ACTA ARITHMETICA
Online First version

An unconditional Montgomery theorem for pair correlation of zeros of the Riemann zeta-function

by

Siegfred Alan C. Baluyot (Pasadena, CA),
Daniel Alan Goldston (San José, CA),
Ade Irma Suriajaya (Fukuoka) and
Caroline L. Turnage-Butterbaugh (Northfield, MN)

Dedicated to Henryk Iwaniec on the occasion of his 75th birthday

Abstract. Assuming the Riemann Hypothesis (RH), Montgomery proved a theorem concerning pair correlation of zeros of the Riemann zeta-function. One consequence of this theorem is that, assuming RH, at least 67.9% of the nontrivial zeros are simple. Here we obtain an unconditional form of Montgomery's theorem and show how to apply it to prove the following result on simple zeros: If all the zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function such that $T^{3/8} < \gamma \le T$ satisfy $|\beta - 1/2| < 1/(2\log T)$, then, as T tends to infinity, at least 61.7% of these zeros are simple. The method of proof neither requires nor provides any information on whether any of these zeros are or are not on the critical line where $\beta = 1/2$. We also obtain the same result under the weaker assumption of a strong zero-density hypothesis.

1. Introduction and statement of results. Let $\rho = \beta + i\gamma$ denote a nontrivial zero of the Riemann zeta-function $\zeta(s)$ with $\beta, \gamma \in \mathbb{R}$, that is, a zero satisfying $\beta > 0$. The Riemann Hypothesis (RH) states that $\beta = 1/2$ for all ρ. To study the pair correlation of zeros of the zeta-function, Montgomery [Mon73] assumed RH and defined, for $x > 0$ and $T \ge 3$,

$$F(x,T) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} w(\gamma-\gamma'), \quad \text{where} \quad w(u) = \frac{4}{4+u^2}. \tag{1.1}$$

2020 *Mathematics Subject Classification*: Primary 11M06; Secondary 11M26.
Key words and phrases: Riemann zeta-function, zeros, pair correlation, simple zeros, zero-density.
Received 12 June 2023; revised 28 February 2024.
Published online 22 April 2024.

DOI: 10.4064/aa230612-20-3

Our first goal is to generalize Montgomery's pair correlation method so that it is unconditional. To this end, define, for $x > 0$ and $T \geq 3$,

$$F(x,T) := \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\leq T}} x^{\rho-\rho'} W(\rho-\rho'), \quad \text{where} \quad W(u) := \frac{4}{4-u^2}. \tag{1.2}$$

Here and throughout the paper, zeros are counted with multiplicity. Note that if RH holds then $\rho - \rho' = i(\gamma - \gamma')$ and $W(i(\gamma-\gamma')) = w(\gamma-\gamma')$, so that (1.2) agrees with (1.1).

Following Montgomery, we normalize $F(x,T)$ by defining, for real α,

$$F(\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} F(T^\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\leq T}} T^{\alpha(\rho-\rho')} W(\rho-\rho'). \tag{1.3}$$

The first result of this paper is the following unconditional theorem.

Theorem 1. *The function $F(\alpha)$ is real, even, and nonnegative. Moreover, as $T \to \infty$, we have*

$$F(\alpha) = (T^{-2\alpha}\log T + \alpha)\left(1 + O\left(\frac{1}{\sqrt{\log T}}\right)\right) \tag{1.4}$$

uniformly for $0 \leq \alpha \leq 1$.

Theorem 1 is nearly identical to Montgomery's theorem in [Mon73] and [GM87, Lemma 8] except it does not assume RH, and it includes the improvements from [GM87, Lemma 8] where (1.4) holds up to $\alpha = 1$ with explicit error terms. The proof, which we give in Section 2, is also nearly identical. See also [IK04]. A simple application of Theorem 1 concerns the multiplicities of the zeros of $\zeta(s)$. We use a slight modification of a kernel due to Tsang [Tsa93] to prove the following result.

Theorem 2. *Suppose that all the zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function with $T^{3/8} < \gamma \leq T$ lie within the thin box*

$$\frac{1}{2} - \frac{1}{2\log T} < \beta < \frac{1}{2} + \frac{1}{2\log T}. \tag{1.5}$$

Then for any sufficiently large $T > 0$, at least 61.7% of the nontrivial zeros are simple.

Remark. The pair correlation method developed in this paper neither requires nor provides any information as to whether or not the nontrivial zeros of $\zeta(s)$ satisfy $\beta = 1/2$.

