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EFFECTIVE ENERGY INTEGRAL FUNCTIONALS FOR THIN FILMS ON CURL-FREE VECTOR FIELDS IN THE ORLICZ–SOBOLEV SPACE SETTING

WŁODZIMIERZ LASKOWSKI

School of Mathematics, West Pomeranian University of Technology
Al. Piastów 48, 70-311 Szczecin, Poland
E-mail: wlaskowski@zut.edu.pl

HONG THAI NGUYEN

Institute of Mathematics,
Center for N-Physics and N-quantized Mathematics, Szczecin University
ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail: hong_thai.nguyen@usz.edu.pl

Abstract. We consider an elastic thin film $\omega \subset \mathbb{R}^2$ with three dimensional bending moment. The effective energy functional defined on the Orlicz–Sobolev space over ω is obtained by Γ-convergence and $3D$-$2D$ dimension reduction techniques in the case when the energy density function is cross-quasiconvex. In the case when the energy density function is not cross-quasiconvex we obtained both upper and lower bounds for the Γ-limit. These results are proved in the case when the energy density function has the growth prescribed by an Orlicz convex function M. Here M, M^* are assumed to be non-power-growth-type and to satisfy the condition Δ_2^{glob} (that imply the reflexivity of Orlicz and Orlicz–Sobolev spaces generated by M), and M^* denotes the complementary (conjugate) Orlicz N-function of M.

1. Introduction. In this paper we consider an elastic thin film as a bounded open subset $\omega \subset \mathbb{R}^2$ with Lipschitz boundary. The set $\Omega_\varepsilon := \omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \mathbb{R}^3$ for a small thickness ε is considered as an elastic cylinder approximate to the film ω. We consider

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the variational integral functional (the re-scaled kinetic energy of the elastic cylinder Ω_ε) defined by

$$G_\varepsilon(H) := \begin{cases} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(H(x))\,dx & \text{if } H \in \mathcal{V}_\varepsilon \\ +\infty & \text{otherwise,} \end{cases} \tag{1}$$

where

$$\mathcal{V}_\varepsilon := \{H \in L^M(\Omega_\varepsilon;\mathbb{R}^{3\times 3}) : \operatorname{curl} H = 0 \text{ (distributionally)}\}.$$

The purpose of this type of research is to investigate, as the thickness ε goes to zero, the Γ-convergence limit of the sequence of the above re-scaled energy functional.

Let the energy density function $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfy the growth and coercivity conditions

$$\frac{1}{C}(M(|F|) - 1) \le W(F) \le C(1 + M(|F|)) \quad (\forall F \in \mathbb{R}^{3\times 3}) \tag{2}$$

for some $C > 0$. We assume that $M\colon \mathbb{R} \to [0,\infty)$ is some Orlicz convex N-function of the non-power growth and M, M^* satisfies the condition Δ_2^{glob}. Here M^* denotes the complementary (conjugate) Orlicz N-function of M. Examples of Orlicz N-functions M with these properties are $M(t) = |t|^p(\log(1 + |t|))^q$, where $p > 1$ and $q > 1$ or $M(t) = |t|^p(\log(1+|t|))^{q_1}\cdot(\log(\log(1+|t|)))^{q_2}$, where $p > 1$ and $q_1, q_2 > 1$. Many other examples of the N-function M can be found in [35, Theorem 7.1, pp. 58–59], [52, 59, 60] and [47, 48].

In our previous papers (see [40, 41, 42, 39]) we extend to the reflexive Orlicz–Sobolev space setting $W^{1,M}$ the results established respectively by H. Le Dret and A. Raoult in 1993–1995 [43, 44] and by G. Bouchitté, I. Fonseca and M. L. Mascarenhas in 2004 and 2009 [5, 6] in the case of the re-scaled total energy functionals (containing both the bulk and surface energies) for thin films in the reflexive Sobolev space setting with $M(t) = |t|^p$ for some $p \in (1,\infty)$. It is important to note that the papers [43, 44] of H. Le Dret and A. Raoult in 1993–1995 contain the first precise convergence results for the re-scaled energy functionals in the nonlinear theory of thin membranes through the use of Γ-convergence. Their work was further developed by G. Bouchitté, I. Fonseca and M. L. Mascarenhas in 2004 [5] and in 2009 [6] for the re-scaled elastic total energy functionals with the additional two- and three-dimensional bending moment in the nonlinear membrane theory in the reflexive Sobolev spaces $W^{1,p}$.

In [37] C. Kreisbeck and F. Rindler extend the results established in 2009 [6] by G. Bouchitté, I. Fonseca and M. L. Mascarenhas to the setting of $\mathcal{A}$-free vector fields in the reflexive Sobolev spaces $W^{1,p}$. Results on variational problems within $\mathcal{A}$-free framework (the notion of $\mathcal{A}$-quasiconvexity) in the Sobolev space setting $W^{1,p}$ were obtained by I. Fonseca and S. Müller in 1999 [22] and were advanced by A. Braides, I. Fonseca, and G. Leoni in 2000 [9] and by I. Fonseca, G. Leoni, and S. Müller in 2004 [21]. References on recent results about $\mathcal{A}$-free vector fields in the reflexive Sobolev spaces $W^{1,p}$ can be found in [61, 38, 36, 37, 14, 15].

The main purpose of the present paper (see Theorem 5.5) is to extend to the reflexive Orlicz–Sobolev space setting $W^{1,M}$ the results (in the special case $\mathcal{A} = \operatorname{curl}$) established by C. Kreisbeck and F. Rindler in 2015 [37] for the case of the above re-scaled energy functional and thin films in the reflexive Sobolev space setting with $M(t) = |t|^p$ for some

$p \in (1, \infty)$. Here we present a different approach for the proof of [37, Proposition 4.1], by the explicit use of special test function (15) from Step 1 of Proposition 6.5 defined by means of the function from Lemma 6.4. We consider only the case $\mathcal{A} = \mathrm{curl}$ since the general case requires a new study, for examples in order to find an explicit special test function of the type (15) for the general operator $\mathcal{A}$.

The technical scheme for proving the upper bound in Theorem 5.5 comes back to the proofs of similar results [12, Theorem 9.1], [23], [58, Chapter 8] for the classical case for the integral energy functional of an elastic body in Sobolev spaces. The technical use of Young measures for proving the lower bound in Theorem 5.5 is originated from the proofs of similar results [32], [58, Chapter 8] for the classical case for this energy functional in Sobolev spaces (cf. [22, Proposition 3.8], [37, Proposition 3.5] and [30, 19]). The coercivity condition $W(F) \geq \frac{1}{C}(M(|F|) - 1)$ is crucial for proving the upper bound (see the arguments after the inequality (16) up to the use of the Moore Lemma in the proof of Theorem 5.5).

We would like to point out that our results and the results established by C. Kreisbeck and F. Rindler do not involve any boundary conditions of thin films. Further, the re-scaled energy functional (1) does not contain the surface part of the total free energy. It is important to consider curl-free thin films with boundary conditions and with the surface part of the energy. In the connection with these open problems, as in our previous papers (see [40, 41, 42]) it would require the use of specific trace theorems for Orlicz–Sobolev spaces (see A. Kałamajska and M. Krbec [29] and references therein). Recent results on dimension reduction problems involving thin structures in the Orlicz–Sobolev space setting can be found in [33, 34, 24]. References of various recent papers on variational multiple integral functionals and partial differential equations from nonlinear elasticity and non-newtonian mechanics in Orlicz–Sobolev spaces can be found in [55, 11, 50, 57].

2. Some terminology and notation. From now on, unless stated to the contrary, $M\colon \mathbb{R} \to [0, \infty)$ is assumed to be a non-power-growth-type Orlicz N-function (i.e., even convex function satisfying $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{M(t)}{t} = +\infty$). Let M^* be the complementary (conjugate) Orlicz N-function of M defined by $M^*(\tau) := \sup\{t\tau - M(t) : t \in \mathbb{R}\}$. We assume $M, M^* \in \Delta_2^{\mathrm{glob}}$. Here the condition $M \in \Delta_2^{\mathrm{glob}}$ means that $M(2t) \leq cM(t)$ $(t \geq 0)$ for some $c \in (0, \infty)$. The condition $M \in \Delta_2$ means that $M(2t) \leq cM(t)$ $(t \geq t_0)$ for some $t_0 \in [0, \infty)$ and $c \in (0, \infty)$.

Denote by $|v|$ the Euclidean norm of v and by (u, v) the scalar product. Given an open bounded subset $G \subset \mathbb{R}^N$ with Lipschitz (e.g., C^2-smooth) boundary ∂G equipped with the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$, denote by $L^M(G; \mathbb{R}^m)$ the Orlicz space of all (equivalent classes of) measurable functions $u\colon G \to \mathbb{R}^m$ equipped with the Luxemburg norm $\|u\|_{L^M(G;\mathbb{R}^m)} := \inf\{\lambda > 0 : \int_G M(|u(x)|/\lambda)\, dx \leq 1\}$. It is known that $M, M^* \in \Delta_2$ is equivalent to the reflexivity of $L^M(G; \mathbb{R}^m)$.

