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ON GENERALIZED YOUNG'S INEQUALITY

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Abstract. We generalize Young's inequality to Orlicz functions. The Young's inequality is widely used not only in Mathematics but also in Mechanics and Risk Management. We show that for Orlicz function Φ, its Young complementary function $\widetilde{\Phi}$ and dual complementary function Φ^* coincide.

1. Introduction. In the 1930's Young's Inequality was proved [HLP]. That is, for $f : [0, +\infty) \to [0, +\infty)$ a continuous and strictly increasing function with $f(0) = 0$, for all nonnegative u, v,

$$\int_0^u f(s)\,ds + \int_0^v f^{-1}(s)\,ds \geq uv$$

and the inequality turns into equality if and only if $v = f(u)$. After that, in the last century, Jensen's Inequality was proved [BO, Jj, MPF]. That is, if $p : [0, +\infty) \to [0, +\infty)$ a right continuous and nondecreasing function with

(1) $p(0) = 0$;
(2) $p(s) > 0$ if $s > 0$;
(3) $\lim_{s\to 0} p(s) = 0$ and $\lim_{s\to +\infty} p(s) = +\infty$

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then for all $u, v \in \mathbb{R}$

$$\int_0^u p(s)\,ds + \int_0^v p^*(t)\,dt \geq uv$$

and the inequality turns into equality if and only if $v = p(u)$ or $u = p^*(v)$, where $p^*(t) = \sup\{s : p(s) \leq t\}$. There is a proof using graphs [KR] with details in [WS]. These inequalities play not only a fundamental role in many fields of Mathematics [BB, Bz, MO], but also an important role in other fields [BDP]. The developing of Mechanics [FL, FS] and Risk Management [BF] lead the more functions involved. For example, in [BF], it is needed that the functions take the value of $+\infty$. In this paper, removing the above restrictions (1)–(3), we prove Young's Inequality in every detail for a right continuous and nondecreasing function $p : [0, +\infty) \to [0, +\infty]$ whose value can be $+\infty$. Such kind of functions are widely adopted [BF, FL, FS], especially in Orlicz spaces theory [HW, FHS, Mj, HM, Ml]. We generalize the results of [KR, Cs, WW, WWCW]. We refer the reader to see [Cs, WW, WWCW, RR] for more details.

Definition 1.1 ([KR]). $\Phi : \mathbb{R} \to [0, +\infty]$, where $+\infty$ can be a possible value, is called an *Orlicz function*, provided that it is even, convex and left continuous on $[0, +\infty)$ with $\Phi(0) = 0$. Set

$$\alpha_\Phi := \sup\{s \geq 0 : \Phi(s) = 0\}; \qquad \beta_\Phi := \sup\{s \geq 0 : \Phi(s) < \infty\}$$

where $\mathbb{R}$ is the set of all real numbers. An interval (a, b) is called a *Structure Affine Interval* (SAI) of Φ provided that $\Phi(s)$ is affine on (a, b), and for all $\varepsilon > 0$, $\Phi(s)$ is not affine on $(a - \varepsilon, b)$ or $(a, b + \varepsilon)$. Set $S_\Phi := \mathbb{R} \setminus \bigcup_{i=0}^{\infty}(a_i, b_i)$, where (a_i, b_i) is a SAI of Φ and $b_0 = +\infty$.

Definition 1.2 ([KR]). Let Ω be a set in $\mathbb{R}^n$ and (Ω, Σ, μ) be a measure space [DU]. For a real valued measurable function $u(t)$ on Ω, let $\rho_\Phi(u) := \int_\Omega \Phi(u(t))\,d\mu$. We define the Orlicz function spaces L_Φ

$$L_\Phi := \{u : \rho_\Phi(\lambda u) < \infty \ \text{ for some } \lambda > 0\},$$

equipped with the Luxemburg norm

$$\|u\|_{(\Phi)} := \inf\Big\{\lambda > 0 : \rho_\Phi\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}$$

or the Orlicz norm

$$\|u\|_\Phi := \sup_{\rho_{\Phi^*}(v) \leq 1} \int_\Omega |u(t)v(t)|\,d\mu = \inf_{k>0} \frac{1}{k}[1 + \rho_\Phi(ku)], \quad \text{where } v \in L_{\Phi^*}$$

L_Φ is a Banach space.

Lemma 1.3 ([KR]). *For an Orlicz function Φ, its right derivative $\Phi'_+(s)$ exists for all $s \in \mathbb{R}$, and $\Phi'_+(s)$ is nonnegative, nondecreasing and right continuous in $[0, +\infty)$. Moreover, for each $u \in \mathbb{R}$,*

$$\Phi(u) = \int_0^{|u|} \Phi'_+(s)\,ds.$$

Proof. In the paper, for a convex function Φ, if $s' < s''$, $\Phi(s') = \Phi(s'') = +\infty$, we always assume

$$\Phi(s'') - \Phi(s') = +\infty$$

(that is to say, $\Phi'_+(s) = \infty$ for $s \geq \beta_\Phi$).

First, for $0 \le s_1 < s_2 < s_3$, since Φ is convex and $\Phi(0) = 0$, we see that

$$\frac{\Phi(s_2) - \Phi(s_1)}{s_2 - s_1} \le \frac{\Phi(s_3) - \Phi(s_1)}{s_3 - s_1} \le \frac{\Phi(s_3) - \Phi(s_2)}{s_3 - s_2}. \tag{$*$}$$

In fact, if $\Phi(s_3) < +\infty$, by [KR], we get $(*)$. If $\Phi(s_2) < +\infty$, and $\Phi(s_3) = +\infty$, by the convexity of Φ, we get

$$\frac{\Phi(s_2) - \Phi(s_1)}{s_2 - s_1} \le \frac{\Phi(s_3) - \Phi(s_1)}{s_3 - s_1},$$

and

$$\frac{\Phi(s_2) - \Phi(s_1)}{s_2 - s_1} < +\infty = \frac{\Phi(s_3) - \Phi(s_2)}{s_3 - s_2},$$

so $(*)$ is true. If $\Phi(s_1) < +\infty$, and $\Phi(s_2) = +\infty = \Phi(s_3)$, then by the assumption at the beginning of the proof

$$\frac{\Phi(s_2) - \Phi(s_1)}{s_2 - s_1} = +\infty = \frac{\Phi(s_3) - \Phi(s_1)}{s_3 - s_1} = \frac{\Phi(s_3) - \Phi(s_2)}{s_3 - s_2},$$

so $(*)$ is true. If $\Phi(s_1) = \Phi(s_2) = \Phi(s_3) = +\infty$, then by the same assumption

$$\frac{\Phi(s_2) - \Phi(s_1)}{s_2 - s_1} = +\infty = \frac{\Phi(s_3) - \Phi(s_1)}{s_3 - s_1} = \frac{\Phi(s_3) - \Phi(s_2)}{s_3 - s_2},$$

so $(*)$ is true.

Summarizing, for $0 \le s_1 < s_2 < s_3$, $(*)$ holds.

Secondly, for all $h > 0$, by $(*)$, $f(h) = \frac{\Phi(s+h) - \Phi(s)}{h}$ is nondecreasing, so

$$\Phi'_+(s) := \lim_{h \to 0_+} \frac{\Phi(s+h) - \Phi(s)}{h},$$

exists for all $s \in [0, +\infty)$.

