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DISCRETE MORREY SPACES ARE CLOSED SUBSPACES OF THEIR CONTINUOUS COUNTERPARTS

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Abstract. The Banach space structure of the discrete Morrey spaces follows from the fact that they are closed subspaces of the classical Morrey spaces. In this paper, we present the complete proof of this statement. Some inclusion properties of the discrete Morrey spaces are also discussed.

1. Introduction. The discrete Morrey spaces are introduced in [1]. The paper [1] was presented as a contributed talk at the conference *Function Spaces XII* in Kraków, July 2018. In an answer to a question posed by Professor Y. Sawano, it was noted that the Banach space structure of the discrete Morrey spaces (cf. [1, Proposition 2.2]) follows from the fact that they are closed subspaces of the classical Morrey spaces (cf. [1, Remark, p. 1285]). This fact is not trivial to prove, and such a proof was not presented in the original paper [1]. In this paper, we present the complete proof of this statement; this is presented in Section 2. We also provide a similar result for the weak discrete Morrey spaces, which were also introduced in [1].

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We remark that sequences in the discrete Morrey spaces introduced in [1] are defined on the set of integers. An extension to $\mathbb{Z}^d$ are given in the paper [2]. Haroske and Skrzypczak also considered such an extension in their paper [3], with a different definition, and a different name—Morrey sequence space. The paper [3] is presented as a plenary lecture by Professor D. Haroske at *Function Spaces XII*. There are some inclusion properties that were considered in [3], which (nontrivially) coincide with a few results in [1]. We present this observation in Section 3.

2. Main results. Given $1 \le p \le q < \infty$, the classical Morrey space on the real line, which we denote by $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R})$, is a Banach space with respect to the norm

$$\|f\|_{\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}, r>0} (2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r} |f(t)|^p\, dt\Big)^{1/p} \quad (f \in \mathcal{M}_q^p).$$

We remark that if $p = q$ then $\mathcal{M}_p^p = L^p$.

We denote by ω the set $\mathbb{N} \cup \{0\}$. For any $m \in \mathbb{Z}$ and $N \in \omega$, we define

$$S_{m,N} := \{m - N, \dots, m, \dots, m + N\}.$$

Note that the cardinality of $S_{m,N}$, which we denote by $|S_{m,N}|$, is $2N+1$. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and fix $1 \le p \le q < \infty$. The discrete Morrey space $\ell_q^p = \ell_q^p(\mathbb{Z})$ is the space of sequences $x = (x_k)_{k\in\mathbb{Z}}$ taking values in $\mathbb{K}$ such that

$$\|x\|_{\ell_q^p} := \sup_{m\in\mathbb{Z},\ N\in\omega} |S_{m,N}|^{1/q-1/p}\Big(\sum_{k\in S_{m,N}} |x_k|^p\Big)^{1/p} < \infty.$$

We remark that if $p = q$ then $\ell_p^p = \ell^p$.

Any sequence x defined on $\mathbb{Z}$ can be naturally identified with a function $\bar{x} : \mathbb{R} \to \mathbb{R}$ defined by

$$\bar{x}(t) = \Big(\sum_{k\in\mathbb{Z}} |x_k|^p \chi_{[k,k+1)}(t)\Big)^{1/p} \quad (t \in \mathbb{R}),$$

and we use this notation throughout the paper.

We next prove that $\bar{x} \in \mathcal{M}_q^p$ whenever $x \in \ell_q^p$, showing that ℓ_q^p can be considered as a subspace of $\mathcal{M}_q^p$. The following notation is used in the proof: Let $a, b \in \mathbb{R}$, we denote by $\lfloor a\rfloor$, the greatest integer less than or equal to x; we denote by $\lceil a\rceil$, the least integer greater than or equal to a; and we denote by $a \vee b$, the least upper bound of a and b.

THEOREM 2.1. *Let $1 \le p \le q < \infty$. Then ℓ_q^p can be considered as a closed subspace of $\mathcal{M}_q^p$. Moreover, there exist constants $B, C > 0$ such that for every $x \in \ell_q^p$,*

$$B\|x\|_{\ell_q^p} \le \|\bar{x}\|_{\mathcal{M}_q^p} \le C\|x\|_{\ell_q^p}.$$

Proof. Let $x \in \ell_q^p$. We wish to show that

$$\|\bar{x}\|_{\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}, r>0} (2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\, dt\Big)^{1/p} < \infty.$$

To this end, let $a \in \mathbb{R}$, and put $r \in (0, 1)$. We consider five mutually disjoint cases.

