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A NOTE ON TYPE OF WEAK-L^1 AND WEAK-ℓ^1 SPACES

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Abstract. We present a direct proof of the fact that the weak-L^1 and weak-ℓ^1 spaces do not have type 1.

It has been known for some time that the weak-L^1 space is not normable, that is, there does not exist a norm equivalent to the standard quasi-norm $\|f\|_{1,\infty}$ in the weak-L^1 space [2]. In [4, Proposition 2.3] it was proved more, namely that the weak-L^1 space does not have type 1. This was obtained indirectly as a corollary of more general investigations. Here we present a direct proof by constructing suitable sequences of functions that contradicts type 1 property in weak-L^1 or weak-ℓ^1 spaces.

Let $r_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, be Rademacher functions, that is $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$. A quasi-Banach space $(X, \|\cdot\|)$ has type 1 [3, 7] if there is a constant $K > 0$ such that, for any choice of finitely many vectors $x_1, \dots, x_n$ from X,

$$\int_0^1 \Big\|\sum_{k=1}^n r_k(t) x_k\Big\| dt \le K \sum_{k=1}^n \|x_k\|.$$

Clearly if X is a normable space then X has type 1. For theory of quasi-Banach spaces see [5].

If f is a real-valued measurable function on I, where $I = (0,1)$ or $I = (0,\infty)$, then we define the *distribution function* of f by $d_f(\lambda) = \big|\{x \in \mathbb{R} : |f(x)| > \lambda\}\big|$ for each $\lambda \ge 0$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$, and the *decreasing rearrangement* of f is defined as

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}, \quad t \in I.$$

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The *weak-L^1* space on I, called also the *Marcinkiewicz* space and denoted by $L_{1,\infty}(I)$ [1, 6], is the collection of all real valued measurable functions on I such that

$$\|f\|_{1,\infty} = \sup_{t\in I} t f^*(t) < \infty.$$

The space $L_{1,\infty}(I)$ equipped with the quasi-norm $\|\cdot\|_{1,\infty}$ is complete.

Analogously we define a sequence weak-ℓ^1 space. Given a bounded real-valued sequence $x = \{x(n)\}$, consider the function $f(t) = \sum_{n=1}^{\infty} x(n)\chi_{[n-1,n)}(t)$, $t \geq 0$, and define a decreasing rearrangement $x^* = \{x^*(n)\}$ of the sequence x as follows

$$x^*(n) = f^*(n-1), \qquad n \in \mathbb{N}.$$

Then the weak-ℓ^1 space denoted as $\ell_{1,\infty}$ consists of all sequences $x = \{x(n)\}$ such that

$$\|x\|_{1,\infty} = \sup_n n x^*(n) < \infty,$$

and $\ell_{1,\infty}$ equipped with $\|\cdot\|_{1,\infty}$ is a quasi-Banach space.

Lemma 1. *For every $n \in \mathbb{N}$ there exists a sequence $(g_{jn})_{j=1}^n \subset L_{1,\infty}(0,1)$ such that*

$$\|g_{j(n-1)}\|_{1,\infty} \leq 1, \qquad n \in \mathbb{N}, \quad j = 1,\ldots,n,$$

and for sufficiently large $n \in \mathbb{N}$ and every choice of signs $\eta_j = \pm 1$, $j = 1,\ldots,n$,

$$\frac{1}{2} n \log n \leq \Big\|\sum_{j=1}^{n-1} \eta_j g_{j(n-1)}\Big\|_{1,\infty} \leq n \log n.$$

Proof. Let $k, n \in \mathbb{N}$ and $i = 1,\ldots,n-1$. Define for $t \in (0,1)$,

$$f_{ki}(t) = \frac{1}{t + n^{1-k} - i\,n^{-k}} \chi_{(n^{-k}, in^{-k}]}(t) + \frac{1}{t - (i-1)n^{-k}} \chi_{(in^{-k}, n^{1-k}]}(t).$$

Setting

$$F_k = \sum_{j=1}^{n-1} f_{kj},$$

we have

$$F_k(t) = \sum_{i=1}^{n-1} \Big(\sum_{j=1}^{n-1} \frac{1}{t + (j-i)n^{-k}}\Big) \chi_{(in^{-k},(i+1)n^{-k}]}(t), \qquad t \in (0,1).$$

We will show that for all $k \in \mathbb{N}$ and all $n \geq 10$,

$$\frac{1}{2} n \log n \leq \|F_k\|_{1,\infty} \leq n \log n.$$

In fact, if $in^{-k} < t \leq (i+1)n^{-k}$, $i = 1,\ldots,n-1$, then

$$n^k(\log n - 1) \leq F_k(t) = \sum_{j=1}^{n-1} \frac{1}{t + (j-i)n^{-k}} \leq n^k \log n.$$

Hence for all $0 < t \leq n^{1-k}(1 - 1/n)$,

$$n^k(\log n - 1) \leq F_k^*(t) \leq n^k \log n,$$

and so

$$\|F_k\|_{1,\infty} \leq \frac{1}{n^{k-1}} \Big(1 - \frac{1}{n}\Big) n^k \log n \leq n \log n$$

and for all $n \geq 10$,

$$\|F_k\|_{1,\infty} \geq \frac{1}{n^{k-1}}\left(1-\frac{1}{n}\right)n^k(\log n - 1) \geq \frac{1}{2}\, n\log n.$$