We claim that $\Phi'_+(s)$ is nondecreasing on $[0, +\infty)$. In fact, for $0 \le s_1 < s_2$, and $h > 0$ small enough, by $(*)$

$$\begin{aligned}\frac{\Phi(s_1 + h) - \Phi(s_1)}{h} &\le \frac{\Phi(s_2 - h) - \Phi(s_1 + h)}{s_2 - s_1 - 2h} \\ &\le \frac{\Phi(s_2) - \Phi(s_2 - h)}{h} \le \frac{\Phi(s_2 + h) - \Phi(s_2)}{h},\end{aligned}$$

so

$$\frac{\Phi(s_1 + h) - \Phi(s_1)}{h} \le \frac{\Phi(s_2) - \Phi(s_2 - h)}{h} \le \frac{\Phi(s_2 + h) - \Phi(s_2)}{h}.$$

Let $h \to 0$. We get

$$\Phi'_+(s_1) \le \Phi'_-(s_2) \le \Phi'_+(s_2).$$

Finally, $\Phi'_+(s)$ is right continuous on $[0, +\infty)$. Since $\Phi'_+(s)$ is nondecreasing, we get

$$\lim_{s' \to s^+} \Phi'_+(s') \ge \Phi'_+(s).$$

If $\Phi'_+(s) = +\infty$, we have $+\infty \ge \lim_{s' \to s^+} \Phi'_+(s') \ge \Phi'_+(s) = +\infty$, i.e. $\lim_{s' \to s^+} \Phi'_+(s') = \Phi'_+(s)$.

If $\Phi'_+(s) < +\infty$, for any $\varepsilon > 0$, there exists $h > 0$ such that

$$\Phi'_+(s) \le \frac{\Phi(s+h) - \Phi(s)}{h} \le \Phi'_+(s) + \varepsilon.$$

For $s < s' < s+h$

$$\Phi'_+(s) \le \Phi'_+(s') < \Phi'_-(s+h) \le \frac{\Phi(s+h)-\Phi(s)}{h} + \varepsilon \le \Phi'_+(s) + 2\varepsilon$$

so $\Phi'_+(s) \le \lim_{s'\to s_+} \Phi'_+(s') \le \Phi'_+(s) + 2\varepsilon$, since ε is arbitrary, we get $\lim_{s'\to s_+} \Phi'_+(s') = \Phi'_+(s)$.

Summarizing, for all $s \in [0,+\infty)$, $\lim_{s'\to s_+} \Phi'_+(s') = \Phi'_+(s)$, i.e. $\Phi'_+(s)$ is right continuous on $[0,+\infty)$.

For each $u \in \mathbb{R}$, since $\Phi(u)$ is even, we can assume $u \ge 0$.

As $u < \beta_\Phi$, then there exists a positive number M such that $\Phi'_+(s) \le M$, for all $s \le u$. By [Ni], we get

$$\Phi(u) = \int_0^u \Phi'_+(s)\, ds.$$

As $u = \beta_\Phi$, since $\Phi'_+(s)$ is nondecreasing and nonnegative, by the left continuity of $\Phi(u)$ and the Levy Theorem, we have

$$\Phi(\beta_\Phi) = \lim_{u\to\beta_\Phi^-} \Phi(u) = \lim_{u\to\beta_\Phi^-} \int_0^{|u|} \Phi'_+(s)\, ds = \int_0^{\beta_\Phi} \Phi'_+(s)\, ds.$$

As $u > \beta_\Phi$, $\Phi(u) = +\infty$. By the assumption, we get $+\infty = \int_{\beta_\Phi}^u \Phi'_+(s)\, ds \le \int_0^u \Phi'_+(s)\, ds$, so $\Phi(u) = +\infty = \int_{\beta_\Phi}^u \Phi'_+(s)\, ds \le \int_0^u \Phi'_+(s)\, ds$.

In summary, for all $u \in \mathbb{R}$,

$$\Phi(u) = \int_0^{|u|} \Phi'_+(s)\, ds. \;\blacksquare$$

Remark 1.4 ([KR]). For an Orlicz function Φ, there exists a nonnegative, nondecreasing and right continuous function p on $[0,+\infty)$ such that for each $u \in \mathbb{R}$,

$$\Phi(u) = \int_0^{|u|} p(s)\, ds.$$

For simplicity, we rewrite $\Phi'_+(s)$ as $p(s)$ and $\Phi'_-(s)$ as $p_-(s)$.

Obviously, we have

Remark 1.5 ([Cs]). An interval (a,b) is SAI of Φ if and only if $p(s)$ is constant on (a,b).

Definition 1.6 ([KR]). For $p : [0,+\infty) \to [0,+\infty]$ a nondecreasing function, set

$$p^*(t) := \sup\{s \ge 0 : p(s) \le t\} = \inf\{s \ge 0 : p(s) > t\},$$

$$p^*_-(t) := \sup\{s \ge 0 : p(s) < t\} = \inf\{s \ge 0 : p(s) \ge t\},$$

$$\Phi^*(v) := \int_0^{|v|} p^*(t)\, dt.$$

Remark 1.7. If $p(s) \equiv 0$, $\Phi(u) \equiv 0$, $L_\Phi = L_0 :=$ {all measurable functions}, but for all $u \in L_\Phi$, $\|u\|_{(\Phi)} = \inf\left\{\lambda > 0 : \rho_\Phi(\frac{u}{\lambda}) \le 1\right\} = 0$, so $(L_\Phi, \|.\|)$ is not a normed space. If $p^*(t) = \sup\{s \ge 0 : p(s) \le t\} = +\infty$, $\Phi^*(v) \equiv +\infty$, then $L_{\Phi^*} = \{\theta\}$, a trivial space. Also the converse is true, i.e. if $p \equiv +\infty$, $p^* \equiv 0$.

Hence we further assume $p \not\equiv 0$ and $p \not\equiv +\infty$, i.e. $\alpha_\Phi < +\infty$ and $\beta_\Phi > 0$.

Definition 1.8 (Young's sense complementary function [KR]).

$$\tilde{\Phi}(v) := \sup\{u|v| - \Phi(u) : u \in \mathbb{R}\}.$$

2. Main results

LEMMA 2.1 ([KR]). *If* $p : [0, +\infty) \to [0, +\infty]$ *is nondecreasing and right continuous, then* $p^* : [0, +\infty) \to [0, +\infty]$ *is nondecreasing and right continuous, and for all* $\varepsilon > 0$,

(1) $p^*(p(s)) \geq s$;
(2) $p^*(p(s) + \varepsilon) > s$, *if* $p(s) < +\infty$;
(3) $p^*(p(s) - \varepsilon) \leq s$, *if* $p(s) < +\infty$.

Proof. For $t \in [0, +\infty)$, first by the definition $p^*(t) = \sup\{s \geq 0 : p(s) \leq t\} \geq 0$. Next for $0 \leq t' < t''$, we see that

$$p^*(t') = \sup\{s \geq 0 : p(s) \leq t'\} \leq \sup\{s \geq 0 : p(s) \leq t''\} = p^*(t'').$$

Suppose that for some $t \geq 0$, $p^*(t) < p^*(t_+) := \lim_{h\to 0, h>0} p^*(t + h)$. Take $p^*(t) < s' < s'' < p^*(t_+)$. By $p^*(t_+) \leq p^*(t + h)$ for all $h > 0$, from the definition of p^* and since p is nondecreasing, we see $p(s'') \leq t + h$. Since h is arbitrary, $p(s'') \leq t$, we obtain a contradiction: $t < p(s') \leq p(s'') \leq t$.