There are many results concerning the proportion of nontrivial zeros of the Riemann zeta-function that are simple. Pratt, Robles, Zaharescu, and Zeindler [PRZZ20] have proved that more than 41.7% of the zeros are on the critical line, and also that more than 40.7% of the zeros are on the critical line and are simple. Conrey, Iwaniec, and Soundararajan [CIS13] have proved that more than $14/25 = 56\%$ of the nontrivial zeros of all Dirichlet L-functions are on the critical line and are simple. Going back to the case of the Riemann zeta-function, assuming RH, Montgomery [Mon73] deduced from Theorem 1 that more than $2/3 = 66.\overline{6}\%$ of the zeros are simple, and soon after, Montgomery and Taylor [Mon75] improved this to more than 67.2%. Recently Chirre, Gonçalves, and de Laat [CGdL20] obtained by this method 67.9%. By a mollifier method, Conrey, Ghosh, and Gonek [CGG98] showed on RH and an additional hypothesis that at least $19/27 = 70.\overline{3703}\%$ of the zeros are simple, and later Bui and Heath-Brown [BHB13] showed that this result holds on RH alone.

We can weaken the assumption that there are no zeros outside the box (1.5) by using a strong zero-density hypothesis. Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ with $\beta \geq \sigma$ and $0 < \gamma \leq T$.

THEOREM 3. *Assume that*

$$N(\sigma, T) = o(T^{2(1-\sigma)}(\log T)^{-1}) \quad \text{for} \quad \frac{1}{2} + \frac{1}{2\log T} \leq \sigma \leq \frac{25}{32} + \eta, \tag{1.6}$$

for any fixed $\eta > 0$. Then as $T \to \infty$, at least 61.7% of the nontrivial zeros of $\zeta(s)$ are simple.

Selberg [Sel91] made the conjecture that for all $\sigma > 1/2$ we have

$$N(\sigma, T) = O\bigg(T^{1-c(\sigma-1/2)}\frac{\log T}{\sqrt{\log\log T}}\bigg), \tag{1.7}$$

where $c > 0$ is some constant, and he stated that (1.7) can often be used as a replacement for RH in the Selberg class. To see this, note that the conjecture is expected to hold for *all* $\sigma > 1/2$ and thus implies that almost all the nontrivial zeros are on the critical line $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$. In a recent paper, Aryan [Ary22] used this type of conjecture as a replacement of RH to obtain Montgomery's result on simple zeros. Our density conjecture (1.6) implies that all except $o(T/\log T)$ of the zeros are in the box (1.5), and if we extend this conjecture to all $\sigma > 1/2 + \varepsilon/\log T$ for any $\varepsilon > 0$, we can also obtain Montgomery's simple zero result. Iwaniec and Kowalski [IK04, p. 249] made a weaker density conjecture that $N(\sigma, T) \ll T^{2(1-\sigma)}\log T$ for $1/2 \leq \sigma \leq 1$ and $T \geq 3$, which, however, is too weak to be used in Theorem 3.

2. Proof of Theorem 1. Recall that if ρ is a zero of $\zeta(s)$, then $1-\rho$, $\overline{\rho}$, and $1-\overline{\rho}$ are also zeros. Write

$$\rho = \beta + i\gamma := 1/2 + \delta + i\gamma,$$

where $-1/2 < \delta < 1/2$. If $\delta \neq 0$ and $\gamma > 0$, then there is another zero in the upper half-plane given by

$$1 - \overline{\rho} = 1/2 - \delta + i\gamma.$$

Therefore we may rewrite (1.2) as

$$\begin{aligned} F(x,T) &= \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} x^{\rho+\overline{\rho'}-1} W(\rho+\overline{\rho'}-1) \\ &= \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} x^{\delta+\delta'+i(\gamma-\gamma')} W(\delta+\delta'+i(\gamma-\gamma')). \end{aligned} \tag{2.1}$$

Lemma 1 (Montgomery). *Let $\rho = 1/2+\delta+i\gamma$. Then for $x \ge 1$ and all t we have*

$$\begin{aligned} &\sum_{\rho} \frac{2x^{\delta+i(\gamma-t)}}{1+((t-\gamma)+i\delta)^2} \\ &\qquad = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2+it}} \min\left\{\frac{n}{x}, \frac{x}{n}\right\} + x^{-1}(\log(|t|+2)+O(1)) \\ &\qquad\quad + O\left(\frac{x^{1/2}}{1+t^2}\right) + O\left(\frac{x^{-5/2}}{|t|+2}\right). \end{aligned} \tag{2.2}$$