Case 1: $a \in \mathbb{Z}$. Then

$$(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p} = (2r)^{1/q-1/p}\Big(|x_{a-1}|^p r + |x_a|^p r\Big)^{1/p}$$
$$\le (2r)^{1/q}\Big(|x_{a-1}|^p + |x_a|^p\Big)^{1/p} \le 2^{1/p}(2\cdot 1)^{1/q-1/p}\Big(|x_{a-1}|^p + |x_a|^p\Big)^{1/p}$$
$$= 2^{1/p}(2\cdot 1)^{1/q-1/p}\Big(\int_{a-1}^{a+1} |\bar{x}(t)|^p\,dt\Big)^{1/p}.$$

Case 2: $a \notin \mathbb{Z}$ and $r > a - \lfloor a \rfloor$ and $r > \lceil a \rceil - a$. Then $r > \frac{1}{2}$, and thus $(2r)^{1/q-1/p} \le 1$. Moreover, we also have $\lfloor a-r \rfloor = \lfloor a-1 \rfloor$ and $\lfloor a+r \rfloor = \lfloor a+1 \rfloor$. Thus, since $\bar{x}$ is a step function, we have

$$(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p}$$
$$= (2r)^{1/q-1/p}\big(|x_{\lfloor a-r\rfloor}|^p(\lfloor a\rfloor - (a-r)) + |x_{\lfloor a\rfloor}|^p + |x_{\lfloor a+r\rfloor}|^p(a+r-\lceil a\rceil)\big)^{1/p}$$
$$\le \big(|x_{\lfloor a-r\rfloor}|^p(\lfloor a\rfloor - (a-1)) + |x_{\lfloor a\rfloor}|^p + |x_{\lfloor a+r\rfloor}|^p(a+1-\lceil a\rceil)\big)^{1/p}$$
$$= \big(|x_{\lfloor a-1\rfloor}|^p(\lfloor a\rfloor - (a-1)) + |x_{\lfloor a\rfloor}|^p + |x_{\lfloor a+1\rfloor}|^p(a+1-\lceil a\rceil)\big)^{1/p}$$
$$= 2^{1/p-1/q}(2\cdot 1)^{1/q-1/p}\Big(\int_{a-1}^{a+1} |\bar{x}(t)|^p\,dt\Big)^{1/p}.$$

Case 3: $a \notin \mathbb{Z}$ and $r \le a - \lfloor a \rfloor$ and $r \le \lceil a \rceil - a$. Then

$$(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p} = (2r)^{1/q-1/p}|x_{\lfloor a\rfloor}|(2r)^{1/p}$$
$$= (2r)^{1/q}|x_{\lfloor a\rfloor}|$$
$$\le 2^{1/q}|x_{\lfloor a\rfloor}|$$
$$= 2^{1/p}(2\cdot 1)^{1/q-1/p}|x_{\lfloor a\rfloor}|$$
$$\le 2^{1/p}(2\cdot 1)^{1/q-1/p}\big(|x_{\lfloor a-1\rfloor}|^p(\lfloor a\rfloor - (a-1)) + |x_{\lfloor a\rfloor}|^p + |x_{\lfloor a+1\rfloor}|^p(a+1-\lceil a\rceil)\big)^{1/p}$$
$$= 2^{1/p}(2\cdot 1)^{1/q-1/p}\Big(\int_{a-1}^{a+1} |\bar{x}(t)|^p\,dt\Big)^{1/p}.$$

Case 4: $a \notin \mathbb{Z}$ and $a - \lfloor a \rfloor < r \le \lceil a \rceil - a$. We note that $\lfloor a-r \rfloor = \lfloor a-1 \rfloor$. Then

$$(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p}$$
$$= (2r)^{1/q-1/p}\big(|x_{\lfloor a-r\rfloor}|^p\big(\lfloor a\rfloor - (a-r)\big) + |x_{\lfloor a\rfloor}|^p(a+r-\lfloor a\rfloor)\big)^{1/p}$$
$$= (2r)^{1/q-1/p}\big(|x_{\lfloor a-1\rfloor}|^p\big(\lfloor a\rfloor - (a-r)\big) + |x_{\lfloor a\rfloor}|^p(a+r-\lfloor a\rfloor)\big)^{1/p}$$
$$\le (2r)^{1/q-1/p}\big(2r\max\{|x_{\lfloor a-1\rfloor}|^p, |x_{\lfloor a\rfloor}|^p\}\big)^{1/p}$$
$$= (2r)^{1/q}\big(\max\{|x_{\lfloor a-1\rfloor}|^p, |x_{\lfloor a\rfloor}|^p\}\big)^{1/p}$$
$$\le 2^{1/p}(2\cdot 1)^{1/q-1/p}\max\{|x_{\lfloor a-1\rfloor}|, |x_{\lfloor a\rfloor}|\}.$$