We see that $p^*(p(s)) = \sup\{s' \geq 0 : p(s') \leq p(s)\} \geq s$, hence (1) is true.

For any $s \geq 0$, $\varepsilon > 0$, $p(s) < +\infty$, since $p(s)$ is right continuous, there exists $s' > s$ such that $p(s') < p(s) + \varepsilon$, so $p^*(p(s) + \varepsilon) = \sup\{s' \geq 0 : p(s') \leq p(s) + \varepsilon\} \geq s' > s$, thus (2) holds.

For $p(s) < +\infty$, $p^*(p(s) - \varepsilon) = \sup\{s' \geq 0 : p(s') \leq p(s) - \varepsilon\} = \inf\{s' \geq 0 : p(s') > p(s) - \varepsilon\} \leq s$. Thus (3) is satisfied. ■

By the Levy Theorem and p^* being nondecreasing and nonnegative on $[0, +\infty)$, it is easy to see the following

REMARK 2.2 ([KR]). For p^* of Lemma 2.1, $\Phi^*(v) = \int_0^{|v|} p^*(t)\,dt$ is an Orlicz function, i.e. $\Phi^* : \mathbb{R} \to [0, +\infty]$ is even, convex and left continuous on $[0, +\infty)$ with $\Phi^*(0) = 0$.

PROPOSITION 2.3. *For an Orlicz function* Φ

$$\alpha_{\Phi^*} = p(0_-) := \lim_{s\to 0_+} p_-(s) = \lim_{s\to 0_+} p(s)$$

$$\beta_{\Phi^*} = p(+\infty) := \lim_{s\to +\infty} p_-(s) = \lim_{s\to +\infty} p(s).$$

Proof. Let $\alpha = \lim_{s\to 0_+} p(s) = \lim_{s>0, s\to 0} p(s)$, by the assumption $p \not\equiv \infty$, we get $\alpha < +\infty$. For any $h > 0$, there is $\delta > 0$ such that for all $0 \leq s \leq \delta$, $p(s) < \alpha + h$. From $\alpha_{\Phi^*} = \sup\{v \geq 0 : \Phi^*(v) = 0\} = \inf\{t \geq 0 : p^*(t) > 0\}$ and $p^*(\alpha + h) = \sup\{s \geq 0 : p(s) \leq \alpha + h\} \geq \delta > 0$, we see that $\alpha_{\Phi^*} \leq \alpha + h$. Since $h > 0$ is arbitrary, we deduce that $\alpha_{\Phi^*} \leq \alpha$. If $\alpha = 0$, we get $\alpha_{\Phi^*} = \alpha$. If $\alpha > 0$, since $p(s)$ is nondecreasing, we see that $p^*(t) = \sup\{s \geq 0 : p(s) \leq t\} \leq 0$ for all $0 \leq t < \alpha$, moreover $\alpha_{\Phi^*} = \sup\{v \geq 0 : \Phi^*(v) = 0\} = \sup\{t \geq 0 : p^*(t) \leq 0\} \geq t$. Since t is arbitrary, we deduce $\alpha_{\Phi^*} \geq \alpha$, hence $\alpha_{\Phi^*} = \alpha$.

Put $\beta = \lim_{s\to +\infty} p_-(s) = \lim_{s\to +\infty} p(s)$.

A. Consider the case $\beta_\Phi < +\infty$ and $p(+\infty) = +\infty$. For all $n > 0$, $p^*(n) = \sup\{s \geq 0 : p(s) \leq n\} \leq \beta_\Phi$ and $\beta_{\Phi^*} = \sup\{v \geq 0 : \Phi^*(v) < +\infty\} = \sup\{t \geq 0 : p^*(t) < +\infty\} \geq n$. Since n is arbitrary, we get $\beta_{\Phi^*} = +\infty = p(+\infty)$.

B. The case $\beta_\Phi = +\infty$.

B-1. If $\beta = +\infty$, then for all $n > 0$, there exists $+\infty > s' > 0$ such that $p(s') > n$, so $p^*(n) = \sup\{s \geq 0 : p(s) \leq n\} = \inf\{s \geq 0 : p(s) > n\} \leq s' < +\infty$, moreover $\beta_{\Phi^*} = \sup\{t \geq 0 : p^*(t) < +\infty\} \geq n$. Since n is arbitrary, we get $\beta_{\Phi^*} \geq +\infty = \beta = p(+\infty)$.

B-2. If $\beta < +\infty$, for all $h > 0$, $p^*(\beta + h) = \sup\{s \geq 0 : p(s) \leq \beta + h\} = +\infty$,

$$\beta_{\Phi^*} = \sup\{t \geq 0 : p^*(t) < +\infty\} = \inf\{t \geq 0 : p^*(t) = +\infty\} \leq \beta + h.$$

Since h is arbitrary, we get $\beta_{\Phi^*} \leq \beta$.

B-2-i. If $0 = \beta$, combining $\beta_{\Phi^*} \leq \beta$ and $\beta_{\Phi^*} \geq 0$, we get $\beta_{\Phi^*} = 0 = \beta = p(+\infty)$.

B-2-ii. If $0 < \beta$, for all $h > 0$ there exists $0 \leq s' < +\infty$ such that $+\infty > p(s') > \beta - h$. Thus $p^*(\beta - h) = \inf\{s \geq 0 : p(s) > \beta - h\} \leq s' < +\infty$. Moreover $\beta_{\Phi^*} = \sup\{t \geq 0 : p^*(t) < +\infty\} \geq \beta - h$. Since h is arbitrary, we get $\beta_{\Phi^*} \geq \beta$, and therefore $\beta_{\Phi^*} = \beta = p(+\infty)$. ∎

Lemma 2.4 ([KR]). *If $p : [0, +\infty) \to [0, +\infty]$ is nondecreasing and right continuous, then $p^{**} = p$. Moreover $\Phi^{**} = \Phi$.*

Proof. Let $s \in [0, +\infty)$. If $p(s) < +\infty$, $s < \beta_\Phi$, then for any $\varepsilon > 0$, by Lemma 2.1 $p^*[p(s) - \varepsilon] \leq s$, so

$$p^{**}(s) = \sup\{t \geq 0 : p^*(t) \leq s\} \geq p(s) - \varepsilon.$$

Since ε is arbitrary, we get $p^{**}(s) \geq p(s)$.

On the other hand, by Lemma 2.1 $p^*[p(s) + \varepsilon] > s$, so

$$p^{**}(s) = \sup\{t \geq 0 : p^*(t) \leq s\} = \inf\{t \geq 0 : p^*(t) > s\} \leq p(s) + \varepsilon.$$

Since ε is arbitrary, we get $p^{**}(s) \leq p(s)$. So $p^{**}(s) = p(s)$.