Proof. This is the Lemma from [Mon73] if one takes $\sigma = 3/2$ and $\delta = 0$. The starting point for proving this lemma is the explicit formula due to Landau [Lan09] that, for $x > 1$ and $x \neq p^m$,

$$\sum_{\rho} \frac{x^{\rho-s}}{s-\rho} = \sum_{n\le x} \frac{\Lambda(n)}{n^s} + \frac{\zeta'}{\zeta}(s) - \frac{x^{1-s}}{1-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s}, \tag{2.3}$$

provided $s \neq 1$, $s \neq \rho$, $s \neq -2n$, which we henceforth assume. When $s = 0$, this is the usual explicit formula for primes. Writing $s = \sigma + it$ and $\rho = 1/2 + \delta + i\gamma$, we obtain

$$\sum_{\rho=1/2+\delta+i\gamma} \frac{x^{1/2+\delta-\sigma+i(\gamma-t)}}{\sigma-1/2-\delta+i(t-\gamma)} = R(\sigma+it),$$

where $R(\sigma+it)$ is the right-hand side of (2.3), which does not depend on ρ and is treated exactly as in [Mon73]. Multiplying both sides by $x^{\sigma-1/2}$, we obtain

$$\sum_{\rho} \frac{x^{\delta+i(\gamma-t)}}{\sigma-1/2-\delta+i(t-\gamma)} = x^{\sigma-1/2}R(\sigma+it).$$

Next, replace σ with $1-\sigma$ in the equation above, which adds the conditions $s \neq 0$ and $s \neq 2n+1$ and gives

$$\sum_{\rho} \frac{x^{\delta+i(\gamma-t)}}{1/2-\sigma-\delta+i(t-\gamma)} = x^{1/2-\sigma} R(1-\sigma+it).$$

Subtract this equation from the previous one and simplify to obtain

$$\sum_{\rho} \frac{(2\sigma-1)x^{\delta+i(\gamma-t)}}{(\sigma-1/2)^2+((t-\gamma)+i\delta)^2} = x^{\sigma-1/2}R(\sigma+it) - x^{1/2-\sigma}R(1-\sigma+it).$$

Taking $\sigma = 3/2$, we see that all the restrictions on s above are automatically satisfied, which gives the left-hand side of (2.2). The right-hand side is obtained exactly as in the original proof where it is shown that the restrictions $x \neq 1$ and $x \neq p^m$ may be removed. ∎

LEMMA 2. *Letting $N(T)$ denote the number of zeros in the upper half-plane up to height T, we have*

$$N(T) := \sum_{0<\gamma\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{2.4}$$

This is proved in many books, for instance [Ing90, Theorem 25], [Tit86, Theorem 9.4], or [MV07, Corollary 14.3]. In particular, we have

$$N(T) \sim \frac{T}{2\pi}\log T \quad \text{and} \quad N(T+1)-N(T) \ll \log T. \tag{2.5}$$

LEMMA 3. *We have, for $x>0$ and $T \ge 3$,*

$$F(x,T) = \frac{2}{\pi}\int_{-\infty}^{\infty} \Bigg| \sum_{\substack{\rho \\ 0<\gamma\le T}} \frac{x^{\rho-1/2}}{1-(\rho-(1/2+it))^2} \Bigg|^2 dt. \tag{2.6}$$

Proof. We will make use of the formula, with $a \in \mathbb{C}$ and $-1 < \operatorname{Im} a < 1$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(1+(t+a)^2)} \\ &\qquad = 2\pi i\left(\frac{1}{2i(1+(a+i)^2)} + \frac{1}{2i(1+(-a+i)^2)}\right) = \frac{2\pi}{4+a^2}, \end{aligned} \tag{2.7}$$

which is easily obtained by the residue theorem or Mathematica. Now, multiplying out the right-hand side of (2.6), we obtain