If $\max\{|x_{\lfloor a-1\rfloor}|, |x_{\lfloor a\rfloor}|\} = |x_{\lfloor a\rfloor}|$ then

$$\begin{aligned}
&2^{1/p}(2\cdot 1)^{1/q-1/p}\max\{|x_{\lfloor a-1\rfloor}|, |x_{\lfloor a\rfloor}|\} = 2^{1/p}(2\cdot 1)^{1/q-1/p}|x_{\lfloor a\rfloor}|\\
&\quad\leq 2^{1/p}(2\cdot 1)^{1/q-1/p}\big(|x_{\lfloor a-1\rfloor}|^p(\lfloor a\rfloor-(a-1)) + |x_{\lfloor a\rfloor}|^p + |x_{\lfloor a+1\rfloor}|^p(a+1-\lceil a\rceil)\big)^{1/p}\\
&\quad= 2^{1/p}(2\cdot 1)^{1/q-1/p}\Big(\int_{a-1}^{a+1}|\bar{x}(t)|^p\,dt\Big)^{1/p}.
\end{aligned}$$

On the other hand, if $\max\{|x_{\lfloor a-1\rfloor}|, |x_{\lfloor a\rfloor}|\} = |x_{\lfloor a-1\rfloor}|$ then

$$\begin{aligned}
&2^{1/p}(2\cdot 1)^{1/q-1/p}\max\{|x_{\lfloor a-1\rfloor}|, |x_{\lfloor a\rfloor}|\} = 2^{1/p}(2\cdot 1)^{1/q-1/p}|x_{\lfloor a-1\rfloor}|\\
&\leq 2^{1/p}(2\cdot 1)^{1/q-1/p}\big(|x_{\lfloor a-2\rfloor}|^p(\lfloor a-1\rfloor-(a-2)) + |x_{\lfloor a-1\rfloor}|^p + |x_{\lfloor a\rfloor}|^p(a-\lceil a-1\rceil)\big)^{1/p}\\
&= 2^{1/p}(2\cdot 1)^{1/q-1/p}\Big(\int_{(a-1)-1}^{(a-1)+1}|\bar{x}(t)|^p\,dt\Big)^{1/p}.
\end{aligned}$$

Treating Case 5, which is $a\notin\mathbb{Z}$ and $\lceil a\rceil - a < r \leq a - \lfloor a\rfloor$, similarly to Case 4, we see from these five cases that

$$\|\bar{x}\|_{\mathcal{M}_q^p} \leq C\sup_{a\in\mathbb{R}, r\geq 1}(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r}|\bar{x}(t)|^p\,dt\Big)^{1/p}.$$

Furthermore, if $u\geq 1$ then

$$(2u)^{1/q-1/p} \leq 2^{1/p-1/q}\big(2(u+1)\big)^{1/q-1/p}, \tag{1}$$

so for $a\in\mathbb{R}$ and $r\geq 1$ we have

$$\begin{aligned}
(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r}|\bar{x}(t)|^p\,dt\Big)^{1/p} &\leq (2\lfloor r\rfloor)^{1/q-1/p}\Big(\int_{a-\lceil r\rceil}^{a+\lceil r\rceil}|\bar{x}(t)|^p\,dt\Big)^{1/p}\\
&\leq 2^{1/p-1/q}(2\lceil r\rceil)^{1/q-1/p}\big(\int_{a-\lceil r\rceil}^{a+\lceil r\rceil}|\bar{x}(t)|^p\,dt\big)^{1/p}.
\end{aligned}$$

Hence

$$\|\bar{x}\|_{\mathcal{M}_q^p} \leq C\sup_{a\in\mathbb{R}, r\in\mathbb{N}}(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r}|\bar{x}(t)|^p\,dt\Big)^{1/p}.$$