If $p(s) = +\infty$, then $s \geq \beta_\Phi$, for all $n > 0$, $p^*(n) = \sup\{s \geq 0 : p(s) \leq n\} \leq \beta_\Phi < +\infty$. $p^{**}(s) = \sup\{t \geq 0 : p^*(t) \leq s\} \geq \sup\{t \geq 0 : p^*(t) \leq \beta_\Phi\} \geq n$. Since n is arbitrary, we get $p^{**}(s) = +\infty = p(s)$. ∎

Lemma 2.5. *Given a nonnegative and nondecreasing function $p(s)$, $p(s)$ is strictly increasing on $[0, +\infty)$ implies that p^* is continuous on $[0, +\infty)$, and $p(s)$ is continuous on $[0, +\infty)$ implies that p^* is strictly increasing on $[0, p(+\infty)]$.*

Proof. First, we prove that $p(s)$ is strictly increasing implies that $p^*(t)$ is continuous.

Suppose that $p^*(t)$ is not continuous. Since $p^*(t)$ is right continuous, we have $p^*(t_-) < p^*(t)$ for some $t \in (0, +\infty)$. Take $s', s'' \in \mathbb{R}$ such that $p^*(t_-) < s' < s'' < p^*(t)$. By Lemma 2.4, $p(s) = p^{**}(s)$ and by Lemma 2.1, we see that for all $t' < t$, there exists $\varepsilon' > 0$ such that $p(s') = p^{**}(s') = p^{**}(p^*(t') + \varepsilon') > t'$. By the arbitrariness of t', $p(s') \geq t$. On the other hand, by Lemma 2.1, $p(s'') = p^{**}(s'') = \inf\{t \geq 0 : p^*(t) > s''\} \leq t$, so $p(s'') \leq t \leq p(s')$, and since p is nondecreasing, $p(s'') \geq p(s')$, hence $p(s'') = p(s')$. This is a contradiction to $p(s') < p(s'')$.

Secondly, we prove that $p(s)$ is continuous implies $p^*(t)$ is strictly increasing. Suppose that for some $t', t'' \in (0, p(+\infty)]$, $t' < t''$ with $p^*(t') = p^*(t'') := s$. Then

$$p(s) = p^{**}(s) = \sup\{t : p^*(t) \leq s\} \geq t'' > t'.$$

On the other hand, for all $s' < s$, since $p^*(t)$ is nondecreasing,

$$p(s') = p^{**}(s') = \sup\{t \geq 0 : p^*(t) \leq s'\} = \inf\{t \geq 0 : p^*(t) > s'\} \leq t' < t'',$$

so $p(s_-) = \lim_{s' \to s_-} p(s') \leq t' < t'' \leq p(s)$, a contradiction to the continuity of p. ■

Lemma 2.6. *Given a nonnegative and nondecreasing function p, for any $\varepsilon > 0$, if*

$$(1-\varepsilon)p(s) \leq p^\varepsilon(s) \leq (1+\varepsilon)p(s) \qquad \forall\, s \geq 0,$$

then

$$p^*\Big(\frac{t}{1+\varepsilon}\Big) \leq p^{\varepsilon *}(t) \leq p^*\Big(\frac{t}{1-\varepsilon}\Big) \qquad \forall\, t \geq 0,$$

$$(1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big) \leq \Phi^{\varepsilon *}(v) \leq (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big) \qquad \forall\, v \geq 0.$$

Proof. Since $(1-\varepsilon)p(s) \leq p^\varepsilon(s) \leq (1+\varepsilon)p(s)$, we get

$$\begin{aligned} p^{\varepsilon *}(t) &= \sup\{s \geq 0 : p^\varepsilon(s) \leq t\} \leq \sup\{s \geq 0 : (1-\varepsilon)p(s) \leq t\} \\ &= \sup\Big\{s \geq 0 : p(s) \leq \frac{t}{1-\varepsilon}\Big\} = p^*\Big(\frac{t}{1-\varepsilon}\Big) \end{aligned}$$

and

$$\begin{aligned} p^{\varepsilon *}(t) &= \sup\{s \geq 0 : p^\varepsilon(s) \leq t\} \geq \sup\{s \geq 0 : (1+\varepsilon)p(s) \leq t\} \\ &= \sup\Big\{s \geq 0 : p(s) \leq \frac{t}{1+\varepsilon}\Big\} = p^*\Big(\frac{t}{1+\varepsilon}\Big) \end{aligned}$$

Thus $p^*\big(\frac{t}{1+\varepsilon}\big) \leq p^{\varepsilon *}(t) \leq p^*\big(\frac{t}{1-\varepsilon}\big)$. Hence

$$\begin{aligned} \Phi^{\varepsilon *}(v) &= \int_0^{|v|} p^{\varepsilon *}(t)\,dt \leq \int_0^{|v|} p^*\Big(\frac{t}{1-\varepsilon}\Big)\,dt \\ &= (1-\varepsilon)\int_0^{|v|} p^*\Big(\frac{t}{1-\varepsilon}\Big)\,d\frac{t}{1-\varepsilon} = (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big) \\ \Phi^{\varepsilon *}(v) &= \int_0^{|v|} p^{\varepsilon *}(t)\,dt \geq \int_0^{|v|} p^*\Big(\frac{t}{1+\varepsilon}\Big)\,dt \\ &= (1+\varepsilon)\int_0^{|v|} p^*\Big(\frac{t}{1+\varepsilon}\Big)\,d\frac{t}{1+\varepsilon} = (1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big), \end{aligned}$$

so

$$(1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big) \leq \Phi^{\varepsilon *}(v) \leq (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big). \ ■$$

Lemma 2.7 ([Cs]). *Given an Orlicz function Φ, for any $\varepsilon > 0$, there exists a strictly convex Orlicz function Φ^ε with $p^\varepsilon(s)$ strictly increasing such that $\alpha_{\Phi^\varepsilon} = 0$, $p^\varepsilon(0) = 0$ and*

$$\begin{aligned} (1-\varepsilon)p(s) - \varepsilon \leq p^\varepsilon(s) &\leq (1+\varepsilon)p(s) + \varepsilon \qquad \forall\, s \geq 0 \\ (1-\varepsilon)\Phi(u) - \varepsilon \leq \Phi^\varepsilon(u) &\leq (1+\varepsilon)\Phi(u) + \varepsilon \qquad \forall\, u \in \mathbb{R} \end{aligned}$$

$$\textit{and} \quad (1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big) - \varepsilon \leq \Phi^{\varepsilon *}(v) \leq (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big) + \varepsilon \qquad \forall\, v \in \mathbb{R}.$$

If $\alpha_\Phi = 0$ *and* $p(0) = 0$

$$(1-\varepsilon)p(s) \le p^\varepsilon(s) \le (1+\varepsilon)p(s) \qquad \forall\, s \ge 0$$

$$(1-\varepsilon)\Phi(u) \le \Phi^\varepsilon(u) \le (1+\varepsilon)\Phi(u) \qquad \forall\, u \in \mathbb{R}$$

$$\textit{and} \quad (1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big) \le \Phi^{\varepsilon *}(v) \le (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big) \qquad \forall\, v \in \mathbb{R}.$$

Proof.

Case 1. Assume that $\alpha_\Phi = 0$ and $p(0) = 0$.

A. For (a_1, b_1):

A-I. If $p(a_1) = p(b_1)$, by the right continuity of p, take $b_1' > b_1$ such that $p(b_1) < p(b_1') < (1+\varepsilon)p(b_1)$ and take $a_1' = a_1$, define $p^\varepsilon(a_1') = p(a_1)$, $p^\varepsilon(b_1') = p(b_1')$. For $s \in (a_1', b_1')$, $p^\varepsilon(s)$ is defined as a line connecting $(a_1', p(a_1'))$ and $(b_1', p(b_1'))$.

