} \\
&\quad\cup \bigcup_{\mu_1,\mu_2\in\mathbb{Z}} \bigcup_{\lambda=0}^{\infty} \{2^{\lambda/2}\psi_M(2^\lambda x_1-\mu_1) 2^{\lambda/2}\psi_F(2^\lambda x_2-\mu_2)\} \\
&\quad\cup \bigcup_{\mu_1,\mu_2\in\mathbb{Z}} \bigcup_{\lambda=0}^{\infty} \{2^{\lambda/2}\psi_M(2^\lambda x_1-\mu_1) 2^{\lambda/2}\psi_M(2^\lambda x_2-\mu_2)\}
\end{aligned}$$

forms an orthonormal basis of $L^2(\mathbb{R}^2)$. This result is due to Triebel[1] and its proof can be found in Triebel [6].

[1] as confirmed by him during the Function Spaces XII conference.

We denote by $\mathcal{J}$ the set of all pairs (λ, G) such that either $\lambda = 0$ and $G = (F, F)$, or λ is a nonnegative integer and G has the form (F, M), (M, F), or (M, M). For $(\lambda, G) \in \mathcal{J}$ and $(\mu_1, \mu_2) \in \mathbb{Z}^2$ we set

$$\Psi^{\lambda,G}_{\mu_1,\mu_2}(x_1, x_2) = 2^{\lambda/2}\psi_{G_1}(2^\lambda x_1 - \mu_1)2^{\lambda/2}\psi_{G_2}(2^\lambda x_2 - \mu_2).$$

for $(x_1, x_2) \in \mathbb{R}^2$, where $G = (G_1, G_2)$ and $(\lambda, G) \in \mathcal{J}$.

The cancellation of wavelets is manifested in the following result.

LEMMA 2.1. *Let M be a positive integer. Assume that $m \in \mathcal{C}^{M+1}$ is a function on $\mathbb{R}^2$ such that*

$$\sup_{|\alpha|\le M+1} \|\partial^\alpha m\|_{L^\infty} \le C_0 < \infty\,.$$

Then for $(\lambda, G) \in \mathcal{J}$ and $(\mu_1, \mu_2) \in \mathbb{Z}^2$ we have

$$|\langle \Psi^{\lambda,G}_{\mu_1,\mu_2}, m\rangle| \le CC_0 2^{-(M+2)\lambda}\,, \tag{4}$$

provided that ψ_M has M vanishing moments.

This lemma can be easily proved and is essentially a restatement of Lemma 7 in [2]. Note that if $G = (F, F)$ there is no cancellation, however, there is no decay claimed in (4), as $\lambda = 0$ in this case.

3. Proof of Theorem 1.1. To prove the theorem we use the product type wavelets introduced in the previous section. We begin by fixing a large number M to be determined later, which denotes the number of vanishing moments of the mother wavelet.

For $(\lambda, G) \in \mathcal{J}$ and $\mu \in \mathbb{Z}^2$ we denote the wavelet coefficient by

$$b^{\lambda,G}_\mu = \langle \Psi^{\lambda,G}_\mu, m\rangle.$$

By [7, Theorem 1.64] and by the fact that $L^q = F^0_{q,2}$, we obtain

$$\|m\|_{L^q(\mathbb{R}^2)} \approx \Big\|\Big(\sum_{(\lambda,G)\in\mathcal{J}}\sum_{\mu\in\mathbb{Z}^2} |b^{\lambda,G}_\mu 2^\lambda \chi_{Q_{\lambda\mu}}|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^2)}, \tag{5}$$

where $Q_{\lambda\mu}$ is the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{1-\lambda}$.

Now, let us fix $(\lambda, G) \in \mathcal{J}$. For notational simplicity, we write b_μ instead of $b^{\lambda,G}_\mu$ in what follows. We also denote by $\tilde{Q}_{\lambda\mu}$ the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{-\lambda}$. Since these cubes are pairwise disjoint in μ (for the fixed value of λ), the equivalence (5) yields

$$\begin{aligned}\|m\|_{L^q(\mathbb{R}^2)} &\gtrsim 2^\lambda\Big\|\Big(\sum_{\mu\in\mathbb{Z}^2}|b_\mu|^2\chi_{Q_{\lambda\mu}}\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^2)} \ge 2^\lambda\Big\|\Big(\sum_{\mu\in\mathbb{Z}^2}|b_\mu|^2\chi_{\tilde{Q}_{\lambda\mu}}\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^2)}\\ &= 2^\lambda\Big\|\sum_{\mu\in\mathbb{Z}^2}|b_\mu|\chi_{\tilde{Q}_{\lambda\mu}}\Big\|_{L^q(\mathbb{R}^2)} = 2^{\lambda(1-2/q)}\Big(\sum_{\mu\in\mathbb{Z}^2}|b_\mu|^q\Big)^{1/q}.\end{aligned}$$