Recall that the Orlicz–Sobolev space $W^{1,M}(G; \mathbb{R}^m)$ is defined as the Banach space of $\mathbb{R}^m$-valued functions u of $L^M(G; \mathbb{R}^m)$ with the Sobolev–Schwartz distributional derivative $Du \in L^M(G; \mathbb{R}^{m\times N})$ equipped with the norm

$$\|u\|_{W^{1,M}(G;\mathbb{R}^m)} := \|u\|_{L^M(G;\mathbb{R}^m)} + \|Du\|_{L^M(G;\mathbb{R}^{m\times N})} < \infty.$$

Let $\mathcal{M}(\mathbb{R}^m)$ be the Banach space of bounded signed Radon measures on $\mathbb{R}^m$, and $C_0(\mathbb{R}^m)$ be the Banach space of all continuous functions $f : \mathbb{R}^m \to \mathbb{R}$ with $\lim_{|\lambda|\to\infty} f(\lambda) = 0$ equipped with the sup-norm. It is known that $(C_0(\mathbb{R}^m))^* \cong \mathcal{M}(\mathbb{R}^m)$. Let $L^\infty_w(\mathcal{M}(\mathbb{R}^m))$ denote the Banach space (of all equivalence classes) of $C_0(\mathbb{R}^m)$ weakly measurable functions $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ with the norm $\|\nu\|_\infty := \big\|x \mapsto |\nu_x|(\mathbb{R}^m)\big\|_{L^\infty} < \infty$, where $|\nu_x|(\mathbb{R}^m)$ is the total variation of ν_x on $\mathbb{R}^m$. It is known that $L^\infty_w(\mathcal{M}(\mathbb{R}^m))$ can be interpreted as dual space $(L^1(C_0(\mathbb{R}^m)))^*$ via the injection $\nu \mapsto \langle\cdot,\nu\rangle$, where $\langle f,\nu\rangle := \int_\Omega \langle \nu(x), f(x)\rangle\, dx$ for all $f \in L^1(C_0(\mathbb{R}^m))$. Given a measurable function $z : \Omega \to \mathbb{R}^m$, define the parametrized Dirac measure $\delta_z \in L^\infty_w(\mathcal{M}(\mathbb{R}^m))$ by $x \in \Omega \mapsto \delta_z(x) := \delta_{z(x)}$ (the Dirac measure supported at $z(x)$).

Recall [13, 7], [8, Definition 7.1] that a sequence of functions I_ε from a topological space X to $\overline{R}$ is said to Γ-converge to I_0 for the topology of X if the following conditions are satisfied for all $x \in X$:

$$\begin{cases} \forall x_\varepsilon \to x, & I_0(x) \le \liminf I_\varepsilon(x_\varepsilon), \\ \exists y_\varepsilon \to y, & I_\varepsilon(y_\varepsilon) \to I_0(y). \end{cases}$$

3. Setup. Define $I := (-\frac{1}{2}, \frac{1}{2})$ and $\Omega := \omega \times I$. Without loss of generality we may assume that $\omega \subset\subset Q^2$, where $Q^n = (-\frac{1}{2}, \frac{1}{2})^n$. Greek indexes will be used to distinguish the first two components of a vector, for instance (x_α) and (x_α, x_3), designates (x_1, x_2) and (x_1, x_2, x_3), respectively. We denote by $\mathbb{R}^{3\times3}$ and $\mathbb{R}^{3\times2}$ the vector spaces of respectively 3×3 and 3×2 real-valued matrices. Given $\bar{F} \in \mathbb{R}^{3\times2}$ and $b \in \mathbb{R}^3$, denote by $(\bar{F}|b)$ the 3×3 matrix whose first two columns are those of $\bar{F}$ and the last column is b.

Let $W : \mathbb{R}^{3\times3} \to \mathbb{R}$ be a continuous function satisfying the growth conditions (2). We consider the variational integral functional $\widetilde{J}_\varepsilon : W^{1,M}(\Omega_\varepsilon;\mathbb{R}^3) \to \mathbb{R}$, where $\widetilde{J}_\varepsilon(U)$ (the re-scaled kinetic energy of the elastic cylinder Ω_ε under a deformation $U : \Omega_\varepsilon \to \mathbb{R}^3$) is represented by the functional

$$\widetilde{J}_\varepsilon(U) := \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\, d\tilde{x}. \tag{3}$$

A function $g : \mathbb{R}^{m\times n} \times \mathbb{R}^l \to \mathbb{R}$ is called *cross-quasiconvex* (cf. [18, 45]), if

$$g(F,d) \le \int_{Q^n} g(F + D\varphi(x), d + \eta(x))\, dx$$

for all $(F,d) \in \mathbb{R}^{m\times n} \times \mathbb{R}^l$ and for all $\varphi \in W^{1,M}_{\mathrm{per}}(Q^n;\mathbb{R}^m)$ and $\eta \in L^\infty_0(Q^n;\mathbb{R}^l)$, where

$$L^\infty_0(Q^n;\mathbb{R}^l) := \Big\{\eta \in L^\infty(Q^n;\mathbb{R}^l) : \int_{Q^n} \eta\, dx = 0\Big\},$$

and $W^{1,M}_{\mathrm{per}}(Q^n;\mathbb{R}^m)$ denotes the $W^{1,M}$-closure of the set of all C^1-smooth Q^n-periodic functions defined on Q^n with values in $\mathbb{R}$ endowed with $W^{1,M}$-norm. Here the Q^n-periodicity of φ on Q^n means that $\varphi = \tilde{\varphi}|_Q$, where $\tilde{\varphi}$ is defined on $\mathbb{R}^n$ and $\tilde{\varphi}(x) = \tilde{\varphi}(x+k)$ for all $k \in \mathbb{Z}^n$ and all $x \in \mathbb{R}^n$. We will use also $W^{1,M}_{\mathrm{per}}(Q^n;\mathbb{R}^m) := W^{1,M}(\mathbb{T}^n;\mathbb{R}^m)$, where $\mathbb{T}^n$ denotes n-dimensional torus. Similarly, $L^M_{\mathrm{per}}(Q^n;\mathbb{R}^m)$ denotes the L^M-closure of the set of all continuous Q^n-periodic functions defined on Q^n with values in $\mathbb{R}$ endowed with L^M-norm and we use $L^M_{\mathrm{per}}(Q^n;\mathbb{R}^m) := L^M(\mathbb{T}^n;\mathbb{R}^m)$.

4. The formulation of main results. Let $\mathcal{Z}$ be the space of membrane deformations defined by

$$\mathcal{Z} = \{z \in W^{1,M}(\Omega;\mathbb{R}^3) : D_3 z = 0\}.$$

Observe that $\mathcal{Z}$ is canonically isomorphic to $W^{1,M}(\omega;\mathbb{R}^3)$ [49, Theorem 1.1.3/1]. Let $\bar{z}$ denote the element of $W^{1,M}(\omega;\mathbb{R}^3)$ that is associated with $z \in \mathcal{Z}$ through this isomorphism:

$$z(x_\alpha, x_3) = \bar{z}(x_\alpha) \text{ a.e.} \tag{4}$$

Since we want to identify the sequence convergence with the thickness of our domain tending to zero, for simplicity we assume this thickness parameter ε takes its values in a sequence $\varepsilon_n \to 0$.

THEOREM 4.1. *Let $\tilde{J}_\varepsilon$ be defined in* (3). *Assume $M, M^* \in \Delta_2^{\mathrm{glob}}$. Assume that the function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is cross-quasiconvex and satisfies the hypothesis* (2). *Let $U_\varepsilon \in W^{1,M}(\Omega_\varepsilon;\mathbb{R}^3)$. For each $\varepsilon > 0$ and $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ we associate $x = (x_\alpha, x_3) := \big(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\big) \in \Omega$ and we set $u_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3)$. Then:*

(i) (*lower bound*) *if $(D_\alpha u_\varepsilon \,|\, \frac{1}{\varepsilon} D_3 u_\varepsilon) \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$ for $u_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ and $H \in L^M(\Omega;\mathbb{R}^{3\times 3})$, then there exist $\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3)$ and $b \in L^M(\Omega;\mathbb{R}^3)$ such that by the 3D-2D dimension reduction isomorphism* (4) *$H = (D_\alpha \bar{u} | b)$ and*

$$\liminf_{\varepsilon\to 0} \tilde{J}_\varepsilon(U_\varepsilon) \ge \int_\Omega W(D_\alpha \bar{u} \,|\, b)\, dx;$$

(ii) (*upper bound*) *for every $H = (D_\alpha \bar{u} | b)$ with $\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3)$ and $b \in L^M(\Omega;\mathbb{R}^3)$, there exists a sequence $u_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ such that $(D_\alpha u_\varepsilon \,|\, \frac{1}{\varepsilon} D_3 u_\varepsilon) \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$ and*

$$\limsup_{\varepsilon\to 0} \tilde{J}_\varepsilon(U_\varepsilon) \le \int_\Omega W(D_\alpha \bar{u} \,|\, b)\, dx.$$

Theorem 4.1 is a corollary of the following Theorem 4.2.