A-II. If $p(a_1) < p(b_1)$. Take $a_1' = a_1$ and $b_1' = b_1$, define $p^\varepsilon(a_1') = p(a_1)$, $c = \min\{p(b_1'), (1+\varepsilon)p(a_1)\}$ and $p^\varepsilon(s)$ is defined as a line that connects $(a_1', p(a_1'))$ and (b_1', c) for $s \in (a_1', b_1')$.

Thus we see that $p^\varepsilon(s)$ is nonnegative and strictly increasing on (a_1', b_1') and for all $s \in (a_1', b_1')$, $(1-\varepsilon)p(s) \le \frac{p(s)}{(1+\varepsilon)} \le \frac{(1+\varepsilon)}{(1+\varepsilon)}p(a_1) = p(a_1) \le p^\varepsilon(s) \le (1+\varepsilon)p(a_1) \le (1+\varepsilon)p(s)$.

B. For (a_2, b_2):

B-I. In the case of $S_2 = (a_2, b_2) \cap [a_1', b_1') = \emptyset$, repeating arguments as in Case A, we define $p^\varepsilon(s)$.

B-II. In the case of $S_2 = (a_2, b_2) \cap [a_1', b_1') \ne \emptyset$, we see that $a_2 < a_1' < b_2$ or $a_2 < b_1' < b_2$.

B-II-1. Since $(a_1, b_1) \cap (a_2, b_2) = \emptyset$ and $a_1 = a_1'$, the inequality $a_2 < a_1' < b_2$ is impossible.

B-II-2. If $b_1' < b_2$, take $a_2' = b_1'$. Repeating the arguments of Case A, we define $p^\varepsilon(s)$.

C. For (a_3, b_3):

C-I. In the case of $S_3 = (a_3, b_3) \cap ([a_1', b_1') \cup [a_2', b_2')) = \emptyset$, repeating the arguments of Case A, we define $p^\varepsilon(s)$.

C-II. In the case of $S_3 = (a_3, b_3) \cap ([a_1', b_1') \cup [a_2', b_2')) \ne \emptyset$.

C-II-1. If $(a_3, b_3) \subseteq ([a_1', b_1') \cup [a_2', b_2'))$, $p^\varepsilon(s)$ has been well defined.

C-II-2. If $(a_3, b_3) \nsubseteq ([a_1', b_1') \cup [a_2', b_2'))$, combining that (a_1, b_1),(a_2, b_2) and (a_3, b_3) are mutually disjoint, and (a_1', b_1') and (a_2', b_2') are disjoint, we see that there exists one and only one i such that $a_i' < a_3 < b_i' < b_3$. Repeating arguments of Case B-II-2, we define $p^\varepsilon(s)$.

Assume that for (a_k, b_k), there exist mutually disjoint $\{(a_i', b_i')\}_{i=1}^k$ with $\bigcup_{i=1}^k (a_i, b_i) \subseteq \bigcup_{i=1}^k (a_i', b_i')$ such that $p^\varepsilon(s)$ is positive and strictly increasing with

$$(1-\varepsilon)p(s) \le p^\varepsilon(s) \le (1+\varepsilon)p(s) \qquad \forall s \in \bigcup_{i=1}^k (a_i', b_i').$$

For (a_{k+1}, b_{k+1}), we see that

D-I. In the case of $S_{k+1} = (a_{k+1}, b_{k+1}) \cap (\bigcup_{i=1}^k [a_i', b_i')) = \emptyset$, repeating the arguments of Case A, we define $p^\varepsilon(s)$.

D-II. In the case of $S_{k+1} = (a_{k+1}, b_{k+1}) \cap (\bigcup_{i=1}^k [a_i', b_i')) \ne \emptyset$.

D-II-1. If $(a_{k+1}, b_{k+1}) \subseteq (\bigcup_{i=1}^k [a_i', b_i'))$, $p^\varepsilon(s)$ has been defined previously.

D-II-2. If $(a_{k+1}, b_{k+1}) \nsubseteq (\bigcup_{i=1}^k [a_i', b_i'))$, combining that $\{(a_i, b_i)\}_{i=1}^k$ are mutually disjoint and $\{(a_i', b_i')\}_{i=1}^k$ are mutually disjoint, we see that there exists one and only one i such that $a_i' < a_{k+1} < b_i' < b_{k+1}$. Repeating arguments of Case B-II-2, we define $p^\varepsilon(s)$. Hence $p^\varepsilon(s)$ is positive and strictly increasing with

$$(1-\varepsilon)p(s) \le p^\varepsilon(s) \le (1+\varepsilon)p(s) \qquad \forall s \in \bigcup_{i=1}^{k+1}(a_i', b_i').$$

By induction, we define $p^\varepsilon(s)$ on $\bigcup_{i=1}^{\infty}(a_i, b_i)$ such that $p^\varepsilon(s)$ is positive and strictly increasing with

$$(1-\varepsilon)p(s) \le p^\varepsilon(s) \le (1+\varepsilon)p(s) \qquad \forall s \ge 0.$$

In the case of $(a_0, b_0) = (a, +\infty) \ne \emptyset$, if $+\infty > p(a) > 0$, define

$$p^\varepsilon(s) = \begin{cases} (1+(1-\frac{1}{2^n})\varepsilon)p(a), & s = a+n,\ n = 0,1,2,\dots \\ \text{connected by line,} & s \in (a+n, a+n+1),\ n = 0,1,2,\dots \end{cases}$$

then for $s \in (a, +\infty)$, there exists n such that for $s \in [a+n, a+n+1)$, we have

$$\begin{aligned} p(s) &\le \Big(1+\Big(1-\frac{1}{2^n}\Big)\varepsilon\Big)p(a) \le p^\varepsilon(s) \\ &\le \Big(1+\Big(1-\frac{1}{2^{n+1}}\Big)\varepsilon\Big)p(a) \le (1+\varepsilon)p(a) = (1+\varepsilon)p(s). \end{aligned}$$

If $p(a) = +\infty$, we define $p^\varepsilon(s) = p(s) = +\infty$, $s \in [a_0, +\infty)$.

If $p(a) = 0$, by the monotonicity, we see that $(a_0, +\infty) = (0, +\infty)$, then $p \equiv 0$, which contradicts the assumption.

Hence we define $p^\varepsilon(s)$ well as $s \in \bigcup_{i=0}^{\infty}(a_i, b_i)$, and define $p^\varepsilon(s) = p(s)$ as $s \notin \bigcup_{i=0}^{\infty}(a_i, b_i)$. Then for any $\varepsilon > 0$, there exists a strictly convex Orlicz function Φ^ε such that $p^\varepsilon(s)$ is strictly increasing with $\alpha_{\Phi^\varepsilon} = 0$, $p^\varepsilon(0) = 0$ and

$$\begin{gathered} (1-\varepsilon)p(s) \le p^\varepsilon(s) \le (1+\varepsilon)p(s) \qquad \forall\, s \ge 0 \\ (1-\varepsilon)\Phi(u) \le \Phi^\varepsilon(u) \le (1+\varepsilon)\Phi(u) \qquad \forall\, u \in \mathbb{R} \\ \text{and} \quad (1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big) \le \Phi^{\varepsilon*}(v) \le (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big) \qquad \forall\, v \in \mathbb{R}. \end{gathered}$$

Case 2. $\alpha_\Phi > 0$ or $p(0) > 0$.