$$\frac{2}{\pi}\sum_{\substack{\rho,\rho'\\0<\gamma,\gamma'\le T}} x^{\rho+\overline{\rho'}-1}\int_{-\infty}^{\infty}\frac{dt}{\big(1-(\rho-(1/2+it))^2\big)\big(1-(\overline{\rho'}-(1/2-it))^2\big)}$$
$$=\frac{2}{\pi}\sum_{\substack{\rho,\rho'\\0<\gamma,\gamma'\le T}} x^{\rho+\overline{\rho'}-1}\int_{-\infty}^{\infty}\frac{dt}{\big(1+(t+i(\rho-1/2))^2\big)\big(1+(t-i(\overline{\rho'}-1/2))^2\big)}$$
$$=\frac{2}{\pi}\sum_{\substack{\rho,\rho'\\0<\gamma,\gamma'\le T}} x^{\rho+\overline{\rho'}-1}\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)\big(1+(t-i(\rho+\overline{\rho'}-1))^2\big)}$$
$$=\sum_{\substack{\rho,\rho'\\0<\gamma,\gamma'\le T}} x^{\rho+\overline{\rho'}-1}\frac{4}{4+(i(\rho+\overline{\rho'}-1))^2}=F(x,T)$$

by (2.1). ∎

We now rewrite (2.6) with $\rho=1/2+\delta+i\gamma$ so that

$$F(x,T)=\frac{2}{\pi}\int_{-\infty}^{\infty}\Bigg|\sum_{\substack{\rho\\0<\gamma\le T}}\frac{x^{\delta+i\gamma}}{1+((t-\gamma)+i\delta)^2}\Bigg|^2 dt. \tag{2.8}$$

Define

$$\Theta(t):=\max\{\beta:\rho=\beta+i\gamma,\ 0<|\gamma|\le t\}. \tag{2.9}$$

If RH is false then $\Theta(t)$ is a step function, and we often replace it with the "zero-free" region in the complex $s=\sigma+it$ plane

$$\sigma>1-\eta(t)\ge\Theta(t),\qquad t\ge 3, \tag{2.10}$$

where $0<\eta(t)\le 1/2$ and $\eta(t)$ is a continuous decreasing but not necessarily strictly decreasing function. Clearly we can make $\eta(t)$ as close to $1-\Theta(t)$ as we wish pointwise except at the jumps of $\Theta(t)$. We will often make use of this with $\rho=1/2+\delta+i\gamma$ in the form

$$\delta\le\Theta(t)-1/2\le 1/2-\eta(t)\quad\text{ for } 0<\gamma\le t. \tag{2.11}$$

Our next lemma gives the unconditional version of Montgomery's result that relates $F(x,T)$ to the explicit formula in Lemma 1.

Lemma 4. *For $x\ge 1$ and $T\ge 3$, let*

$$L(x,T):=\int_{0}^{T}\Bigg|\sum_{\rho}\frac{2x^{\delta+i\gamma}}{1+((t-\gamma)+i\delta)^2}\Bigg|^2 dt. \tag{2.12}$$

Then

$$F(x,T)=\frac{1}{2\pi}L(x,T)+O(x^{1-2\eta(T\log^2 T)}\log^3 T)+O(x). \tag{2.13}$$

Proof. We first truncate the sum over zeros in (2.12) using trivial estimates, which then allows us to apply Montgomery's argument without modification to the truncated sum. For $x \geq 1$, since $|\delta| < 1/2$, we have the trivial estimate

$$\left|\sum_{\rho} \frac{2x^{\delta+i\gamma}}{1+((t-\gamma)+i\delta)^2}\right| \ll x^{1/2} \sum_{\gamma} \frac{1}{1+(t-\gamma)^2} \ll x^{1/2} \log(|t|+2), \tag{2.14}$$

where the last estimate is well known and follows from the second estimate in (2.5). In the same way we have, for $0 \leq t \leq T$ and $Z \geq 2T$,

$$\left|\sum_{\substack{\rho \\ |\gamma|\geq Z}} \frac{2x^{\delta+i\gamma}}{1+((t-\gamma)+i\delta)^2}\right| \ll x^{1/2} \sum_{\gamma \geq Z} \frac{1}{\gamma^2} \ll \frac{x^{1/2}\log Z}{Z}. \tag{2.15}$$

Next, on squaring we have

$$L(x,T) = 4\sum_{\rho,\rho'} x^{\delta+\delta'+i(\gamma-\gamma')} \int_0^T \frac{dt}{\big(1+((t-\gamma)+i\delta)^2\big)\big(1+((t-\gamma')-i\delta')^2\big)}.$$

By (2.14) and (2.15) we can exclude the terms with $\gamma \notin [-Z,Z]$ in the sum above with an error

$$\ll \int_0^T \left|\sum_{\substack{\rho \\ |\gamma|\geq Z}} \frac{x^{\delta+i\gamma}}{1+((t-\gamma)+i\delta))^2}\right| \left|\sum_{\rho'} \frac{x^{\delta'-i\gamma'}}{1+((t-\gamma')-i\delta')^2}\right| dt \ll \frac{xT\log^2 Z}{Z}.$$