We define the $\operatorname{curl}_\varepsilon$ operator of a matrix-valued function $F : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ as

$$\operatorname{curl}_\varepsilon F := \begin{bmatrix} \operatorname{curl}_\varepsilon F^1 \\ \operatorname{curl}_\varepsilon F^2 \\ \operatorname{curl}_\varepsilon F^3 \end{bmatrix}, \quad \operatorname{curl}_\varepsilon F^i := \left[\frac{\partial F_{i3}}{\partial x_2} - \frac{1}{\varepsilon}\frac{\partial F_{i2}}{\partial x_3}, \frac{1}{\varepsilon}\frac{\partial F_{i1}}{\partial x_3} - \frac{\partial F_{i3}}{\partial x_1}, \frac{\partial F_{i2}}{\partial x_1} - \frac{\partial F_{i1}}{\partial x_2}\right]$$

where F^i denotes the i-th row of F ($i = 1, 2, 3$), i.e. $F^i = (F_{i1}, F_{i2}, F_{i3})$. Note that $\operatorname{curl}_1 F = \operatorname{curl} F$. We define

$$\operatorname{curl}_0 F := \begin{bmatrix} \operatorname{curl}_0 F^1 \\ \operatorname{curl}_0 F^2 \\ \operatorname{curl}_0 F^3 \end{bmatrix}, \quad \operatorname{curl}_0 F^i := \left[-\frac{\partial F_{i2}}{\partial x_3}, \frac{\partial F_{i1}}{\partial x_3}, \frac{\partial F_{i2}}{\partial x_1} - \frac{\partial F_{i1}}{\partial x_2}\right].$$

By the expression $\operatorname{curl}_\varepsilon F = 0$ in Ω (distributionally) we mean

$$\begin{aligned}
\left(\frac{\partial F_{i3}}{\partial x_2} - \frac{1}{\varepsilon}\frac{\partial F_{i2}}{\partial x_3}\right)(\varphi) &:= -\int_\Omega \left(F_{i3}\frac{\partial \varphi}{\partial x_2} - F_{i2}\frac{1}{\varepsilon}\frac{\partial \varphi}{\partial x_3}\right) dx = 0,\\
\left(\frac{1}{\varepsilon}\frac{\partial F_{i1}}{\partial x_3} - \frac{\partial F_{i3}}{\partial x_1}\right)(\varphi) &:= -\int_\Omega \left(F_{i1}\frac{1}{\varepsilon}\frac{\partial \varphi}{\partial x_3} - F_{i3}\frac{\partial \varphi}{\partial x_1}\right) dx = 0,\\
\left(\frac{\partial F_{i2}}{\partial x_1} - \frac{\partial F_{i1}}{\partial x_2}\right)(\varphi) &:= -\int_\Omega \left(F_{i2}\frac{\partial \varphi}{\partial x_1} - F_{i1}\frac{\partial \varphi}{\partial x_2}\right) dx = 0
\end{aligned} \tag{5}$$

for all $\varphi \in C_0^\infty(\Omega)$ $(i = 1, 2, 3)$. Analogously by $\operatorname{curl}_0 F = 0$ in Ω (distributionally) we mean

$$\begin{aligned} \Big(-\frac{\partial F_{i2}}{\partial x_3}\Big)(\varphi) &:= -\int_\Omega -F_{i2}\frac{\partial \varphi}{\partial x_3}\,dx = 0,\\ \Big(\frac{\partial F_{i1}}{\partial x_3}\Big)(\varphi) &:= -\int_\Omega F_{i1}\frac{\partial \varphi}{\partial x_3}\,dx = 0,\\ \Big(\frac{\partial F_{i2}}{\partial x_1} - \frac{\partial F_{i1}}{\partial x_2}\Big)(\varphi) &:= -\int_\Omega \Big(F_{i2}\frac{\partial \varphi}{\partial x_1} - F_{i1}\frac{\partial \varphi}{\partial x_2}\Big)\,dx = 0 \end{aligned} \tag{6}$$

for all $\varphi \in C_0^\infty(\Omega)$ $(i = 1, 2, 3)$.

A function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is said to be *curl-quasiconvex* (cf. [22, 9, 21]) if

$$W(F) \le \int_Q W(F + H(y))\,dy$$

for all $F \in \mathbb{R}^{3\times 3}$ and all $H \in W^{1,M}_{\mathrm{per}}(Q; \mathbb{R}^{3\times 3})$ such that $\int_Q H(y)\,dy = 0$ and $\operatorname{curl} H = 0$ in Q, with $Q = Q^3 = (-\frac{1}{2}, \frac{1}{2})^3$. Let $\mathcal{Q}_{\mathrm{curl}} W$ denote *the curl-quasiconvex envelope* of W

$$\begin{aligned} \mathcal{Q}_{\mathrm{curl}} W(F) := \inf\Big\{\int_Q W(F + H(y))\,dy : H \in W^{1,M}_{\mathrm{per}}(Q; \mathbb{R}^{3\times 3}),\\ \operatorname{curl} H = 0 \text{ in } Q, \int_Q H(y)\,dy = 0\Big\} \end{aligned} \tag{7}$$

for $F \in \mathbb{R}^{3\times 3}$. By $\mathcal{Q}^\infty_{\mathrm{curl}_0} W$ we denote *the asymptotic* curl_0*-quasiconvex envelope* of W

$$\mathcal{Q}^\infty_{\mathrm{curl}_0} W(F) := \lim_{\eta\to\infty} \mathcal{Q}^\eta_{\mathrm{curl}_0} W(F) = \sup_{\eta>0} \mathcal{Q}^\eta_{\mathrm{curl}_0} W(F), \tag{8}$$

where

$$\begin{aligned} \mathcal{Q}^\eta_{\mathrm{curl}_0} W(F) := \inf\Big\{\int_Q W(F + H(y))\,dy : H \in W^{1,M}_{\mathrm{per}}(Q; \mathbb{R}^{3\times 3}),\\ \|\operatorname{curl}_0 H\|_{W^{-1,1}(Q;\mathbb{R}^{3\times 3})} \le \frac{1}{\eta}, \int_Q H(y)\,dy = 0\Big\} \end{aligned}$$

with $W^{-1,1}(Q; \mathbb{R}^{3\times 3})$ denoting the dual of $W^{1,\infty}_0(Q; \mathbb{R}^{3\times 3})$. Here $W^{1,\infty}_0(Q; \mathbb{R}^m)$ denotes the closure of $C_0^\infty(Q; \mathbb{R}^m)$ in $W^{1,\infty}(Q; \mathbb{R}^m)$-norm.

THEOREM 4.2. *Let $\tilde{J}_\varepsilon$ be defined in* (3), *$\mathcal{Q}_{\mathrm{curl}} W$ in* (7) *and $\mathcal{Q}^\infty_{\mathrm{curl}_0} W$ in* (8). *Assume $M, M^* \in \Delta_2^{\mathrm{glob}}$. Assume that the continuous function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the hypothesis* (2). *Let $U_\varepsilon \in W^{1,M}(\Omega_\varepsilon; \mathbb{R}^3)$. For each $\varepsilon > 0$ and $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ we associate $x = (x_\alpha, x_3) := \big(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\big) \in \Omega$ and we set $u_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3)$. Then:*

(i) (*lower bound*) *if $(D_\alpha u_\varepsilon \,|\, \frac{1}{\varepsilon} D_3 u_\varepsilon) \rightharpoonup H$ weakly in $L^M(\Omega; \mathbb{R}^{3\times 3})$ for $u_\varepsilon \in W^{1,M}(\Omega; \mathbb{R}^3)$ and $H \in L^M(\Omega; \mathbb{R}^{3\times 3})$, then there exist $\bar{u} \in W^{1,M}(\omega; \mathbb{R}^3)$ and $b \in L^M(\Omega; \mathbb{R}^3)$ such that by the 3D-2D dimension reduction isomorphism* (4) *$H = (D_\alpha \bar{u} \,|\, b)$ and*

$$\liminf_{\varepsilon\to 0} \tilde{J}_\varepsilon(U_\varepsilon) \ge \int_\Omega \mathcal{Q}^\infty_{\mathrm{curl}_0} W(D_\alpha \bar{u} \,|\, b)\,dx;$$

(ii) (*upper bound*) *for every* $H = (D_\alpha \bar{u}\,|\,b)$ *with* $\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3)$ *and* $b \in L^M(\Omega;\mathbb{R}^3)$, *there exists a sequence* $u_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ *such that* $(D_\alpha u_\varepsilon\,|\,\frac{1}{\varepsilon}D_3 u_\varepsilon) \rightharpoonup H$ *weakly in* $L^M(\Omega;\mathbb{R}^{3\times 3})$ *and*

$$\limsup_{\varepsilon\to 0} \tilde{J}_\varepsilon(U_\varepsilon) \le \int_\Omega \mathcal{Q}_{\mathrm{curl}} W(D_\alpha \bar{u}\,|\,b)\,dx.$$

5. The equivalent formulation of main results. Notice that after the change of variables as in previous theorems with the association

$$x = (x_\alpha, x_3) := \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\,\tilde{x}_3\right),\ u_\varepsilon(x_\alpha, x_3) := U_\varepsilon(\tilde{x}_\alpha, \tilde{x}_3),$$

the re-scaled energy $\widetilde{J}_\varepsilon(U)$ in (3) can be rewritten in the equivalent form

$$J_\varepsilon(u) := \int_\Omega W\left(D_\alpha u\,|\,\frac{1}{\varepsilon}\,D_3 u\right)dx.$$

In order to individualize this new sequence $\frac{1}{\varepsilon}D_3 u$ and since the direct consideration of J_ε would imply the study involving the weak topology of the Orlicz–Sobolev space $W^{1,M}(\Omega;\mathbb{R}^3)$ which is non-metrizable on unbounded sets, it is needed to consider the new functional $\bar{J}_\varepsilon : W^{1,M}(\Omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\bar{J}_\varepsilon(u,\bar{b}) := \begin{cases} \int_\Omega W(D_\alpha u\,|\,\frac{1}{\varepsilon}D_3 u)\,dx & \text{if } \frac{1}{\varepsilon}D_3 u = \bar{b} \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

Theorem 4.2 is equivalent to Theorem 5.1.