2-I. If $\alpha_\Phi > 0$, then without loss of generality, assume $p(s)$ is strictly increasing for $s > \alpha_\Phi$. We can make the arguments in Case 1 once more if $p(s)$ is not strictly increasing for $s > \alpha_\Phi$. By the assumption $\alpha_\Phi < +\infty$, since p is right continuous, there exists a number α, $\alpha > \alpha_\Phi > 0$ such that $\alpha c \le \varepsilon$ for some $c \le \min\{p(\alpha), \varepsilon\}$. Define $p^\varepsilon(s) = \frac{c}{\alpha}s$ as $s \le \alpha$ and $p^\varepsilon(s) = p(s)$ as $s > \alpha$. Then $p^\varepsilon(s)$ is strictly increasing with $\alpha_{\Phi^\varepsilon} = 0$, $p^\varepsilon(0) = 0$. And as $0 \le t \le c$, $p^{\varepsilon*}(t) = \sup\{s \ge 0 : p^\varepsilon(s) \le t\} = \sup\{s \ge 0 : \frac{c}{\alpha}s \le t\} = \frac{\alpha}{c}t$; as $t > c$, $p^{\varepsilon*}(t) = \sup\{s \ge 0 : p^\varepsilon(s) \le t\} = \sup\{s \ge 0 : p(s) \le t\} = p^*(t)$. Thus $\alpha_{\Phi^*} = 0$, and

$$\begin{aligned} \Phi^\varepsilon(u) \le \Phi^\varepsilon(\alpha) &= \frac{c}{\alpha}\frac{\alpha^2}{2} = \frac{\alpha c}{2} \le \frac{\varepsilon}{2}, \qquad 0 \le u \le \alpha, \\ \Phi^{\varepsilon*}(v) \le \Phi^{\varepsilon*}(c) &= \frac{\alpha}{c}\frac{c^2}{2} = \frac{\alpha c}{2} \le \frac{\varepsilon}{2}, \qquad 0 \le v \le c. \end{aligned}$$

In summary, $p^\varepsilon(s)$ is strictly increasing with $\alpha_{\Phi^\varepsilon}=0$, $p^\varepsilon(0)=0$ and

$$\begin{aligned} (1-\varepsilon)p(s)-\varepsilon \le p^\varepsilon(s) \le (1+\varepsilon)p(s)+\varepsilon \quad & \forall\, s\ge 0\\ (1-\varepsilon)\Phi(u)-\varepsilon \le \Phi^\varepsilon(u) \le (1+\varepsilon)\Phi(u)+\varepsilon \quad & \forall\, u\in\mathbb{R}\\ \text{and}\quad (1+\varepsilon)\Phi^*\Big(\frac{v}{1+\varepsilon}\Big)-\varepsilon \le \Phi^{\varepsilon *}(v) \le (1-\varepsilon)\Phi^*\Big(\frac{v}{1-\varepsilon}\Big)+\varepsilon \quad & \forall\, v\in\mathbb{R}. \end{aligned}$$

2-II. If $p(0)>0$ then since p is nondecreasing, $\alpha_\Phi=0$. Since $\beta_\Phi>0$, take $\beta_\Phi>\alpha>\alpha_\Phi=0$ such that $\alpha p(\alpha)\le\varepsilon$. Repeating the arguments of Case 2-I, we get $p^\varepsilon(s)$ of 2-I. ∎

THEOREM 2.8 (Young's Inequality [KR]). *If Φ is an Orlicz function, then for all $u,v\in\mathbb{R}$,*

$$\Phi(u)+\Phi^*(v)\ge |u|\,|v|\ge uv.$$

Proof. Let $u,v\in\mathbb{R}$.

A. $\Phi:\mathbb{R}\to[0,+\infty]$ and $p(s)$ is continuous and strictly increasing on $[0,+\infty)$. Let $t=p(s)\leftrightarrow s=p^{-1}(t)$, then $p:[0,+\infty)\to[0,+\infty]$, $s=0\leftrightarrow t=p(0):=\alpha^*$ and $s=u\leftrightarrow t=p(u)$. Therefore $p^{-1}:p([0,+\infty))\to[0,+\infty)$, i.e. $p^{-1}:[p(0),p(+\infty))\to[0,+\infty)$. Then $p^*:[0,+\infty)\to[0,+\infty]$, and for $0\le t<\alpha^*$, $p^*(t)=\sup\{s\ge 0: p(s)\le t\}\le \sup\{s\ge 0: p(s)<\alpha^*\}=0$. Hence for all $u,v\in\mathbb{R}$

$$\int_0^{|u|} p(s)\,ds = sp(s)\big|^{|u|} - \int_0^{|u|} s\,dp(s) = |u|p(u) - \int_{\alpha^*}^{p(u)} p^{-1}(t)\,dt$$

$$\int_0^{|v|} p^*(t)\,dt = \int_0^{\alpha^*} p^*(t)\,dt + \int_{\alpha^*}^{|v|} p^*(t)\,dt = \int_{\alpha^*}^{|v|} p^{-1}(t)\,dt$$

$$\begin{aligned} \Phi(u)+\Phi^*(v) &= \int_0^{|u|} p(s)\,ds + \int_0^{|v|} p^*(t)\,dt\\ &= |u|p(u) - \int_{\alpha^*}^{p(u)} p^{-1}(t)\,dt + \int_{\alpha^*}^{|v|} p^{-1}(t)\,dt = |u|p(u) + \int_{p(u)}^{|v|} p^{-1}(t)\,dt\\ &= |u|\,|v| + |u|(p(u)-|v|) + \int_{p(u)}^{|v|} p^{-1}(t)\,dt \end{aligned}$$

If $|v|=p(u)$,

$$\Phi(u)+\Phi^*(v)=|u|\,|v|.$$

If $|v|>p(u)$,

$$\begin{aligned} \Phi(u)+\Phi^*(v) &= |u|\,|v| + |u|(p(u)-|v|) + \int_{p(u)}^{|v|} p^{-1}(t)\,dt\\ &\ge |u|\,|v| + |u|(p(u)-|v|) + p^{-1}(p(u))(|v|-p(u)) = |u|\,|v|. \end{aligned}$$

If $|v|<p(u)$,

$$\begin{aligned} \Phi(u)+\Phi^*(v) &= |u|\,|v| + |u|(p(u)-|v|) + \int_{p(u)}^{|v|} p^{-1}(t)\,dt\\ &= |u|\,|v| + |u|(p(u)-|v|) - \int_{|v|}^{p(u)} p^{-1}(t)\,dt\\ &\ge |u|\,|v| + |u|(p(u)-|v|) - p^{-1}(p(u))(p(u)-|v|) = |u|\,|v|. \end{aligned}$$