Let $m = 1,\dots,2^{n-1}$ and $\varepsilon^m = (\varepsilon_1^m,\dots,\varepsilon_{n-1}^m)$ be a sequence of signs $\varepsilon_j^m = \pm 1$, $j = 1,\dots,n-1$. We assume here that $\varepsilon^{m_1} \neq \varepsilon^{m_2}$ if $m_1 \neq m_2$. Define now the functions

$$G_m = \sum_{j=1}^{n-1} \varepsilon_j^m f_{mj}.$$

Notice that the supports of G_m are disjoint and that $G_{m_0} = F_{m_0}$ whenever $\varepsilon_j^{m_0} = 1$ for $j = 1,\dots,n-1$. Therefore

$$\Big\|\sum_{m=1}^{2^{n-1}} G_m\Big\|_{1,\infty} \geq \|G_{m_0}\|_{1,\infty} = \|F_{m_0}\|_{1,\infty} \geq \frac{1}{2}\, n\log n.$$

On the other hand observe that

$$\sum_{m=1}^{2^{n-1}} F_m \leq \sum_{m=1}^{2^{n-1}} n^m \log n\, \chi_{(n^{-m},n^{-m+1}]}.$$

Then

$$\Big(\sum_{m=1}^{2^{n-1}} F_m\Big)^* \leq \sum_{m=1}^{2^{n-1}} n^m \log n\, \chi_{(n^{-m}-n^{-2^{n-1}},\,n^{-m+1}-2^{-2^{n-1}}]},$$

and so

$$\Big\|\sum_{m=1}^{2^{n-1}} F_m\Big\|_{1,\infty} \leq \max_{m=1,\dots,2^{n-1}} \sup_{t\in(n^{-m}-n^{-2^{n-1}},\,n^{-m+1}-2^{-2^{n-1}}]} t n^m \log n \leq n\log n.$$

Hence

$$\Big\|\sum_{m=1}^{2^{n-1}} G_m\Big\|_{1,\infty} \leq \Big\|\sum_{m=1}^{2^{n-1}} F_m\Big\|_{1,\infty} \leq n\log n.$$

Now, let for $j = 1,\dots,n-1$, $n \in \mathbb{N}$,

$$g_{j(n-1)} = \sum_{m=1}^{2^{n-1}} \varepsilon_j^m f_{mj}.$$

For any $\eta_j = \pm 1$, $j = 1,\dots,n-1$, we have

$$\sum_{j=1}^{n-1} \eta_j g_{j(n-1)} = \sum_{j=1}^{n-1} \eta_j \Big(\sum_{m=1}^{2^{n-1}} \varepsilon_j^m f_{mj}\Big) = \sum_{m=1}^{2^{n-1}} \Big(\sum_{j=1}^{n-1} \eta_j \varepsilon_j^m f_{mj}\Big).$$

Setting now $\alpha_j^m = \eta_j \varepsilon_j^m$, $m = 1,\dots,2^{n-1}$, $j = 1,\dots,n-1$, we get

$$\sum_{j=1}^{n-1} \eta_j g_{j(n-1)} = \sum_{m=1}^{2^{n-1}} \Big(\sum_{j=1}^{n-1} \alpha_j^m f_{mj}\Big) = \sum_{m=1}^{2^{n-1}} G_m.$$

Hence by the previous inequalities, for every choice of signs $\eta_j = \pm 1$ and for sufficiently large n, we have

$$\frac{1}{2}\, n \log n \leq \Big\| \sum_{j=1}^{n-1} \eta_j g_{j(n-1)} \Big\|_{1,\infty} \leq n \log n. \tag{1}$$

The functions f_{mj} have disjoint supports with respect to $m = 1, \ldots, 2^{n-1}$ for each $j = 1, \ldots, n-1$. Hence

$$|g_{j(n-1)}| = \sum_{m=1}^{2^{n-1}} f_{mj} \quad \text{and} \quad \operatorname{supp} |g_{j(n-1)}| = (n^{-2^{n-1}}, 1].$$

It follows in view of the construction of the sequence (f_{mj}) that for $j = 1, \ldots, n-1$, $t \in (0,1)$ and $n \in \mathbb{N}$ we have

$$g^*_{j(n-1)}(t) = \frac{1}{t + n^{-2^{n-1}}}\, \chi_{(0, 1 - n^{-2^{n-1}}]}(t).$$

Hence for all $j = 1, \ldots, n-1$ and $n \in \mathbb{N}$,

$$\|g_{j(n-1)}\|_{1,\infty} = \sup_{t \in (0,1)} t g^*_{j(n-1)}(t) = \sup_{t \in (0,1)} \frac{t}{t + n^{-2^{n-1}}}\, \chi_{(0, 1 - n^{-2^{n-1}}]}(t) \leq 1. \tag{2}$$

In view of (1) and (2) the proof is completed. ■

Remark 2. Lemma 1 remains also true for the sequence space $\ell_{1,\infty}$.

Theorem 3. *The spaces $L_{1,\infty}(I)$ and $\ell_{1,\infty}$ do not have type* 1. *In particular, $L_{1,\infty}(I)$ and $\ell_{1,\infty}$ are not normable.*

Proof. Applying Lemma 1 we get

$$\frac{\int_0^1 \big\| \sum_{j=1}^n r_j(t) g_{jn} \big\|_{1,\infty} dt}{\sum_{j=1}^n \|g_{jn}\|_{1,\infty}} = \frac{2^{-n} \sum_{\eta_j = \pm 1} \big\| \sum_{j=1}^n \eta_j g_{jn} \big\|_{1,\infty}}{\sum_{j=1}^n \|g_{jn}\|_{1,\infty}} \geq \frac{(n+1) \log (n+1)}{2n} \to \infty,$$

as $n \to \infty$, which shows that the space $L_{1,\infty}(I)$ does not have type 1. By Remark 2 the proof also holds for sequence case. ■

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