THEOREM 5.1. *Let* $\bar{J}_\varepsilon$ *be defined in* (9), $\mathcal{Q}_{\mathrm{curl}} W$ *in* (7) *and* $\mathcal{Q}^\infty_{\mathrm{curl}_0} W$ *in* (8). *Assume* $M, M^* \in \Delta_2^{\mathrm{glob}}$. *Assume that the continuous function* $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ *satisfies hypothesis* (2). *Then*:

(i) (*lower bound*) *if* $(D_\alpha u_\varepsilon\,|\,\frac{1}{\varepsilon}D_3 u_\varepsilon) \rightharpoonup H$ *weakly in* $L^M(\Omega;\mathbb{R}^{3\times 3})$ *for* $u_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ *and* $H \in L^M(\Omega;\mathbb{R}^{3\times 3})$, *then there exist* $\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3)$ *and* $b \in L^M(\Omega;\mathbb{R}^3)$ *such that by the* 3*D*-2*D dimension reduction isomorphism* (4) $H = (D_\alpha \bar{u}\,|\,b)$ *and*

$$\liminf_{\varepsilon\to 0} \bar{J}_\varepsilon(u_\varepsilon, \tfrac{1}{\varepsilon}D_3 u_\varepsilon) \ge \int_\Omega \mathcal{Q}^\infty_{\mathrm{curl}_0} W(D_\alpha \bar{u}\,|\,b)\,dx;$$

(ii) (*upper bound*) *for every* $H = (D_\alpha \bar{u}\,|\,b)$ *with* $\bar{u} \in W^{1,M}(\omega;\mathbb{R}^3)$ *and* $b \in L^M(\Omega;\mathbb{R}^3)$, *there exists a sequence* $u_\varepsilon \in W^{1,M}(\Omega;\mathbb{R}^3)$ *such that* $(D_\alpha u_\varepsilon\,|\,\frac{1}{\varepsilon}D_3 u_\varepsilon) \rightharpoonup H$ *weakly in* $L^M(\Omega;\mathbb{R}^{3\times 3})$ *and*

$$\limsup_{\varepsilon\to 0} \bar{J}_\varepsilon(u_\varepsilon, \tfrac{1}{\varepsilon}D_3 u_\varepsilon) \le \int_\Omega \mathcal{Q}_{\mathrm{curl}} W(D_\alpha \bar{u}\,|\,b)\,dx.$$

Define $I_0 : W^{1,M}(\omega;\mathbb{R}^3) \times L^M(\Omega;\mathbb{R}^3) \to \mathbb{R}$ by

$$I_0(\bar{u}, b) := \int_0^1 \int_\omega W(D_\alpha \bar{u}\,|\,b)\,dx_\alpha\,dx_3.$$

By [37, Lemma 6.3] Theorem 5.1 implies:

COROLLARY 5.2. *If* W *is asymptotically* curl_0*-quasiconvex, i.e.* $W = \mathcal{Q}^\infty_{\mathrm{curl}_0} W$ (*in particular, if* W *is cross-quasiconvex*), *then* $\bar{J}_\varepsilon$ *converges to the functional* I_0 *in the above sense of* Γ*-convergence with respect to the weak topology in* $L^M(\Omega;\mathbb{R}^{3\times 3})$.

We consider the equivalent to the functional $\bar{J}_\varepsilon$ re-scaled integral functional that is

$$F_\varepsilon(H) := \begin{cases} \int_\Omega W(H(x))\,dx & \text{if } \operatorname{curl}_\varepsilon H = 0 \text{ in } \Omega \text{ (distributionally)} \\ +\infty & \text{otherwise,} \end{cases} \tag{10}$$

where $H \in L^M(\Omega;\mathbb{R}^{3\times 3})$.

Define

$$\mathcal{U}_\varepsilon := \{H \in L^M(\Omega;\mathbb{R}^{3\times 3}) : \operatorname{curl}_\varepsilon H = 0 \text{ (distributionally)}\}$$

and

$$\mathcal{U}_0 := \{H \in L^M(\Omega;\mathbb{R}^{3\times 3}) : \operatorname{curl}_0 H = 0 \text{ (distributionally)}\}.$$

FACT 5.3. *If $H = (D_\alpha u \,|\, \frac{1}{\varepsilon} D_3 u)$ for $u \in W^{1,M}(\Omega;\mathbb{R}^3)$, then $H \in \mathcal{U}_\varepsilon$.*

FACT 5.4. *If $H \in \mathcal{U}_\varepsilon$, then there exists $u \in W^{1,M}(\Omega;\mathbb{R}^3)$ such that $H = (D_\alpha u \,|\, \frac{1}{\varepsilon} D_3 u)$.*

The proof of Fact 5.4 is analogous to the proof of [62, Regularity Theorem, p. 3], [28, Theorem 10.5.1, Sections 10.4–10.6], [51, Theorem 3.8].

Theorem 5.1 is equivalent to Theorem 5.5.

THEOREM 5.5. *Let F_ε be defined in (10), $\mathcal{Q}_{\mathrm{curl}}W$ in (7) and $\mathcal{Q}^\infty_{\mathrm{curl}_0}W$ in (8). Assume $M, M^* \in \Delta_2^{\mathrm{glob}}$. Assume that the continuous function $W\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies hypothesis (2). Then:*

(i) (*lower bound*) *if $H_\varepsilon \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$ for $H_\varepsilon \in \mathcal{U}_\varepsilon$ and $H \in L^M(\Omega;\mathbb{R}^{3\times 3})$, then $H \in \mathcal{U}_0$ and*
$$\liminf_{\varepsilon\to 0} F_\varepsilon(H_\varepsilon) \ge \int_\Omega \mathcal{Q}^\infty_{\mathrm{curl}_0} W(H)\,dx;$$

(ii) (*upper bound*) *for every $H \in \mathcal{U}_0$, there exists a sequence $H_\varepsilon \in \mathcal{U}_\varepsilon$ such that $H_\varepsilon \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$ and*
$$\limsup_{\varepsilon\to 0} F_\varepsilon(H_\varepsilon) \le \int_\Omega \mathcal{Q}_{\mathrm{curl}} W(H)\,dx.$$

Define

$$F_0(H) := \begin{cases} \int_\Omega W(H(x))\,dx & \text{if } \operatorname{curl}_0 H = 0 \text{ in } \Omega \text{ (distributionally)} \\ +\infty & \text{otherwise,} \end{cases}$$

where $H \in L^M(\Omega;\mathbb{R}^{3\times 3})$.

COROLLARY 5.6. *If W is asymptotically curl_0-quasiconvex, i.e. $W = \mathcal{Q}^\infty_{\mathrm{curl}_0}W$ (in particular, if W is cross-quasiconvex), then F_ε converges to the functional F_0 in the sense of Γ-convergence with respect to the weak topology in $L^M(\Omega;\mathbb{R}^{3\times 3})$.*

6. The proof of Theorem 5.5

LEMMA 6.1. *Let $\mathcal{Q}_{\mathrm{curl}}W$ be the curl-quasiconvex envelope of W. Then $-\mathcal{Q}_{\mathrm{curl}}W$ is a normal integrand.*

LEMMA 6.2. *For all $\varepsilon > 0$ and $\delta > 0$*

$$\mathcal{Q}_{\mathrm{curl}_\varepsilon} W = \mathcal{Q}_{\mathrm{curl}_\delta} W. \tag{11}$$

The proofs of Lemmas 6.1 and 6.2 are the same as in [37, Lemmas 2.14 and 2.12].