Taking $Z = T\log^2 T$, we conclude that

$$\begin{aligned} L(x,T) = 4 \sum_{\substack{\rho,\rho' \\ |\gamma|,|\gamma'|\leq Z}} & x^{\delta+\delta'+i(\gamma-\gamma')} \\ & \times \int_0^T \frac{dt}{\big(1+((t-\gamma)+i\delta)^2\big)\big(1+((t-\gamma')-i\delta')^2\big)} + O(x). \end{aligned}$$

Montgomery, arguing unconditionally except for taking $\delta = \delta' = 0$ in $L(x,T)$ and with no truncation, showed that the terms with $\gamma \notin [0,T]$ can be excluded with an error $O(\log^3 T)$, and then the range of integration can be extended to $\mathbb{R}$ with an error $O(\log^2 T)$. Here we apply the same argument where we need to include the factor

$$x^{\delta+\delta'+i(\gamma-\gamma')} \ll x^{2\Theta(Z)-1} \leq x^{1-2\eta(T\log^2 T)} \tag{2.16}$$

in the error term, at which point the bound for these error terms is majorized by dropping the truncation at Z, which then exactly matches Montgomery's argument. Thus by (2.8) we obtain

$$\begin{aligned} L(x,T) = 4 &\sum_{\substack{\rho,\rho' \\ 0\le\gamma,\gamma'\le T}} x^{\delta+\delta'+i(\gamma-\gamma')} \\ &\qquad\times \int_{-\infty}^{\infty} \frac{dt}{\big(1+((t-\gamma)+i\delta)^2\big)\big(1+((t-\gamma')-i\delta')^2\big)} \\ &+ O(x^{1-2\eta(T\log^2 T)}\log^3 T) + O(x) \\ &= 2\pi F(x,T) + O(x^{1-2\eta(T\log^2 T)}\log^3 T) + O(x). \ \blacksquare \end{aligned}$$

Proof of Theorem 1. Since $W(u)$ is even, we see from (1.2) that $F(1/x,T) = F(x,T)$, and therefore $F(\alpha)$ is even. That $F(\alpha)$ is real and nonnegative follows immediately from Lemma 3.

We write (2.2) as $l(x,t) = r(x,t)$, and define

$$L(x,T) := \int_0^T |l(x,t)|^2\,dt = \int_0^T |r(x,t)|^2\,dt =: R(x,T). \tag{2.17}$$

As we just saw in Lemma 4,

$$L(x,T) = 2\pi F(x,T) + O(x^{1-2\eta(T\log^2 T)}\log^3 T) + O(x).$$

The current widest known zero-free region $\sigma \ge 1-\eta(t)$ was obtained independently by Korobov and Vinogradov with

$$\eta(t) = \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}} \qquad \text{for } t \ge 3,$$

for some constant $c > 0$. Thus we see that, for $T^{1/2} \le x \le T$,

$$x^{1-2\eta(T\log^2 T)}\log^3 T \ll x\exp\bigg(-c\frac{\log x}{(\log x)^{2/3}(\log\log x)^{1/3}}\bigg)\log^3 x \ll x,$$

while, for $1 \le x \le T^{1/2}$,

$$x^{1-2\eta(T\log^2 T)}\log^3 T \ll T^{1/2},$$

since this error term is increasing in x and thus we may take $x = T^{1/2}$ in the previous bound. We conclude for $1 \le x \le T$ that

$$L(x,T) = 2\pi F(x,T) + O(T^{1/2}) + O(x). \tag{2.18}$$

Next, $R(x,T)$ does not depend on RH, and Montgomery [Mon73] proved unconditionally that

$$R(x,T) = (1+o(1))Tx^{-2}\log^2 T + T(\log x + O(1)) + O(x\log x).$$

In [GM87, Lemma 8] this was improved, so that for $1 \le x \le T$,

$$R(x,T) = (x^{-2}T\log^2 T + T\log x)\bigg(1 + O\bigg(\frac{1}{\sqrt{\log T}}\bigg)\bigg), \tag{2.19}$$

where we have removed an extraneous factor of $\log\log T$ which can be avoided by using Lemma 6 there in place of Lemma 7. Since $L(x,T) = R(x,T)$, from (2.18) and (2.19) we conclude

$$F(x,T) = (x^{-2}T\log^2 T + T\log x)\left(\frac{1}{2\pi} + O\left(\frac{1}{\sqrt{\log T}}\right)\right) + O(T^{1/2}) + O(x).$$