If we set $b = (b_\mu)_{\mu\in\mathbb{Z}^2}$, the preceding sequence of inequalities yields

$$\|b\|_{\ell^q} \le C2^{-\lambda(1-2/q)}\|m\|_{L^q} \tag{6}$$

Also, Lemma 2.1 implies that

$$\|b\|_{\ell^\infty} \le CC_0 2^{-\lambda(M+2)}, \tag{7}$$

where M is the number of vanishing moments of ψ_M.

We have an infinite $\times$ infinite matrix of wavelet coefficients indexed by $\mathbb{Z}^2$. To better organize these coefficients, define

$$U_r = \big\{(k,l) \in \mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2 : 2^{-r-1}\|b\|_{\ell^\infty} < |b_{(k,l)}| \le 2^{-r}\|b\|_{\ell^\infty}\big\},$$

where r is a nonnegative integer. Also, we write U_r as a union of the following two disjoint sets:

$$U_r^1 = \big\{(k,l) \in U_r : \operatorname{card}\{s : (k,s) \in U_r\} \ge K\big\};$$
$$U_r^2 = \big\{(k,l) \in U_r : \operatorname{card}\{s : (k,s) \in U_r\} < K\big\},$$

where K is a positive number to be determined. Thinking of U_r an infinite $\times$ infinite matrix with integers entries, in this splitting, we placed in U_r^1 all columns of U^r that have size greater than or equal to K and in U_r^2 the remaining ones. We call U_r^1 the long columns of U_r and U_r^1 the short columns. Let us define

$$E = \{k \in \mathbb{Z} : (k,l) \in U_r^1 \text{ for some } l \in \mathbb{Z}\}.$$

This set is exactly the set of projections of all long columns. Then

$$(\operatorname{card} E)\, K\big[2^{-(r+1)}\|b\|_{\ell^\infty}\big]^q \le \sum_{(k,l)\in U_r^1} |b_{(k,l)}|^q \le \|b\|_{\ell^q}^q,$$

and therefore

$$\operatorname{card} E \le K^{-1}\big[2^{-(r+1)}\|b\|_{\ell^\infty}\big]^{-q}\|b\|_{\ell^q}^q. \tag{8}$$

Having separated the wavelet coefficients in groups we proceed with the analysis of the sums of the decomposition associated to these groups. Given $(k,l) \in \mathbb{Z}\times\mathbb{Z}$, it follows from the definition of $\Psi^{\lambda,G}_{(k,l)}$ that $\Psi^{\lambda,G}_{(k,l)}$ can be written in the tensor product form

$$\Psi^{\lambda,G}_{(k,l)}(x_1,x_2) = \omega_{1,k}(x_1)\omega_{2,l}(x_2)$$

and

$$\|\omega_{1,k}\|_{L^\infty} \approx \|\omega_{2,l}\|_{L^\infty} = 2^{\lambda/2}.$$

Define

$$m^{r,1} = \sum_{(k,l)\in U_r^1} b_{(k,l)}\Psi^{\lambda,G}_{(k,l)} = \sum_{(k,l)\in U_r^1} b_{(k,l)}\omega_{1,k}\omega_{2,l}.$$

Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Then

$$\begin{aligned}
\big\|T_{m^{r,1}}(f,g)\big\|_{L^1} &\le \Big\|\sum_{(k,l)\in U_r^1} b_{(k,l)}\mathcal{F}^{-1}(\omega_{1,k}\widehat{f}\,)\mathcal{F}^{-1}(\omega_{2,l}\widehat{g}\,)\Big\|_{L^1}\\
&\le \sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}\Big\|\sum_{l:(k,l)\in U_r^1} b_{(k,l)}\omega_{2,l}\widehat{g}\,\Big\|_{L^2}\\
&\le C\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}2^{\lambda/2}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\le C\Big(\sum_{k\in E}1\Big)^{1/2}\Big(\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\Big)^{1/2}2^{\lambda/2}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\le C\big\{K^{-1/2}\big[2^{-(r+1)}\|b\|_{\ell^\infty}\big]^{-q/2}\|b\|_{\ell^q}^{q/2}\big\}\big\{2^{\lambda/2}2^{-r}\|b\|_{\ell^\infty}\big\}2^{\lambda/2}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}$$

where we used estimate (8) and the property that the supports of the functions $\omega_{1,k}$ and $\omega_{2,l}$ have finite overlap.