Let $f \in L^M(Q)$ be a Q-periodic function, where $Q = [-\frac{1}{2}; \frac{1}{2}]^3$. The triple Fourier series of a complex-valued function f is defined as

$$S[f](x_1, x_2, x_3) = \sum_{m,n,k=-\infty}^{\infty} c_{mnk}\, \mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}, \tag{12}$$

$$\text{where} \quad c_{mnk} = \iiint_Q f(x_1, x_2, x_3)\, \mathrm{e}^{-2\pi i(mx_1+nx_2+kx_3)}\, dx_1\, dx_2\, dx_3.$$

For $N \in \mathbb{N}$, let $S_N[f](x_1, x_2, x_3)$ denote the partial sums of (12), i.e.

$$S_N[f](x_1, x_2, x_3) = \sum_{|m|,|n|,|k|=0}^{N} c_{mnk}\, \mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}.$$

LEMMA 6.3. *Assume $M, M^* \in \Delta_2^{\mathrm{glob}}$ and let $f \in L^M(\mathbb{T}^3) = L^M_{\mathrm{per}}(Q)$. Then $S_N[f] \to [f]$ in $L^M(\mathbb{T}^3)$-norm as $N \to \infty$.*

The proof is analogous to its special case for $f \in L^M(\mathbb{T})$ (see, e.g., [60, Corollary 9, p. 197]). It suffices to use also corresponding auxiliary results for functions over $\mathbb{T}^3$ in [25, Proof of Theorem 3.5.7] and [25, Proofs of Theorem 3.5.1 and Corollary 3.5.2].

Define

$$\begin{aligned}
I_N(-\tfrac{1}{2}, x_3)[f] :=& \sum_{|m|,|n|,|k|=1}^{N} c_{mnk} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2+kt)}\, dt \\
&+ \sum_{|m|,|n|=1}^{N} c_{mn0} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2)}\, dt + \sum_{|m|,|k|=1}^{N} c_{m0k} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+kt)}\, dt \\
&+ \sum_{|n|,|k|=1}^{N} c_{0nk} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(nx_2+kt)}\, dt + \sum_{|m|=1}^{N} c_{m00} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1)}\, dt \\
&+ \sum_{|n|=1}^{N} c_{0n0} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(nx_2)}\, dt + \sum_{|k|=1}^{N} c_{00k} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(kt)}\, dt + c_{000} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(0)}\, dt \\
:=& I_{N(1,1,1)}(-\tfrac{1}{2}, x_3)[f] + I_{N(1,1,0)}(-\tfrac{1}{2}, x_3)[f] + I_{N(1,0,1)}(-\tfrac{1}{2}, x_3)[f] \\
&+ I_{N(0,1,1)}(-\tfrac{1}{2}, x_3)[f] + I_{N(1,0,0)}(-\tfrac{1}{2}, x_3)[f] + I_{N(0,1,0)}(-\tfrac{1}{2}, x_3)[f] \\
&+ I_{N(0,0,1)}(-\tfrac{1}{2}, x_3)[f] + I_{N(0,0,0)}(-\tfrac{1}{2}, x_3)[f].
\end{aligned}$$

LEMMA 6.4. *Let $f \in C^5(Q)$ be a Q-periodic function. Then there exists $I_\infty(-\frac{1}{2}, x_3)[f] \in L^M(Q)$ such that $I_N(-\frac{1}{2}, x_3)[f] \to I_\infty(-\frac{1}{2}, x_3)[f]$ in $L^M(Q)$-norm and almost everywhere on Q. Moreover*

$$\frac{\partial}{\partial x_3}\big(I_\infty(-\tfrac{1}{2}, x_3)[f]\big) = [f] \tag{13}$$

$$\frac{\partial}{\partial x_i}\big(I_\infty(-\tfrac{1}{2}, x_3)[f]\big) = \left[I_\infty(-\tfrac{1}{2}, x_3)\left[\frac{\partial f}{\partial x_i}\right]\right] \qquad (i = 1, 2) \tag{14}$$

in the sense of Sobolev–Schwartz distributions of $\mathcal{D}'(Q)$, where $[f]$ and $\big[I_\infty(-\frac{1}{2}, x_3)[\frac{\partial f}{\partial x_i}]\big]$ denote the equivalent class of f and $I_\infty(-\frac{1}{2}, x_3)[\frac{\partial f}{\partial x_i}]$ $(i = 1, 2)$, respectively.

Proof. We divide the proof into three steps.

Step 1. By [25, Theorem 3.2.9], for c_{mnk} for the Q-periodic function $f \in C^4(Q)$ and for $I_{N(1,1,1)}(-\frac{1}{2}, x_3)[f]$

$$\Big|c_{mnk} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2+kt)}\, dt\Big| \le \Big|c_{mnk} \int_{-1/2}^{x_3} 1\, dt\Big| \le \frac{C}{(\sqrt{m^2+n^2+k^2})^4} \le \frac{C}{m^{4/3}\cdot n^{4/3}\cdot k^{4/3}}.$$

Therefore

$$\begin{aligned}\sum_{|m|,|n|,|k|=0}^{N} &\Big\|c_{mnk} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2+kt)}\, dt\Big\|_{L^M(Q)} \\ &\le |c_{000}| + \sum_{|m|,|n|,|k|=1}^{N} \Big\|\frac{C}{m^{4/3}\cdot n^{4/3}\cdot k^{4/3}} \cdot \chi_Q(x)\Big\|_{L^M(Q)} \\ &\le |c_{000}| + \frac{1}{M^{-1}(1/\mathrm{meas}(Q))} \sum_{m,n,k=-\infty}^{\infty} \frac{C}{m^{4/3}\cdot n^{4/3}\cdot k^{4/3}} < \infty.\end{aligned}$$

Using the same arguments as above we obtain a similar estimation for the rest of components $I_{N(1,1,0)}(-\frac{1}{2}, x_3)[f], I_{N(1,0,1)}(-\frac{1}{2}, x_3)[f], \dots, I_{N(0,0,0)}(-\frac{1}{2}, x_3)[f]$.

By the Riesz–Fischer L^M-theorem (see [46], cf. [63, Theorem 3.2.1], [53, Proposition 4.A]) we deduce that the sequence $I_N(-\frac{1}{2}, x_3)[f]$ is convergent in L^M-norm and almost everywhere to some function $I_\infty(-\frac{1}{2}, x_3)[f] \in L^M(Q)$.

Step 2. From the result of Step 1 we deduce that for $\varphi \in C_0^\infty(Q) \subset L^{M^*}(Q)$

$$\begin{aligned}\frac{\partial}{\partial x_3}&\Big(I_\infty\Big(-\frac{1}{2}, x_3\Big)[f]\Big)(\varphi) := -\Big\langle I_\infty\Big(-\frac{1}{2}, x_3\Big)[f], \frac{\partial \varphi}{\partial x_3}\Big\rangle \\ &= -\lim_{N\to\infty} \Big\langle I_N\Big(-\frac{1}{2}, x_3\Big)[f], \frac{\partial \varphi}{\partial x_3}\Big\rangle \\ &= -\lim_{N\to\infty} \Big\langle \sum_{|m|,|n|,|k|=1}^{N} c_{mnk} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2+kt)}\, dt \\ &\quad + \sum_{|m|,|n|=1}^{N} c_{mn0} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2)}\, dt + \ldots + c_{000} \int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(0)}\, dt, \frac{\partial \varphi}{\partial x_3}\Big\rangle \\ &= -\lim_{N\to\infty} \Big\langle \sum_{|m|,|n|,|k|=1}^{N} \frac{c_{mnk}}{2\pi i k}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)} - \mathrm{e}^{2\pi i(mx_1+nx_2-k/2)}\big) \\ &\quad + \sum_{|m|,|n|=1}^{N} c_{mn0}\, \mathrm{e}^{2\pi i(mx_1+nx_2)}\Big(x_3 + \frac{1}{2}\Big) + \ldots + c_{000}\Big(x_3 + \frac{1}{2}\Big), \frac{\partial \varphi}{\partial x_3}\Big\rangle.\end{aligned}$$