The last two error terms may be absorbed into the error term

$$(x^{-2}T\log^2 T + T\log x)O\left(\frac{1}{\sqrt{\log T}}\right) = O(x^{-2}T\log^{3/2} T) + O\left(T\frac{\log x}{\sqrt{\log T}}\right),$$

since, for $1 \le x \le T^{1/4}$,

$$T^{1/2} + x \ll T^{1/2} \ll x^{-2}T\log^{3/2} T,$$

while, for $T^{1/4} \le x \le T$,

$$T^{1/2} + x \ll T\frac{\log x}{\sqrt{\log T}}.$$

This proves Theorem 1 on using (1.3) to convert $F(x,T)$ to $F(\alpha)$. ∎

Remark. For more details on the proof of (2.19), see also [Gol81] and [LPZ17]. Montgomery and Vaughan in the forthcoming book *Multiplicative Number Theory II* have obtained a significantly refined version of Montgomery's theorem.

3. Sums over differences of zeros. For $z = x + iy$, $x, y \in \mathbb{R}$, define formally the Fourier transform $\widehat{g}(z)$ of $g(\alpha)$ by

$$\widehat{g}(z) = \int_{-\infty}^{\infty} g(\alpha)e(-z\alpha)\,d\alpha, \quad \text{where} \quad e(w) = e^{2\pi i w}. \tag{3.1}$$

Since

$$\widehat{g}(z) = \int_{-\infty}^{\infty} g(\alpha)e^{2\pi y\alpha}e(-x\alpha)\,d\alpha,$$

we see that if $|y| \le c$ and $g(\alpha)e^{2\pi c|\alpha|} \in L^1$ then $\widehat{g}(z)$ is an analytic function for all z with $|y| < c$. In this paper we take $g(\alpha)$ to have compact support and therefore $\widehat{g}(z)$ is an analytic function for all z. Taking $z = i(\rho - \rho')\frac{\log T}{2\pi}$, we have

$$\widehat{g}\left(i(\rho - \rho')\frac{\log T}{2\pi}\right) = \int_{-\infty}^{\infty} g(\alpha)T^{\alpha(\rho-\rho')}\,d\alpha.$$

Multiplying both sides of this equation by $W(\rho-\rho')$ and summing over $0<\gamma,\gamma'\le T$, we obtain

$$\sum_{\substack{\rho,\rho'\\0<\gamma,\gamma'\le T}} \widehat{g}\Big(i(\rho-\rho')\frac{\log T}{2\pi}\Big)W(\rho-\rho') = \Big(\frac{T}{2\pi}\log T\Big)\int_{-\infty}^{\infty} F(\alpha)g(\alpha)\,d\alpha. \tag{3.2}$$

We now apply Theorem 1 to obtain the following unconditional version of Montgomery's result on evaluating sums over pairs of zeros with even Fourier transform kernels obtained from functions supported in $[-1,1]$. If we assume RH, this agrees with the earlier version in [Mon73].

LEMMA 5. *Suppose $\alpha\in\mathbb{R}$ and $z\in\mathbb{C}$. Suppose $r(\alpha)$ is a real-valued even function in $L^1(\mathbb{R})$ with support in $[-1,1]$, and also that $r(\alpha)$ is Lipschitz continuous* (1) *at $\alpha=0$. Then $\widehat{r}(z)$ is an even analytic function,*

$$\widehat{r}(z) = 2\int_0^1 r(\alpha)\cos(2\pi z\alpha)\,d\alpha, \tag{3.3}$$

and we have

$$\sum_{\substack{\rho,\rho'\\0<\gamma,\gamma'\le T}} \widehat{r}\Big(i(\rho-\rho')\frac{\log T}{2\pi}\Big)W(\rho-\rho') = \frac{T}{2\pi}\log T\Big(r(0)+2\int_0^1 \alpha r(\alpha)\,d\alpha + O\Big(\frac{1}{\sqrt{\log T}}\Big)\Big). \tag{3.4}$$

Proof. In (3.2) we take $g(\alpha)=r(\alpha)$, and see that (3.3) follows from (3.1) by the evenness of r. For the integral in (3.2), applying Theorem 1 we have

$$\int_{-\infty}^{\infty} F(\alpha)r(\alpha)\,d\alpha = \Big(1+O\Big(\frac{1}{\sqrt{\log T}}\Big)\Big)\Big(2\int_0^1 (T^{-2\alpha}\log T)r(\alpha)\,d\alpha + 2\int_0^1 \alpha r(\alpha)\,d\alpha\Big).$$