Next let $a\notin\mathbb{Z}$, and fix $r\in\mathbb{N}$. Then $\lfloor a\pm r\rfloor = \lfloor a\rfloor \pm r$ and $\lceil a\pm r\rceil = \lceil a\rceil \pm r$. Thus

$$\begin{aligned}
&(2r)^{1/q-1/p}\Big(\int_{a-r}^{a+r}|\bar{x}(t)|^p\,dt\Big)^{1/p}\\
&\quad= (2r)^{1/q-1/p}\Big(\sum_{i=\lceil a-r\rceil}^{\lfloor a+r-1\rfloor}|x_i|^p + |x_{\lfloor a-r\rfloor}|^p\big(\lceil a-r\rceil-(a-r)\big)\\
&\qquad\quad + |x_{\lfloor a+r\rfloor}|^p\big(a+r-\lfloor a+r\rfloor\big)\Big)^{1/p}\\
&\quad= (2r)^{1/q-1/p}\Big(\sum_{i=\lceil a\rceil-r}^{\lfloor a\rfloor+r-1}|x_i|^p + |x_{\lfloor a\rfloor-r}|^p\big(\lceil a\rceil-a\big) + |x_{\lfloor a\rfloor+r}|^p\big(a-\lfloor a\rfloor\big)\Big)^{1/p}.
\end{aligned}$$

Define the continuous function $g : [\lfloor a\rfloor, \lceil a\rceil] \to \mathbb{R}$ by

$$g(u) = (2r)^Q\Big(\sum_{i=\lceil a\rceil-r}^{\lfloor a\rfloor+r-1} |x_i|^p + |x_{\lfloor a\rfloor-r}|^p\big(\lceil a\rceil - u\big) + |x_{\lfloor a\rfloor+r}|^p\big(u-\lfloor a\rfloor\big)\Big)^{1/p},$$

where $Q = 1/q - 1/p$. Observe that g is monotone on $[\lfloor a\rfloor, \lceil a\rceil]$. It follows that

$$(2r)^Q\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p} \le \max\Big\{(2r)^Q\Big(\int_{\lfloor a\rfloor-r}^{\lfloor a\rfloor+r} |\bar{x}(t)|^p\,dt\Big)^{1/p}, (2r)^Q\Big(\int_{\lceil a\rceil-r}^{\lceil a\rceil+r} |\bar{x}(t)|^p\,dt\Big)^{1/p}\Big\}.$$

Therefore,

$$\|\bar{x}\|_{\mathcal{M}^p_q} \le C \sup_{a\in\mathbb{Z}, r\in\mathbb{N}} (2r)^Q\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p}.$$

It follows that

$$\begin{aligned}\|\bar{x}\|_{\mathcal{M}^p_q} &\le C \sup_{a\in\mathbb{Z}, r\in\mathbb{N}} (2r)^Q\Big(\int_{a-r}^{a+r} |\bar{x}(t)|^p\,dt\Big)^{1/p} = C \sup_{a\in\mathbb{Z}, r\in\mathbb{N}} (2r)^Q\Big(\sum_{k\in S_{a,r}\setminus\{a+r\}} |x_k|^p\Big)^{1/p}\\ &\le 2^{-Q}C\Big[\sup_{a\in\mathbb{Z}, r\in\omega}(2r+2)^Q\Big(\sum_{k\in S_{a,r}} |x_k|^p\Big)^{1/p}\Big]\\ &\le 2^{-Q}C\Big[\sup_{a\in\mathbb{Z}, r\in\omega}(2r+1)^Q\Big(\sum_{k\in S_{a,r}} |x_k|^p\Big)^{1/p}\Big] < \infty.\end{aligned}$$

Note the use of (1) in the second to last step. Hence, ℓ^p_q can be considered as a subset of $\mathcal{M}^p_q$.