Using integration by parts we have

$$\Big\langle \frac{c_{mnk}}{2\pi ik}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-k/2)}\big)$$
$$+\,c_{mn0}\,\mathrm{e}^{2\pi i(mx_1+nx_2)}\Big(x_3+\frac{1}{2}\Big)+\ldots+c_{000}\Big(x_3+\frac{1}{2}\Big),\frac{\partial\varphi}{\partial x_3}\Big\rangle$$
$$=\iint\limits_{[-1/2;1/2]^2}\Big(\int_{-1/2}^{1/2}\frac{c_{mnk}}{2\pi ik}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-k/2)}\big)\frac{\partial\varphi}{\partial x_3}\,dx_3$$
$$+\,c_{mn0}\int_{-1/2}^{1/2}\mathrm{e}^{2\pi i(mx_1+nx_2)}\Big(x_3+\frac{1}{2}\Big)\frac{\partial\varphi}{\partial x_3}\,dx_3$$
$$+\ldots+c_{000}\int_{-1/2}^{1/2}\Big(x_3+\frac{1}{2}\Big)\frac{\partial\varphi}{\partial x_3}\,dx_3\Big)\,dx_1\,dx_2$$
$$=\frac{c_{mnk}}{2\pi ik}\iint\limits_{[-1/2;1/2]^2}\mathrm{e}^{2\pi i(mx_1+nx_2)}\Big(\mathrm{e}^{2\pi ikx_3}\varphi\Big|_{-1/2}^{1/2}-2\pi ik\int_{-1/2}^{1/2}\mathrm{e}^{2\pi ikx_3}\varphi\,dx_3\Big)\,dx_1\,dx_2$$
$$+\,c_{mn0}\iint\limits_{[-1/2;1/2]^2}\Big(\mathrm{e}^{2\pi i(mx_1+nx_2)}\Big(x_3+\frac{1}{2}\Big)\varphi\Big|_{-1/2}^{1/2}-\int_{-1/2}^{1/2}\mathrm{e}^{2\pi i(mx_1+nx_2)}\varphi\,dx_3\Big)\,dx_1\,dx_2$$
$$+\ldots+c_{000}\iint\limits_{[-1/2;1/2]^2}\Big(\Big(x_3+\frac{1}{2}\Big)\varphi\Big|_{-1/2}^{1/2}-\int_{-1/2}^{1/2}\varphi\,dx_3\Big)\,dx_1\,dx_2$$
$$=-c_{mnk}\iiint\limits_{Q}\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}\varphi\,dx_1\,dx_2\,dx_3-c_{mn0}\iiint\limits_{Q}\mathrm{e}^{2\pi i(mx_1+nx_2)}\varphi\,dx_1\,dx_2\,dx_3$$
$$-\ldots-c_{000}\iiint\limits_{Q}\varphi\,dx_1\,dx_2\,dx_3.$$

Hence

$$\frac{\partial}{\partial x_3}\Big(I_\infty\Big(-\frac{1}{2},x_3\Big)[f]\Big)(\varphi)=\lim_{N\to\infty}\langle S_N[f],\varphi\rangle.$$

By Lemma 6.3 for $f\in C^4_{\mathrm{per}}(Q)\subset L^M_{\mathrm{per}}(Q)=L^M(\mathbb{T}^3)$ we see that $S_N[f]\to[f]$ in $L^M(\mathbb{T}^3)$-norm. Then $S_N[f]\rightharpoonup[f]$ weakly in $L^M(Q)$. Since $C_0^\infty(Q)\subset L^{M^*}(Q)$, then

$$\frac{\partial}{\partial x_3}\Big(I_\infty\Big(-\frac{1}{2},x_3\Big)[f]\Big)(\varphi)=\langle[f],\varphi\rangle.$$

So $\frac{\partial}{\partial x_3}(I_\infty(-\frac{1}{2},x_3)[f])=[f]$ (regular distribution generated by $[f]$), that means, (13) is satisfied.

Step 3. Now, we want to prove (14). Here we restrict ourselves to the case $i=1$. The proof is analogous to that of (13) in Step 2. It is enough to observe that for $\varphi\in C_0^\infty(Q)$

$$\frac{\partial}{\partial x_1}\Big(I_\infty\Big(-\frac{1}{2},x_3\Big)[f]\Big)(\varphi):=-\Big\langle I_\infty\Big(-\frac{1}{2},x_3\Big)[f],\frac{\partial\varphi}{\partial x_1}\Big\rangle$$

$$= -\lim_{N\to\infty}\left\langle I_N\left(-\frac{1}{2},x_3\right)[f],\frac{\partial\varphi}{\partial x_1}\right\rangle = -\lim_{N\to\infty}\left\langle \sum_{|m|,|n|,|k|=1}^{N} c_{mnk}\int_{-1/2}^{x_3} \mathrm{e}^{2\pi i(mx_1+nx_2+kt)}\,dt\right.$$

$$\left. + \sum_{|m|,|n|=1}^{N} c_{mn0}\int_{-1/2}^{x_3}\mathrm{e}^{2\pi i(mx_1+nx_2)}\,dt + \ldots + c_{000}\int_{-1/2}^{x_3}\mathrm{e}^{2\pi i(0)}\,dt,\frac{\partial\varphi}{\partial x_1}\right\rangle$$

$$= -\lim_{N\to\infty}\left\langle \sum_{|m|,|n|,|k|=1}^{N}\frac{c_{mnk}}{2\pi ik}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-k/2)}\big)\right.$$

$$\left. + \sum_{|m|,|n|=1}^{N} c_{mn0}\,\mathrm{e}^{2\pi i(mx_1+nx_2)}\left(x_3+\frac{1}{2}\right)+\ldots+c_{000}\left(x_3+\frac{1}{2}\right),\frac{\partial\varphi}{\partial x_1}\right\rangle.$$

Using integration by parts we obtain

$$\left\langle\frac{c_{mnk}}{2\pi ik}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-k/2)}\big)\right.$$

$$\left. + c_{mn0}\,\mathrm{e}^{2\pi i(mx_1+nx_2)}\left(x_3+\frac{1}{2}\right)+\ldots+c_{000}\left(x_3+\frac{1}{2}\right),\frac{\partial\varphi}{\partial x_1}\right\rangle$$

$$= \iint\limits_{[-1/2;1/2]^2}\left(\int_{-1/2}^{1/2}\frac{c_{mnk}}{2\pi ik}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-k/2)}\big)\frac{\partial\varphi}{\partial x_1}\,dx_1\right.$$

$$+ c_{mn0}\int_{-1/2}^{1/2}\mathrm{e}^{2\pi i(mx_1+nx_2)}\left(x_3+\frac{1}{2}\right)\frac{\partial\varphi}{\partial x_1}\,dx_1$$

$$\left. +\ldots+c_{000}\int_{-1/2}^{1/2}\left(x_3+\frac{1}{2}\right)\frac{\partial\varphi}{\partial x_1}\,dx_1\right)dx_2\,dx_3$$

$$= \frac{c_{mnk}}{2\pi ik}\iint\limits_{[-1/2;1/2]^2}\left(\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-\frac{1}{2}k)}\big)\varphi\Big|_{-1/2}^{1/2}\right.$$

$$\left. - 2\pi im\int_{-1/2}^{1/2}\big(\mathrm{e}^{2\pi i(mx_1+nx_2+kx_3)}-\mathrm{e}^{2\pi i(mx_1+nx_2-\frac{1}{2}k)}\big)\varphi\,dx_1\right)dx_2\,dx_3$$

$$+ c_{mn0}\iint\limits_{[-1/2;1/2]^2}\left(\mathrm{e}^{2\pi i(mx_1+nx_2)}\left(x_3+\frac{1}{2}\right)\varphi\Big|_{-1/2}^{1/2}\right.$$

$$\left. - 2\pi im\int_{-1/2}^{1/2}\mathrm{e}^{2\pi i(mx_1+nx_2)}\left(x_3+\frac{1}{2}\right)\varphi\,dx_1\right)dx_2\,dx_3$$

$$+ c_{000}\iint\limits_{[-1/2;1/2]^2}\left(\left(x_3+\frac{1}{2}\right)\varphi\Big|_{-1/2}^{1/2}-0\right)dx_2\,dx_3$$

$$= -c_{mnk}\iiint\limits_{Q}2\pi im\int_{-1/2}^{x_3}\mathrm{e}^{2\pi i(mx_1+nx_2+kt)}\,dt\,\varphi\,dx_1\,dx_2\,dx_3$$

$$- c_{mn0}\iiint\limits_{Q}2\pi im\int_{-1/2}^{x_3}\mathrm{e}^{2\pi i(mx_1+nx_2)}\,dt\,\varphi\,dx_1\,dx_2\,dx_3+\ldots+c_{000}\cdot 0.$$

Hence
$$\frac{\partial}{\partial x_1}\Big(I_\infty\Big(-\frac{1}{2},x_3\Big)[f]\Big)(\varphi)=\lim_{N\to\infty}\Big\langle I_N\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial f}{\partial x_1}\Big],\varphi\Big\rangle.$$
From the result of Step 1 (by using the Riesz–Fischer L^M-theorem) we see that $I_N(-\frac{1}{2},x_3)[\frac{\partial f}{\partial x_1}]\to I_\infty(-\frac{1}{2},x_3)[\frac{\partial f}{\partial x_1}]$ in $L^M(Q)$-norm for $\frac{\partial f}{\partial x_1}\in C^4_{\mathrm{per}}(Q)$. Therefore $I_N(-\frac{1}{2},x_3)[\frac{\partial f}{\partial x_1}]\rightharpoonup I_\infty(-\frac{1}{2},x_3)[\frac{\partial f}{\partial x_1}]$ weakly in $L^M(Q)$. Thus
$$\frac{\partial}{\partial x_1}\Big(I_\infty\Big(-\frac{1}{2},x_3\Big)[f]\Big)(\varphi)=\Big\langle I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial f}{\partial x_1}\Big],\varphi\Big\rangle,$$
since $C_0^\infty(Q)\subset L^{M^*}(Q)$. So $\frac{\partial}{\partial x_1}(I_\infty(-\frac{1}{2},x_3)[f])=\big[I_\infty(-\frac{1}{2},x_3)\big[\frac{\partial f}{\partial x_1}\big]\big]$ (regular distribution generated by $\big[I_\infty(-\frac{1}{2},x_3)\big[\frac{\partial f}{\partial x_1}\big]\big]$), i.e. (14) is proved for $i=1$. ∎

Proposition 6.5. *For $H\in\mathcal{U}_0$, there exists a sequence $H_n\in\mathcal{U}_{\varepsilon_n}$ such that $H_n\to H$ in $L^M(\Omega;\mathbb{R}^{3\times 3})$-norm.*

Proof. We divide the proof into two steps.