Now define

$$m^{r,2} = \sum_{(k,l)\in U_r^2} b_{(k,l)}\omega_{1,k}\omega_{2,l}.$$

Then

$$\begin{aligned}
\|T_{m^{r,2}}(f,g)\|_{L^1} &\le \Big\| \sum_{(k,l)\in U_r^2} b_{(k,l)}\mathcal{F}^{-1}(\omega_{1,k}\widehat{f}\,)\mathcal{F}^{-1}(\omega_{2,l}\widehat{g}\,)\Big\|_{L^1} \\
&\le \sum_{l:\,\exists k\,(k,l)\in U_r^2} \big\|\omega_{2,l}\widehat{g}\,\big\|_{L^2} \Big\| \sum_{k:(k,l)\in U_r^2} b_{(k,l)}\omega_{1,k}\widehat{f}\,\Big\|_{L^2} \\
&\le \Big(\sum_{l\in\mathbb{Z}} \big\|\omega_{2,l}\widehat{g}\,\big\|_{L^2}^2\Big)^{1/2} \Big(\sum_{l:\,\exists k\,(k,l)\in U_r^2} \Big\| \sum_{k:(k,l)\in U_r^2} b_{(k,l)}\omega_{1,k}\widehat{f}\,\Big\|_{L^2}^2\Big)^{1/2} \\
&\le C\,2^{\lambda/2}\|g\|_{L^2}\Big(\sum_{k:\,\exists l\,(k,l)\in U_r^2} \big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2 \sum_{l:(k,l)\in U_r^2} |b_{(k,l)}|^2\Big)^{1/2} \\
&\le C\,2^{\lambda/2}\|g\|_{L^2}2^{-r}\|b\|_{\ell^\infty}K^{1/2}\Big(\sum_{k\in\mathbb{Z}} \big\|\omega_{1,k}\widehat{f}\,\big\|_{L^2}^2\Big)^{1/2} \\
&\le C\,2^{\lambda/2}2^{-r}\|b\|_{\ell^\infty}K^{1/2}2^{\lambda/2}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

We have now obtained the estimates for an unknown quantity K:

$$\begin{aligned}
\|T_{\sigma_1^r}(f,g)\|_{L^1} &\le CK^{-1/2}\big[2^{-(r+1)}\|b\|_{\ell^\infty}\big]^{-q/2}\|b\|_{\ell^q}^{q/2}2^{\lambda}2^{-r}\|b\|_{\ell^\infty}\|f\|_{L^2}\|g\|_{L^2} \\
\|T_{\sigma_2^r}(f,g)\|_{L^1} &\le C2^{\lambda}2^{-r}\|b\|_{\ell^\infty}K^{1/2}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

We choose K optimally so that the two quantities on the right above are equal. The optimal choice of K is

$$K = \Big(\frac{2^r\|b\|_{\ell^q}}{\|b\|_{\ell^\infty}}\Big)^{q/2}$$

which yields for

$$m^r = \sum_{(k,l)\in U_r} b_{(k,l)}\omega_{1,k}\omega_{2,l} = m^{r,1} + m^{r,2}$$

the estimate

$$\|T_{m^r}\|_{L^2\times L^2\to L^1} \le C\,2^{\lambda}\,2^{-r(1-q/4)}\|b\|_{\ell^\infty}^{1-q/4}\|b\|_{\ell^q}^{q/4}.$$

Using (6) and (7) we obtain

$$\|T_{m^r}\|_{L^2\times L^2\to L^1} \le CC_0^{1-q/4}2^{\lambda-\lambda(1-q/4)(M+2)+(2/q-1)\lambda q/4}2^{-r(1-q/4)}\|m\|_{L^q}^{q/4}.$$

But

$$2^{\lambda-\lambda(1-q/4)(M+2)+(2/q-1)\lambda q/4} = 2^{\lambda[1/2-((4-q)/4)(M+1)]}$$

and the exponent is negative only when $M+1 > \frac{2}{4-q}$. Thus, if we choose $M = \lfloor\frac{2}{4-q}\rfloor$, we can sum first over r and then over (λ, G) in $\mathcal{J}$, obtaining (3). This completes the proof of Theorem 1.1. ∎

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