For the last statement of the proof, set $R_{m,N} = S_{m,N} \setminus \{m+N\}$ and observe that

$$\begin{aligned}\|x\|_{\ell^p_q} &= \sup_{m\in\mathbb{Z}, N\in\omega}(2N+1)^Q\Big(\sum_{k\in S_{m,N}} |x_k|^p\Big)^{1/p}\\ &= |x_m| \vee \sup_{m\in\mathbb{Z}, N\in\mathbb{N}}(2N+1)^Q\Big(\sum_{k\in R_{m,N}} |x_k|^p + |x_{m+N}|^p\Big)^{1/p}\\ &\le 2^{-Q}\|\bar{x}\|_{\mathcal{M}^p_q} \vee \sup_{m\in\mathbb{Z}, N\in\mathbb{N}}(2N)^Q\Big(\Big(\sum_{k\in R_{m,N}} |x_k|^p\Big)^{1/p} + |x_{m+N}|\Big)\\ &\le 2^{-Q}\|\bar{x}\|_{\mathcal{M}^p_q} \vee 2\sup_{m\in\mathbb{Z}, N\in\mathbb{N}}(2N)^Q\max\Big\{\Big(\sum_{k\in R_{m,N}} |x_k|^p\Big)^{1/p}, |x_{m+N}|\Big\}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\sup_{m\in\mathbb{Z}, N\in\mathbb{N}}(2N)^Q\max\Big\{\Big(\sum_{k\in R_{m,N}} |x_k|^p\Big)^{1/p}, |x_{m+N}|\Big\}\\ &\qquad\le 2^{-Q}\sup_{m\in\mathbb{Z}, N\in\mathbb{N}}\max\Big\{(2N)^Q\Big(\sum_{k\in R_{m,N}} |x_k|^p\Big)^{1/p}, (4N)^Q\Big(\sum_{k\in R_{m,2N}} |x_k|^p\Big)^{1/p}\Big\},\end{aligned}$$

and

$$\begin{aligned}
&\sup_{m\in\mathbb{Z},N\in\mathbb{N}} \max\Big\{(2N)^Q\Big(\sum_{k\in R_{m,N}}|x_k|^p\Big)^{1/p}, (4N)^Q\Big(\sum_{k\in R_{m,2N}}|x_k|^p\Big)^{1/p}\Big\}\\
&\quad\le \max\Big\{\sup_{m\in\mathbb{Z},N\in\mathbb{N}}(2N)^Q\Big(\sum_{k\in R_{m,N}}|x_k|^p\Big)^{1/p}, \sup_{m\in\mathbb{Z},N\in\mathbb{N}}(4N)^Q\Big(\sum_{k\in R_{m,2N}}|x_k|^p\Big)^{1/p}\Big\}\\
&\quad= \sup_{m\in\mathbb{Z},N\in\mathbb{N}}(2N)^Q\Big(\sum_{k\in R_{m,N}}|x_k|^p\Big)^{1/p} = \sup_{m\in\mathbb{Z},N\in\mathbb{N}}(2N)^Q\Big(\int_{m-N}^{m+N}|\bar{x}(t)|^p\,dt\Big)^{1/p}\\
&\quad\le \sup_{a\in\mathbb{R},r>0}(2r)^Q\Big(\int_{a-r}^{a+r}|\bar{x}(t)|^p\,dt\Big)^{1/p} = \|\bar{x}\|_{\mathcal{M}^p_q}.
\end{aligned}$$

That ℓ^p_q can be considered as a closed subspace of $\mathcal{M}^p_q$ now follows from [1, Proposition 2.2]. ∎

As a consequence of Theorem 2.1, we obtain the following well-known results, for $p=q$.

Remark 2.2. Let $1\le p<\infty$. Then ℓ^p can be considered as a closed subspace of L^p. Moreover, there exist constants $B, C>0$ such that for every $x\in\ell^p$,

$$B\|x\|_{\ell^p}\le\|\bar{x}\|_{L^p}\le C\|x\|_{\ell^p}.$$

A similar result holds for the weak discrete Morrey spaces. Given $1\le p\le q<\infty$, the classical weak Morrey space on the real line, which will be denoted here by $w\mathcal{M}^p_q = w\mathcal{M}^p_q(\mathbb{R})$, is a quasi-Banach space with respect to the quasi-norm

$$\|f\|_{w\mathcal{M}^p_q} := \sup_{a\in\mathbb{R},r>0,\gamma>0}(2r)^Q\gamma\left(m\{t\in(a-r,a+r):|f(t)|>\gamma\}\right)^{1/p},$$

where m denotes the Lebesgue measure on $\mathbb{R}$ and $Q=1/q-1/p$. For $1\le p\le q<\infty$, the weak type discrete Morrey space $w\ell^p_q = w\ell^p_q(\mathbb{Z})$ is the space of sequences $x=(x_k)_{k\in\mathbb{Z}}$ taking values in $\mathbb{K}$ such that

$$\|x\|_{w\ell^p_q} := \sup_{m\in\mathbb{Z},N\in\omega,\gamma>0}|S_{m,N}|^Q\gamma\big|\{k\in S_{m,N}:|x_k|>\gamma\}\big|^{1/p}<\infty.$$