Step 1. Let $H\in\mathcal{U}_0$. Assume in addition that the function $H\in C^6(\Omega;\mathbb{R}^{3\times3})$ is Q-periodic. Set $H=[H^1,H^2,H^3]^{\mathrm{T}}$, where H^i denotes the i-th row of H ($i=1,2,3$), i.e. $H^i=(H_{i1},H_{i2},H_{i3})$. We can define
$$H^i_\varepsilon=\Big(H_{i1}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial H_{i3}}{\partial x_1}\Big],H_{i2}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial H_{i3}}{\partial x_2}\Big],H_{i3}\Big),\tag{15}$$
where H^i_ε denotes the i-th row of H_ε ($i=1,2,3$) with $H_\varepsilon=[H^1_\varepsilon,H^2_\varepsilon,H^3_\varepsilon]^{\mathrm{T}}$. By Lemma 6.4, $H^i_\varepsilon\in L^M(\Omega;\mathbb{R}^3)$ and
$$\|H^i_\varepsilon-H^i\|_{L^M(\Omega;\mathbb{R}^3)}=\varepsilon\Big\|\Big(I_\infty(-\tfrac{1}{2},x_3)\Big[\frac{\partial H_{i3}}{\partial x_1}\Big],I_\infty(-\tfrac{1}{2},x_3)\Big[\frac{\partial H_{i3}}{\partial x_2}\Big],0\Big)\Big\|_{L^M(\Omega;\mathbb{R}^3)}\longrightarrow 0,$$
as $\varepsilon\to0$ ($i=1,2,3$) and so $H_\varepsilon\to H$ in $L^M(\Omega;\mathbb{R}^{3\times3})$. By Lemma 6.4 we calculate $\operatorname{curl}_\varepsilon H^i_\varepsilon$ for $i=1,2,3$ in $\mathcal{D}'(Q)$ sense (see (5)), namely for every $\varphi\in C_0^\infty(\Omega)$
$$\begin{aligned}&\Big(\frac{\partial H_{i3}}{\partial x_2}-\frac{1}{\varepsilon}\frac{\partial}{\partial x_3}\Big(H_{i2}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial H_{i3}}{\partial x_2}\Big]\Big)\Big)(\varphi)\\&\qquad=\int_\Omega\Big(\frac{\partial H_{i3}}{\partial x_2}-\frac{1}{\varepsilon}\frac{\partial H_{i2}}{\partial x_3}-\Big[\frac{\partial H_{i3}}{\partial x_2}\Big]\Big)\cdot\varphi\,dx=\int_\Omega\Big(-\frac{1}{\varepsilon}\frac{\partial H_{i2}}{\partial x_3}\Big)\cdot\varphi\,dx;\\&\Big(\frac{1}{\varepsilon}\frac{\partial}{\partial x_3}\Big(H_{i1}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial H_{i3}}{\partial x_1}\Big]\Big)-\frac{\partial H_{i3}}{\partial x_1}\Big)(\varphi)\\&\qquad=\int_\Omega\Big(\frac{1}{\varepsilon}\frac{\partial H_{i1}}{\partial x_3}+\Big[\frac{\partial H_{i3}}{\partial x_1}\Big]-\frac{\partial H_{i3}}{\partial x_1}\Big)\cdot\varphi\,dx=\int_\Omega\Big(\frac{1}{\varepsilon}\frac{\partial H_{i1}}{\partial x_3}\Big)\cdot\varphi\,dx;\end{aligned}$$
$$\begin{aligned}&\Big(\frac{\partial}{\partial x_1}\Big(H_{i2}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial H_{i3}}{\partial x_2}\Big]\Big)-\frac{\partial}{\partial x_2}\Big(H_{i1}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial H_{i3}}{\partial x_1}\Big]\Big)\Big)(\varphi)\\&=\int_\Omega\Big(\frac{\partial H_{i2}}{\partial x_1}+\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial^2 H_{i3}}{\partial x_1\partial x_2}\Big]-\frac{\partial F_{i1}}{\partial x_2}-\varepsilon\cdot I_\infty\Big(-\frac{1}{2},x_3\Big)\Big[\frac{\partial^2 H_{i3}}{\partial x_2\partial x_1}\Big]\Big)\cdot\varphi\,dx\\&=\int_\Omega\Big(\frac{\partial H_{i2}}{\partial x_1}-\frac{\partial F_{i1}}{\partial x_2}\Big)\cdot\varphi\,dx.\end{aligned}$$
Then $H_\varepsilon\in\mathcal{U}_\varepsilon$, since $H\in\mathcal{U}_0$ (see (6)).

Step 2. Assume that $H \in \mathcal{U}_0$ is a non-smooth function. By the standard mollifying kernel technique (see, e.g. [37, Step 5, p. 25]) we can define a mollified sequence $H_n \in \mathcal{U}_0$ such that $H_n \in C^\infty_{\text{per}}(Q;\mathbb{R}^{3\times 3})$. By Step 1 for each H_n we may find $H_{n,k} \in \mathcal{U}_{\varepsilon_k}$ such that $H_{n,k} \to H_n$ in $L^M(Q;\mathbb{R}^{3\times 3})$-norm as $k \to \infty$. Since $H_n \to H$ in $L^M(Q;\mathbb{R}^{3\times 3})$-norm (see [16, Theorem 2.1], [1, Theorem 8.21]), by using the Moore Lemma [17, Lemma I.7.6] (on double limits of sequence with respect to metrizable topologies) we can find a subsequence $H_{n(p),k(p)}$ of $H_{n,k}$ such that $H_{n(p),k(p)} \to H$ in $L^M(Q;\mathbb{R}^{3\times 3})$-norm as $p \to \infty$ and $H_{n(p),k(p)} \in \mathcal{U}_{\varepsilon_p}$. ■

THEOREM 6.6 (Decomposition lemma). *Assume $M, M^* \in \Delta_2^{\text{glob}}$. Let $H_n \in \mathcal{U}_{\varepsilon_n}$ and $H_n \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$. Then there exists the decomposition $H_n = W_n + R_n$ for some W_n and R_n, where $W_n \in W^{1,M}_{\text{per}}(Q;\mathbb{R}^{3\times 3})$ is such that $\operatorname{curl}_0 W_n = 0$ in Q, $M(|W_n|)$ is equi-integrable, $\int_Q W_n\,dy = 0$ for all $n \in \mathbb{N}$ and $R_n \in L^M(\Omega;\mathbb{R}^{3\times 3})$ is such that $R_n \to 0$ in $L^{M^*}(\Omega;\mathbb{R}^{3\times 3})$-norm.*

The proof of Theorem 6.6 is analogous to the proof for its $W^{1,p}$-version [22, Lemma 2.14, Lemma 2.15], [37, Theorem 3.2].

THEOREM 6.7 (The Young measure representation; see [3, 2, 10], cf. [20], [54]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain and let z_j be a sequence of measurable functions bounded in $L^1(\Omega,\mathbb{R}^m)$. Then there exist a subsequence z_{j_k} and a weak* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ such that the following statements hold:*

(i) *$\delta_{z_{j_k}}$ is weak* convergent to ν in the sense: for every $f = f(x,\lambda) \in L^1(C_0(\mathbb{R}^m))$*

$$\lim_{k\to\infty} \langle \delta_{z_{j_k}}, f\rangle = \lim_{k\to\infty} \int_\Omega f(x, z_{j_k}(x))\,dx = \int_\Omega \Big(\int_{\mathbb{R}^m} f(x,\lambda)\,d\nu_x(\lambda)\Big)\,dx = \langle \nu, f\rangle;$$

(ii) $\|\nu_x\|_{\mathcal{M}(\Omega;\mathbb{R}^m)} = 1$;

(iii) *if $\psi\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function, bounded from below, then*

$$\liminf_{k\to\infty} \int_\Omega \psi(x, z_{j_k}(x))\,dx \geq \int_\Omega \bar{\psi}(x)\,dx$$

where

$$\bar{\psi}(x) := \langle \nu_x, \psi(x,\cdot)\rangle = \int_{\mathbb{R}^m} \psi(x,\lambda)\,d\nu_x(\lambda);$$

(iv) *if $\psi\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function, bounded from below, then*

$$\lim_{k\to\infty} \int_\Omega \psi(x, z_{j_k}(x))\,dx = \int_\Omega \bar{\psi}(x)\,dx < +\infty$$

if and only if $\psi(\cdot, z_{j_k}(\cdot))$ is equi-integrable. In this case

$$\psi(\cdot, z_{j_k}(\cdot)) \rightharpoonup \bar{\psi} \quad \text{in } L^1(\Omega).$$

The map ν is called *Young measure* generated by the sequence z_j.