In summary, for all $u, v \in \mathbb{R}$

$$\Phi(u) + \Phi^*(v) = |u|\,|v| \geq uv.$$

B. For an Orlicz function Φ, by Lemma 2.7, we get a strictly convex function Φ^ε with a strictly increasing p^ε. By Lemma 2.5, $p^{\varepsilon*}$ is continuous, and by Lemma 2.7 once more we get an Orlicz function $\Phi^{\varepsilon*\varepsilon}$ with a continuous strictly increasing $p^{\varepsilon*\varepsilon}$. Hence for all $u, v \in \mathbb{R}$, let $u' = u(1-\varepsilon)$, $v' = v(1-\varepsilon)$, by the result A, we get

$$\Phi^{\varepsilon*\varepsilon}(v') + \Phi^{\varepsilon*\varepsilon*}(u') \geq |u'|\,|v'|.$$

By Lemmas 2.4, 2.5, 2.7 and 2.6, we get

$$\Phi^{\varepsilon*\varepsilon}(v') \leq (1+\varepsilon)\Phi^{\varepsilon*}(v') + \varepsilon \leq (1+\varepsilon)(1-\varepsilon)\Phi^*\Big(\frac{v'}{1-\varepsilon}\Big) + 2\varepsilon$$

and

$$\Phi^{\varepsilon*\varepsilon*}(u') \leq (1-\varepsilon)\Phi^{\varepsilon**}\Big(\frac{u'}{1-\varepsilon}\Big) = (1-\varepsilon)\Phi^{\varepsilon}\Big(\frac{u'}{1-\varepsilon}\Big) \leq (1-\varepsilon)(1+\varepsilon)\Phi\Big(\frac{u'}{1-\varepsilon}\Big) + \varepsilon.$$

Hence

$$\begin{aligned}(1-\varepsilon^2)\Phi(u) + (1-\varepsilon^2)\Phi^*(v) + 3\varepsilon &= (1-\varepsilon^2)\Phi\Big(\frac{u'}{1-\varepsilon}\Big) + (1-\varepsilon^2)\Phi^*\Big(\frac{v'}{1-\varepsilon}\Big) + 3\varepsilon \\ &\geq |u'|\,|v'| = (1-\varepsilon)^2|u|\,|v|,\end{aligned}$$

so

$$(1-\varepsilon^2)\Phi(u) + (1-\varepsilon^2)\Phi^*(v) + 3\varepsilon \geq (1-\varepsilon)^2|u|\,|v|.$$

Letting $\varepsilon \to 0$, we have

$$\Phi(u) + \Phi^*(v) \geq |u|\,|v| \geq uv. \ \blacksquare$$

PROPOSITION 2.9. *Given an Orlicz function Φ, the following are equivalent: for $u, v \in \mathbb{R}$*

(1) $|v| = p(u)$ *or* $|u| = p^*(v)$;
(2) $|v| \in [p_-(u), p(u)]$;
(3) $|u| \in [p^*_-(v), p^*(v)]$.

Proof. (1) $\Longrightarrow$ (2). Otherwise,

1-I. $|v| = p(u)$ or $|u| = p^*(v)$, $|v| < p_-(u)$ or
1-II. $|v| = p(u)$ or $|u| = p^*(v)$, $|v| > p(u)$.
1-I-i. $|v| = p(u)$, $|v| < p_-(u)$. We deduce $|v| < p_-(u) \leq p(u)$, a contradiction with $|v| = p(u)$.
1-I-ii. $|u| = p^*(v)$, $|v| < p_-(u)$. Since $p_-(u) = \sup\{t \geq 0 : p^*(t) < |u|\} = \inf\{t \geq 0 : p^*(t) \geq |u|\}$, we deduce that $p^*(v) < |u|$, a contradiction with $|u| = p^*(v)$.
1-II-i. $|v| = p(u)$, $|v| > p(u)$. A contradiction.
1-II-ii. $|u| = p^*(v)$, $|v| > p(u)$. By Lemma 2.1, $p^*(v) = p^*(p(u)+\varepsilon) > |u|$, a contradiction with $|u| = p^*(v)$.

(2) $\Longrightarrow$ (1). Otherwise, $|v| \in [p_-(u), p(u)]$, $|v| \neq p(u)$, $|u| \neq p^*(v)$.

2-I. $|v| \in [p_-(u), p(u)]$, $|v| \neq p(u)$, $|u| > p^*(v)$.
2-I-i. $|v| \in [p_-(u), p(u)]$, $|v| > p(u)$, $|u| > p^*(v)$. $v \leq p(u)$ is contradictory to $|v| > p(u)$.

2-I-ii. $|v| \in [p_-(u), p(u)]$, $|v| < p(u)$, $|u| > p^*(v)$. From $p_-(u) = p_-(p^*(v)+\varepsilon)$, since p^* is right continuous, there exists $v' > |v|$ such that $p^*(v') < p^*(v)+\varepsilon$, we deduce $p_-(u) = p_-(p^*(v)+\varepsilon) = \sup\{t \geq 0 : p^*(t) < p^*(v)+\varepsilon\} \geq v' > |v|$, a contradiction to $|v| \geq p_-(u)$).

2-II. $|v| \in [p_-(u), p(u)]$, $|v| \neq p(u)$, $|u| < p^*(v)$.

2-II-i. $|v| \in [p_-(u), p(u)]$, $|v| > p(u)$, $|u| < p^*(v)$. $|v| \leq p(u)$ is contradictory to $|v| > p(u)$.

2-II-ii. $|v| \in [p_-(u), p(u)]$, $|v| < p(u)$, $|u| < p^*(v)$. By Lemma 2.1, we deduce $p(u) = p(p^*(v)-\varepsilon) \leq |v|$, a contradiction to $|v| < p(u)$.

In summary, (1) $\Longleftrightarrow$ (2).

Replacing Φ by Ψ^* and u by v, using Lemma 2.4, repeating the arguments of (1) $\Leftrightarrow$ (2), we get (1) $\Leftrightarrow$ (3). ■

Theorem 2.10 (Young's Equality [KR]). *If Φ is an Orlicz function, then for all $u, v \in \mathbb{R}$*

$$\Phi(u) + \Phi^*(v) = uv \quad \Longleftrightarrow \quad |v| = p(u) \text{ or } |u| = p^*(v).$$

Proof. Necessity. Set $f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $f(u,v) := \Phi(u) + \Phi^*(v) - uv$. By Theorem 2.8, it follows that $f(u,v) \geq 0$. If $f(u_0,v_0) = 0$ then $f(u_0,v_0) = \min f(u,v)$, $\frac{\partial f}{\partial u}_-(u_0,v_0) := \lim_{u\to u_0^-} \frac{f(u,v_0)-f(u_0,v_0)}{u-u_0} \leq 0$ and $\frac{\partial f}{\partial u}_+(u_0,v_0) := \lim_{u\to u_0^+} \frac{f(u,v_0)-f(u_0,v_0)}{u-u_0} \geq 0$. We get

$$p_-(u_0) - v_0 \leq 0, \qquad p(u_0) - v_0 \geq 0,$$

so

$$p_-(u_0) \leq v_0, \qquad p(u_0) \geq v_0$$

i.e. by Proposition 2.9

$$p_-(u_0) \leq v_0 \leq p(u_0) \quad \Longleftrightarrow \quad |v_0| = p(u_0) \text{ or } |u_0| = p^*(v_0).$$

Sufficiency. For $u, v \in \mathbb{R}$, $|v| = p(u)$ or $|u| = p^*(v)$. We shall discuss two cases: A. $|v| = p(u)$ and B. $|u| = p^*(v)$.