Theorem 2.3. *Let $1\le p\le q<\infty$. Then $w\ell^p_q$ can be considered as a closed subspace of $w\mathcal{M}^p_q$. Furthermore, there exist constants $B, C>0$ such that for every $x\in w\ell^p_q$,*

$$B\|x\|_{w\ell^p_q}\le\|\bar{x}\|_{w\mathcal{M}^p_q}\le C\|x\|_{w\ell^p_q}.$$

Proof. Let $x\in w\ell^p_q$. We have

$$\begin{aligned}
\|\bar{x}\|_{w\mathcal{M}^p_q} &= \sup_{a\in\mathbb{R},r>0,\gamma>0}(2r)^Q\gamma\big(m\{t\in(a-r,a+r):|\bar{x}(t)|>\gamma\}\big)^{1/p}\\
&\le \sup_{a\in\mathbb{R},r>0,\gamma>0}(2r)^Q\gamma\big(m\{t\in(\lfloor a\rfloor-\lceil r\rceil,\lfloor a\rfloor+\lceil r\rceil+1):|\bar{x}(t)|>\gamma\}\big)^{1/p}\\
&= \sup_{a\in\mathbb{Z},r\in\mathbb{N},\gamma>0}(2r)^Q\gamma\big(m\{t\in(a-r,a+r+1):|\bar{x}(t)|>\gamma\}\big)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a\in\mathbb{Z}, r\in\mathbb{N}, \gamma>0} (2r)^Q \gamma |\{k \in S_{a,r} : |x_k| > \gamma\}|^{1/p} \\
&\le C \sup_{a\in\mathbb{Z}, r\in\mathbb{N}, \gamma>0} (2r+2)^Q \gamma |\{k \in S_{a,r} : |x_k| > \gamma\}|^{1/p} \\
&\le C \sup_{a\in\mathbb{Z}, r\in\omega, \gamma>0} (2r+1)^Q \gamma |\{k \in S_{a,r} : |x_k| > \gamma\}|^{1/p} = C\|x\|_{w\ell_q^p} < \infty.
\end{aligned}
$$

It now follows from [1, Proposition 3.4] that $w\ell_q^p$ can be considered as a closed subspace of $w\mathcal{M}_q^p$.

Moreover, using the fact that $(2x+1)^Q \le 2^{-Q}(2x+2)^Q$ for $x > 0$, we have

$$
\begin{aligned}
\|x\|_{w\ell_q^p} &= \sup_{a\in\mathbb{Z}, r\in\omega, \gamma>0} (2r+1)^Q \gamma |\{k \in S_{a,r} : |x_k| > \gamma\}|^{1/p} \\
&= \sup_{a\in\mathbb{Z}, r\in\omega, \gamma>0} (2r+1)^Q \gamma \big(m\{t \in (a-r, a+r+1) : |\bar{x}(t)| > \gamma\}\big)^{1/p} \\
&\le \sup_{a\in\mathbb{Z}, r\in\omega, \gamma>0} (2r+1)^Q \gamma \big(m\{t \in (a-(r+1), a+r+1) : |\bar{x}(t)| > \gamma\}\big)^{1/p} \\
&\le 2^{-Q} \sup_{a\in\mathbb{Z}, r\in\omega, \gamma>0} \big(2(r+1)\big)^Q \gamma \big(m\{t \in (a-(r+1), a+r+1) : |\bar{x}(t)| > \gamma\}\big)^{1/p} \\
&= 2^{-Q} \sup_{a\in\mathbb{Z}, r\in\mathbb{N}, \gamma>0} \big(2r\big)^Q \gamma \big(m\{t \in (a-r, a+r) : |\bar{x}(t)| > \gamma\}\big)^{1/p} \\
&\le 2^{-Q} \sup_{a\in\mathbb{R}, r>0, \gamma>0} \big(2r\big)^Q \gamma \big(m\{t \in (a-r, a+r) : |\bar{x}(t)| > \gamma\}\big)^{1/p} \\
&= 2^{-Q} \|\bar{x}\|_{w\mathcal{M}_q^p}. \ \blacksquare
\end{aligned}
$$