PROPOSITION 6.8 (localization). *Assume $M, M^* \in \Delta_2^{\text{glob}}$. Let $H_n \in \mathcal{U}_{\varepsilon_n}$ and $H_n \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$. Let ν_x be the Young measure generated by the sequence H_n. Then for almost every $a \in \Omega$, there exist a subsequence ε_{n_k} of ε_n and a sequence*

$z_k \in W^{1,M}_{\mathrm{per}}(Q;\mathbb{R}^{3\times 3})$ *such that* $\operatorname{curl}_0 z_k = 0$ *in* Q, $M(|z_k|)$ *is equi-integrable,*

$$\int_Q z_k\,dy = H(a) \qquad \textit{for } k \in \mathbb{N}$$

and z_k *generates the homogeneous Young measure* $(\nu_a)_{y\in Q}$.

The proof of Theorem 6.8 is analogous to the proof for its $W^{1,p}$-version [22, Proposition 3.8], [37, Proposition 3.5].

Proof of Theorem 5.5. The upper bound. Let $H \in \mathcal{U}_0$. By Proposition 6.5 there exists a sequence $H_n \in \mathcal{U}_{\varepsilon_n}$ such that $H_n \to H$ in $L^M(\Omega;\mathbb{R}^{3\times 3})$-norm. By Lemma 6.1, $-\mathcal{Q}_{\mathrm{curl}}W$ is a normal integrand. Since $\mathcal{Q}_{\mathrm{curl}}W(F) \le W(F)$ for $F \in \mathbb{R}^{3\times 3}$, by (2) we have

$$-\mathcal{Q}_{\mathrm{curl}}W(F) \ge -C(1+M(|F|)) \qquad (\forall F \in \mathbb{R}^{3\times 3}).$$

By the L^M-generalization [27, Theorem 10] of the L^p-l.s.c. theorem (see [4, 56], cf. [20, Theorem 6.49]) we deduce that the functional $H \mapsto \int_\Omega -\mathcal{Q}_{\mathrm{curl}}W(H)\,dx$ is sequentially lower semicontinuous with respect to strong convergence in $L^M(\Omega;\mathbb{R}^{3\times 3})$. Therefore the functional $u \mapsto \int_\Omega \mathcal{Q}_{\mathrm{curl}}W(u)\,dx$ is upper semicontinuous with respect to strong convergence in $L^M(\Omega;\mathbb{R}^{3\times 3})$, i.e.

$$\limsup_{n\to\infty} \int_\Omega \mathcal{Q}_{\mathrm{curl}}W(H_n)\,dx \le \int_\Omega \mathcal{Q}_{\mathrm{curl}}W(H)\,dx. \tag{16}$$

By the coercivity condition in (2) we deduce from (16) the existence a subsequence (not relabeled) H_n such that $\sup_n \int_\Omega M(|H_n|)\,dx < \infty$. Therefore the sequence H_n is bounded in the Orlicz space $L^M(\Omega;\mathbb{R}^{3\times 3})$.

By the L^M-generalization of the $\mathcal{A}$-free relaxation L^p-theorem (see [9, Theorem 1.1]) and by Lemma 6.2, for each $n \in \mathbb{N}$ there exists a sequence $H_{n,k} \in \mathcal{U}_{\varepsilon_n}$ satisfying $H_{n,k} \rightharpoonup H_n$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$ such that

$$\begin{aligned}\lim_{k\to\infty} F_{\varepsilon_n}(H_{n,k}) &= \lim_{k\to\infty}\int_\Omega W(H_{n,k})\,dx \\ &= \int_\Omega \mathcal{Q}_{\mathrm{curl}_{\varepsilon_n}}W(H_n)\,dx = \int_\Omega \mathcal{Q}_{\mathrm{curl}}W(H_n)\,dx. \end{aligned} \tag{17}$$

By the norm-boundedness of the sequence H_n we can choose $H_{n,k}$ such that the set $\{H_{n,k} : n,k \in \mathbb{N}\}$ is a subset of some closed ball of $L^M(\Omega;\mathbb{R}^{3\times 3})$. By the reflexivity and separability of $L^M(\Omega;\mathbb{R}^{3\times 3})$ [35, 52, 26, 1], the Alaoglu–Bourbaki theorem together with [31, Theorem V.7.6] imply that any closed ball of $L^M(\Omega;\mathbb{R}^{3\times 3})$ equipped with the weak topology is compact and metrizable. Hence by (16) and (17) we obtain

$$\limsup_{n\to\infty}\lim_{k\to\infty} F_{\varepsilon_n}(H_{n,k}) \le \int_\Omega \mathcal{Q}_{\mathrm{curl}}W(H)\,dx.$$

By using the Moore Lemma [17, Lemma I.7.6] (on double limits of sequence with respect to metrizable topologies) we can find a subsequence $H_{n(p),k(p)}$ of $H_{n,k}$ such that $H_{n(p),k(p)} \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$ and

$$\limsup_{p\to\infty} F_{\varepsilon_{n(p)}}(H_{n(p),k(p)}) \le \int_\Omega \mathcal{Q}_{\mathrm{curl}}W(H)\,dx.$$

The lower bound. Let $H_n \in \mathcal{U}_{\varepsilon_n}$ and $H_n \rightharpoonup H$ weakly in $L^M(\Omega;\mathbb{R}^{3\times 3})$. By (5) and (6) we deduce that $H \in \mathcal{U}_0$.

We may extract a subsequence H_{n_k} such that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(H_n) = \lim_{k\to\infty} F_{\varepsilon_{n_k}}(H_{n_k}) = \lim_{k\to\infty} \int_\Omega W(H_{n_k})\,dx < \infty. \tag{18}$$

By the bound (2) and Theorem 6.7 (iii) we obtain that

$$\liminf_{k\to\infty} \int_\Omega W(H_{n_k})\,dx \geq \int_\Omega \left(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_x(\lambda) \right) dx. \tag{19}$$

By Proposition 6.8 there exists $\Omega_0 \subset \Omega$ with $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$ such that for $a \in \Omega_0$, there exists a sequence $z_k \in W^{1,M}_{\mathrm{per}}(Q;\mathbb{R}^{3\times 3})$ with $\operatorname{curl}_0 z_k = 0$ in Q and $M(|z_k|)$ is equi-integrable, such that z_k generates the homogeneous Young measure $(\nu_a)_{y\in Q}$ and satisfies $\int_Q z_k\,dy = H(a)$. Fix $a \in \Omega_0$. By the growth conditions (2), the sequence $W(z_k)$ is equi-integrable. By Theorem 6.7 (iv) we obtain

$$\lim_{k\to\infty} \int_Q W(z_k)\,dy = \int_Q \left(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_a(\lambda) \right) dy. \tag{20}$$

Using $w(y) = z_k(y) - H(a)$ as a test function in the definition of $\mathcal{Q}^\eta_{\mathrm{curl}_0}$ we have

$$\begin{aligned} \liminf_{k\to\infty} \int_Q W(z_k)\,dy &= \lim_{k\to\infty} \int_Q W(H(a) + z_k - H(a))\,dy \\ &\geq \sup_{\eta>0} \mathcal{Q}^\eta_{\mathrm{curl}_0} W(H(a)) = \mathcal{Q}^\infty_{\mathrm{curl}_0} W(H(a)). \end{aligned} \tag{21}$$

Then (20) and (21) imply that

$$\int_Q \left(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_a(\lambda) \right) dy \geq \mathcal{Q}^\infty_{\mathrm{curl}_0} W(H(a)) \qquad (a \in \Omega_0).$$

Therefore by the Tonelli Theorem (see, e.g. [31, Theorem I.6.12], [20, Theorem 1.121]) and by the bound $W(F) \geq -\frac{1}{C}$ (see (2)) we deduce that

$$\begin{aligned} &\int_\Omega \Big(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_x(\lambda) \Big)\,dx = \operatorname{meas}(Q) \int_\Omega \Big(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_x(\lambda) \Big)\,dx \\ &= \int_Q \Big(\int_\Omega \Big(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_x(\lambda) \Big)\,dx \Big)\,dy = \int_Q \Big(\int_{\Omega_0} \Big(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_a(\lambda) \Big)\,da \Big)\,dy \\ &= \int_{\Omega_0} \Big(\int_Q \Big(\int_{\mathbb{R}^m} W(\lambda)\,d\nu_a(\lambda) \Big)\,dy \Big)\,da \geq \int_{\Omega_0} \mathcal{Q}^\infty_{\mathrm{curl}_0} W(H(a))\,da. \end{aligned}$$

Hence, by (18) and (19), we deduce that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(H_n) = \liminf_{k\to\infty} \int_\Omega W(H_{n_k})\,dx \geq \int_\Omega \mathcal{Q}^\infty_{\mathrm{curl}_0} W(H(x))\,dx. \ \blacksquare$$

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