Using the Lipschitz condition on $r(\alpha)$ at $\alpha=0$, we see that the first integral on the right-hand side is

$$= 2\int_0^{\log\log T/\log T} (T^{-2\alpha}\log T)r(\alpha)\,d\alpha + O\Big(\int_{\log\log T/\log T}^{1} (T^{-2\alpha}\log T)|r(\alpha)|\,d\alpha\Big)$$

(1) Recall that a function $f(x)$ is *Lipschitz continuous* at a point $x=a$ if there are constants $C>0$ and $\delta>0$ such that $|f(x)-f(a)|\le C|x-a|$ for all x in a neighborhood $|x-a|<\delta$ of a.

$$= 2\Big(r(0) + O\Big(\frac{\log\log T}{\log T}\Big)\Big) \int_0^{\log\log T/\log T} T^{-2\alpha} \log T \, d\alpha$$
$$+ O\Big(\frac{1}{\log T}\int_0^1 |r(\alpha)|\, d\alpha\Big)$$
$$= 2\Big(r(0) + O\Big(\frac{\log\log T}{\log T}\Big)\Big)\Big(\frac{1}{2} + O\Big(\frac{1}{\log T}\Big)\Big) + O\Big(\frac{1}{\log T}\Big)$$
$$= r(0) + O\Big(\frac{\log\log T}{\log T}\Big).$$

Lemma 5 now follows from (3.2). ■

4. Tsang's kernel. We define the *Tsang kernel* $K(z)$ through its Fourier transform by

$$\widehat{K}(t) := j(2\pi t)\operatorname{sech}(2\pi t), \tag{4.1}$$

where $j(\alpha)$ is an even, nonnegative, bounded function supported on $|\alpha| \le 1$, and we also assume j is twice differentiable on $[0, 1]$ with one-sided derivatives at the endpoints. We moreover require, for all $w \in \mathbb{R}$,

$$0 \le \widehat{j}(w) \ll \frac{1}{1+w^2}.$$

Thus the Tsang kernel $K(z)$ is

$$K(z) = \int_{-\infty}^{\infty} \widehat{K}(t) e(zt)\, dt = 2\int_0^{\infty} j(2\pi t)\operatorname{sech}(2\pi t)\cos(2\pi z t)\, dt \tag{4.2}$$
$$= \frac{1}{\pi}\int_0^1 j(\alpha)\operatorname{sech}(\alpha)\cos(z\alpha)\, d\alpha.$$

LEMMA 6 (K.-M. Tsang). *The kernel $K(z)$ is an even entire function such that:*

(a) $K(x) > 0$ *for all* $x \in \mathbb{R}$.
(b) *For* $z \in \mathbb{C} - \{0\}$, $K(z) \ll e^{|\operatorname{Im} z|}/|z|^2$.
(c) *For* $z = x + iy$, $x, y \in \mathbb{R}$, *when* $|y| < 1$, *we have* $\operatorname{Re} K(x+iy) > 0$.

Tsang proved this lemma [Tsa93, Lemma 1] with the function $\widehat{j}$ taken to be the Fejér kernel, that is,

$$j_F(\alpha) = \max\{0, 1-|\alpha|\}, \qquad \widehat{j}_F(w) = \Big(\frac{\sin(\pi w)}{\pi w}\Big)^2. \tag{4.3}$$

His proof initially proceeds with an unspecified function j enjoying the properties above except differentiability. The last part of the proof is where it is assumed that $j = j_F$; the proof is unchanged if the differentiability assumptions on j are used in place of using j_F.

At the end of Section 7, we will show how to improve the proportion of simple zeros that we first compute using the Fejér kernel by taking $\widehat{j}$ to be the *Montgomery–Taylor kernel* [Mon75, CG93] given by

$$j_M(\alpha) = \frac{1}{1-\cos\sqrt{2}}\left(\frac{1}{2\sqrt{2}}\sin(\sqrt{2}\, j_F(\alpha)) + \frac{1}{2} j_F(\alpha)\cos(\sqrt{2}\,\alpha)\right), \tag{4.4}$$

where $j_F(\alpha)$ is as in (4.3), and

$$\widehat{j}_M(w) = \frac{1}{1-\cos\sqrt{2}}\left(\frac{\sin\left(\frac{1}{2}(\sqrt{2}-2\pi w)\right)}{\sqrt{2}-2\pi w} + \frac{\sin\left(\frac{1}{2}(\sqrt{2}+2\pi w)\right)}{\sqrt{2}+2\pi w}\right)^2. \tag{4.5}$$

It is easily seen that $j_M(\alpha)$ satisfies all the conditions required for $j(\alpha)$ and therefore we may use (4.1) to obtain a Tsang kernel from j_M satisfying Lemma 6.