A. $|v| = p(u)$, then $p(u) = |v| < +\infty$.

A-1. If $|u| < \beta_\Phi$, $p(u) < +\infty$, for ε small enough $\frac{(1+\varepsilon)u}{1-\varepsilon} < \beta_\Phi$, by Lemmas 2.6, 2.4 and 2.7

$$\begin{aligned} p^{\varepsilon*\varepsilon*}((1+\varepsilon)u) &\leq p^{\varepsilon**}\left(\frac{(1+\varepsilon)u}{1-\varepsilon}\right) \\ &= p^{\varepsilon}\left(\frac{(1+\varepsilon)u}{1-\varepsilon}\right) \leq (1+\varepsilon)p\left(\frac{(1+\varepsilon)u}{1-\varepsilon}\right) + \varepsilon < +\infty, \end{aligned}$$

we get $p^{\varepsilon*\varepsilon*}((1+\varepsilon)u) \in \mathbb{R}$. By Theorem 2.8-A

$$(1+\varepsilon)|u|p^{\varepsilon*\varepsilon*}((1+\varepsilon)u) = \Phi^{\varepsilon*\varepsilon*}((1+\varepsilon)u) + \Phi^{\varepsilon*\varepsilon}(p^{\varepsilon*\varepsilon*}((1+\varepsilon)u)).$$

By Lemmas 2.6, 2.4 and 2.7 again

$$\Phi^{\varepsilon*\varepsilon*}((1+\varepsilon)u) \geq (1+\varepsilon)\Phi^{\varepsilon**}\left[\frac{(1+\varepsilon)u}{1+\varepsilon}\right](1+\varepsilon)\Phi^{\varepsilon}(u) \geq (1-\varepsilon^2)\Phi(u) - \varepsilon$$

and

$$\begin{aligned}
&\Phi^{\varepsilon*\varepsilon}\big(p^{\varepsilon*\varepsilon*}((1+\varepsilon)u)\big)\\
&\ge (1-\varepsilon)\Phi^{\varepsilon*}\big(p^{\varepsilon*\varepsilon*}((1+\varepsilon)u)\big) \ge (1-\varepsilon)(1+\varepsilon)\Phi^*\left[\frac{p^{\varepsilon*\varepsilon*}((1+\varepsilon)u)}{1+\varepsilon}\right]-\varepsilon\\
&\ge (1-\varepsilon^2)\Phi^*\left[\frac{p^{\varepsilon**}(u)}{1+\varepsilon}\right]-\varepsilon = (1-\varepsilon^2)\Phi^*\left[\frac{p^{\varepsilon}(u)}{1+\varepsilon}\right]-\varepsilon\\
&\ge (1-\varepsilon^2)\Phi^*\left[\frac{(1-\varepsilon)p(u)-\varepsilon}{1+\varepsilon}\right]-\varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1+\varepsilon)|u|\left((1+\varepsilon)p\left[\frac{(1+\varepsilon)u}{1-\varepsilon}\right]+\varepsilon\right)\\
&\ge (1-\varepsilon^2)\Phi(u)+(1-\varepsilon^2)\Phi^*\left[\frac{(1-\varepsilon)p(u)-\varepsilon}{1+\varepsilon}\right]-2\varepsilon.
\end{aligned}$$

By the left continuity of Φ^* and the right continuity of p and $\beta_{\Phi^*}>0$, we get

$$|u|p(u) \ge \Phi(u)+\Phi^*[p(u)].$$

By Theorem 2.8, we see that for $u\in\mathbb{R}$, $|u|<\beta_\Phi$

$$|u|p(u) = \Phi(u)+\Phi^*[p(u)].$$

A-2. For $u,v\in\mathbb{R}$, $|u|=\beta_\Phi<+\infty$. By the right continuity of $p(s)$, we get $p(u)=p(\beta_\Phi)=\lim_{s'\to\beta_{\Phi}+}p(s)=+\infty$. By the assumption $\beta_\Phi>0$, we see that $|u|p(u)=+\infty$. Also by the assumption $\alpha_{\Phi^*}<+\infty$, $\Phi^*[p(\beta_\Phi)]=\Phi^*[+\infty]=+\infty$. So $\Phi(\beta_\Phi)+\Phi^*[p(\beta_\Phi)]\ge\Phi^*[p(\beta_\Phi)]=+\infty$, thus for $|u|=\beta_\Phi<+\infty$ we have

$$|u|p(u) = \Phi(u)+\Phi^*[p(u)].$$

A-3. For $u,v\in\mathbb{R}$, $|u|>\beta_\Phi<+\infty$. Then $|u|p(u)=+\infty$ and $\Phi(u)\ge\int_{\beta_\Phi}^{|u|}p(s)\,ds=+\infty$, so

$$|u|p(u) = \Phi(u)+\Phi^*[p(u)].$$

Summarizing for $u\in\mathbb{R}$,

$$|u|p(u) = \Phi(u)+\Phi^*[p(u)].$$

B. $|u|=p^*(v)$. By Lemma 2.4, exchanging positions of v and u, Φ and Φ^* (p and p^*), and repeating the arguments of A, we get for $v\in\mathbb{R}$,

$$|v|p^*(v) = \Phi(p^*(v))+\Phi^*(v). \ \blacksquare$$

Theorem 2.11 ([KR]). *Given an Orlicz function Φ, then for all $v\in\mathbb{R}$*

$$\Phi^*(v)=\tilde{\Phi}(v).$$

Proof. For all $u,v\in\mathbb{R}$. By Theorem 2.8,

$$\Phi(u)+\Phi^*(v)\ge u|v|, \qquad \text{i.e.} \qquad \Phi^*(v)\ge u|v|-\Phi(u),$$

so for all $v\in\mathbb{R}$

$$\Phi^*(v)\ge\sup\{u|v|-\Phi(u): u\in\mathbb{R}\}=\tilde{\Phi}(v).$$

On the other hand, if $0 \le p^*(v) < +\infty$, by Theorem 2.10,

$$\Phi^*(v) = p^*(v)|v| - \Phi(p^*(v)) \le \sup\{u|v| - \Phi(u) : u \in \mathbb{R}\} = \tilde{\Phi}(v).$$

if $p^*(v) = +\infty$, $|v| \ge \beta_{\Phi^*} > 0$, $|v|p^*(v) = +\infty$, by the assumption $\alpha_\Phi < +\infty$ and $\Phi(p^*(v)) = \Phi(+\infty) = +\infty$, so

$$|v|p^*(v) = \Phi^*(v) + \Phi(p^*(v)),$$

thus

$$\Phi^*(v) = \sup\{u|v| - \Phi(u) : u \in \mathbb{R}\} = \tilde{\Phi}(v). \ \blacksquare$$

By Remark 2.2, we get

Remark 2.12 ([KR]). $\tilde{\Phi}$ is an Orlicz function, i.e. $\tilde{\Phi} : \mathbb{R} \to [0, +\infty]$ is even, convex and left continuous on $[0, +\infty)$ with $\tilde{\Phi}(0) = 0$.

By Remark 2.12 and Lemma 1.3, we have

Proposition 2.13 ([KR]). *For an Orlicz function Φ, $\tilde{\tilde{\Phi}} = \Phi$.*

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