3. Remarks on some inclusion properties of the discrete Morrey spaces. We start the section by recalling the following result from [1]: for any $1 \le p_1 \le p_2 \le q < \infty$, we have $\ell_q^{p_2} \subseteq \ell_q^{p_1}$ (see also Proposition 1.3 part (ii) of [3]). We may also fix the parameter p and vary q, and therefore obtain the following result:

Proposition 3.1. *Let $1 \le p \le q_1 \le q_2 < \infty$. Then $\ell_{q_1}^p \subseteq \ell_{q_2}^p$.*

Proof. Let $x \in \ell_{q_1}^p$. For all $m \in \mathbb{Z}$ and $N \in \omega$, since $q_1 \le q_2$, we have

$$|S_{m,N}|^{1/q_2} = (2N+1)^{1/q_2} \le (2N+1)^{1/q_1} = |S_{m,N}|^{1/q_1}$$

and therefore

$$|S_{m,N}|^{1/q_2} \Big(\sum_{k\in S_{m,N}} |x_k|^p\Big)^{1/p} \le |S_{m,N}|^{1/q_1} \Big(\sum_{k\in S_{m,N}} |x_k|^p\Big)^{1/p}$$

Taking supremum over all $m \in \mathbb{Z}$ and $N \in \omega$ gives us

$$\|x\|_{\ell_{q_2}^p} \le \|x\|_{\ell_{q_1}^p},$$

and this completes the proof. $\blacksquare$

Furthermore, we have the result that every sequence in the discrete Morrey spaces is bounded (see also Proposition 1.3 part (iii) of [3]), which illustrates a fundamental difference between ℓ_q^p and $\mathcal{M}_q^p$.

PROPOSITION 3.2. *Let $1 \le p \le q < \infty$. Then $\ell_q^p \subseteq \ell^\infty(\mathbb{Z})$.*

Proof. Let $x \in \ell_q^p$. For all $m \in \mathbb{Z}$ and $N \in \omega$, we have

$$|x_m| = |S_{m,0}|^{1/q-1/p}\Big(\sum_{k\in S_{m,0}} |x_k|^p\Big)^{1/p} \le |S_{m,N}|^{1/q-1/p}\Big(\sum_{k\in S_{m,N}} |x_k|^p\Big)^{1/p}.$$

Taking supremum over all $m \in \mathbb{Z}$ and $N \in \omega$ gives us

$$\|x\|_{\ell^\infty} = \sup_{m\in\mathbb{Z}} |x_m| \le \sup_{m\in\mathbb{Z},\ N\in\omega} |S_{m,N}|^{1/q-1/p}\Big(\sum_{k\in S_{m,N}} |x_k|^p\Big)^{1/p} = \|x\|_{\ell_q^p},$$

and this completes the proof. ■

We may also vary both parameters (in the manner stated below), and obtain the following inclusion property:

PROPOSITION 3.3. *Let $1 \le p_1 \le p_2 \le q_2 \le q_1 < \infty$. Then $\ell_{q_2}^{p_2} \subseteq \ell_{q_1}^{p_1}$.*

Proof. Let $x \in \ell_{q_2}^{p_2}$. Note that since $q_2 \le q_1$, we have $(2N+1)^{\frac{1}{q_1}-\frac{1}{q_2}} \le 1$. We also have (by an application of Hölder's inequality), that for all $m \in \mathbb{Z}$ and $N \in \omega$,

$$\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} |x_k|^{p_1}\Big)^{1/p_1} \le \Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} |x_k|^{p_2}\Big)^{1/p_2}$$

and so

$$(2N+1)^{1/q_1-1/q_2}\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} |x_k|^{p_1}\Big)^{1/p_1} \le \Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} |x_k|^{p_2}\Big)^{1/p_2}$$

and finally

$$(2N+1)^{1/q_1}\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} |x_k|^{p_1}\Big)^{1/p_1} \le (2N+1)^{1/q_2}\Big(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} |x_k|^{p_2}\Big)^{1/p_2}.$$

Taking supremum over $m \in \mathbb{Z}$ and $N \in \omega$ completes the proof. ■

In what follows, we discuss how Proposition 3.3 is a special case of [1, Theorem 4.3] and also in [3, Theorem 2.1].

We recall a result by Haroske and Skrzypczak [3]. We remark that the definition of the Morrey sequence space (or, discrete Morrey space, in our terminology) from [3] is omitted, and we rewrite the theorem from [3] in our notation below, for discrete Morrey space as defined in Section 2 (i.e. on $\mathbb{Z}$). We also replace p_1, p_2, q_1, q_2 in the theorem by p_2, p_1, q_2, q_1, respectively.

THEOREM 3.4 (Haroske and Skrzypczak [3, Theorem 2.1]). *Let $0 < p_1 \le q_1 < \infty$ and $0 < p_2 \le q_2 < \infty$. Then, the embedding $\ell_{q_2}^{p_2} \hookrightarrow \ell_{q_1}^{p_1}$ is continuous if and only if the following conditions hold: $q_2 \le q_1$ and $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$.*

REMARK 3.5. The condition in Proposition 3.3 that $1 \le p_1 \le p_2 \le q_2 \le q_1 < \infty$ immediately provides the first condition in Theorem 3.4 that $q_2 \le q_1$. Furthermore, we also have

$$\frac{p_1}{q_1} \le \frac{p_1}{q_2} \le \frac{p_2}{q_2}$$

which satisfies the second condition in Theorem 3.4, and therefore $\ell^{p_2}_{q_2} \subseteq \ell^{p_1}_{q_1}$, as stated in Proposition 3.3.

We remark that we obtained a more generalised version of Proposition 3.3 in [1], for generalised discrete Morrey spaces, which are defined as follows: $\mathcal{G}_p$ is the set of all functions $\phi : 2\omega + 1 \to (0, \infty)$ such that ϕ is almost decreasing (that is there exists $C > 0$ such that $\phi(2M+1) \geq C\phi(2N+1)$, for $M, N \in \omega$ with $M \leq N$), and the mapping $(2N+1) \mapsto (2N+1)^{1/p}\phi((2N+1))$ is almost increasing (that is, there exists $C > 0$ such that $(2M+1)^{1/p}\phi(2M+1) \leq C(2N+1)^{1/p}\phi(2N+1)$, for $M, N \in \omega$ with $M \leq N$). For $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, the generalised discrete Morrey space $\ell^p_\phi = \ell^p_\phi(\mathbb{Z})$ is defined as the set of all real (or complex) sequences $x = (x_k)_{k\in\mathbb{Z}}$ such that

$$\|x\|_{\ell^p_\phi} = \sup_{m\in\mathbb{Z}, N\in\omega} \frac{1}{\phi(2N+1)} \left(\frac{1}{|S_{m,N}|} \sum_{k\in S_{m,N}} |x_k|^p \right)^{1/p} < \infty.$$

Theorem 3.6 (Gunawan, Kikianty, Schwanke [1, Theorem 4.3]). *Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, and $\phi_2 \in \mathcal{G}_{p_2}$. Then the following statements are equivalent*:

(i) $\phi_2 \lesssim \phi_1$ *(on $2\omega + 1$).*
(ii) $\|\cdot\|_{\ell^{p_1}_{\phi_1}} \lesssim \|\cdot\|_{\ell^{p_2}_{\phi_2}}$ *(on $\ell^{p_2}_{\phi_2}$).*
(iii) $\ell^{p_2}_{\phi_2} \subseteq \ell^{p_1}_{\phi_1}$.

We note that for two functions $f, g\colon X \to \mathbb{R}$ (here $X \neq \emptyset$), we have $f \lesssim g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for every $x \in X$.

Remark 3.7.

1. With the following choice of functions: $\phi_i(2N+1) = (2N+1)^{-\frac{1}{q_i}}$, $N \in \omega$, $i = \{1, 2\}$ in Theorem 3.6, we obtain Proposition 3.3.
2. In the proof of the theorem in [1], it is shown that the inclusion in Theorem 3.6 part (iii) is continuous, which is consistent with Theorem 3.4 (of Haroske and Skrzypczak).

References

[1] H. Gunawan, E. Kikianty, C. Schwanke, *Discrete Morrey spaces and their inclusion properties*, Math. Nachr. 291 (2018), 1283–1296.

[2] H. Gunawan, C. Schwanke, *The Hardy–Littlewood maximal operator on discrete Morrey spaces*, Mediterr. J. Math. 16 (2019), art. 24.

[3] D. Haroske, L. Skrzypczak, *Morrey sequence spaces: Pitt's theorem and compact embeddings*, Constr. Approx., to appear, DOI:10.1007/s00365-019-09460-7.