5. Application to sums over differences of zeros. Applying the Tsang kernel in Lemma 5 we obtain the following result.

LEMMA 7. *We have*

$$\begin{aligned} 2\pi \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<1/\log T}} &\operatorname{Re} K(-i(\rho-\rho')\log T) + \mathcal{S}(T) \\ &= \left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + O\left(\frac{1}{\sqrt{\log T}}\right)\right)\frac{T}{2\pi}\log T, \end{aligned} \tag{5.1}$$

where

$$\mathcal{S}(T) := 2\pi \operatorname{Re} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|\ge 1/\log T}} K(-i(\rho-\rho')\log T) W(\rho-\rho'), \tag{5.2}$$

and K and $\widehat{K}$ are given in (4.2) *and* (4.1). *Here* $\operatorname{Re} K > 0$ *for every term in the sum in* (5.1), *and* $\widehat{K} \ge 0$.

Proof. In Lemma 5 we take, for $\alpha \in \mathbb{R}$,

$$r(\alpha) = \widehat{K}\left(\frac{\alpha}{2\pi}\right) = j(\alpha)\operatorname{sech}(\alpha),$$

so that $\operatorname{supp} r \subset [-1,1]$ and

$$\widehat{r}(z) = 2\pi K(-2\pi z).$$

Thus (3.4) becomes

$$2\pi \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} K\big(-i(\rho-\rho')\log T\big)W(\rho-\rho') = \left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + O\left(\frac{1}{\sqrt{\log T}}\right)\right)\frac{T}{2\pi}\log T. \tag{5.3}$$

From Lemma 6 we have

$$\begin{aligned} &\operatorname{Re} K(-i(\rho-\rho')\log T) \\ &\quad = \operatorname{Re} K\big((\gamma-\gamma'-i(\beta-\beta'))\log T\big) > 0 \quad \text{if} \quad |\beta-\beta'| < 1/\log T. \end{aligned} \tag{5.4}$$

Since $W(\rho-\rho')$ is complex-valued for zeros off the critical line, before applying (5.4) we need to show that we can remove this weight with an acceptable error when $|\beta-\beta'| < \frac{1}{\log T}$. To prove this, note that for $z = x+iy$,

$$|W(z)| = \frac{4}{|4-z^2|} \le \frac{4}{|\operatorname{Re}(4-z^2)|} = \frac{4}{|4-x^2+y^2|},$$

and therefore

$$|W(\rho-\rho')| \le \frac{4}{4-(\beta-\beta')^2+(\gamma-\gamma')^2} \le \frac{4}{3+(\gamma-\gamma')^2} \ll w(\gamma-\gamma'), \tag{5.5}$$

where $w(u) = 4/(4+u^2)$ is from (1.1). Thus we have $|W(\rho-\rho')| \ll 1$, and also

$$\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T}} |W(\rho-\rho')| \ll \sum_{0<\gamma,\gamma'\le T} w(\gamma-\gamma') \ll T\log^2 T, \tag{5.6}$$

which follows from (2.5) as is done for the sum over zeros in (2.14).

Next, since $W(z)-1 = \frac{1}{4}z^2 W(z)$ and $W(0)=1$, Lemma 6(b) yields

$$\begin{aligned} &\sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<1/\log T}} K(-i(\rho-\rho')\log T)(W(\rho-\rho')-1) \\ &= \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<1/\log T}} K(-i(\rho-\rho')\log T)(W(\rho-\rho')-1) \\ &\ll \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|<1/\log T}} \frac{T^{|\beta-\beta'|}}{|\rho-\rho'|^2\log^2 T}\,\frac{|\rho-\rho'|^2}{|4-(\rho-\rho')^2|} \\ &\le \frac{T^{1/\log T}}{4\log^2 T} \sum_{\substack{\rho\ne\rho' \\ 0<\gamma,\gamma'\le T}} |W(\rho-\rho')| \ll \frac{1}{\log^2 T}\sum_{0<\gamma,\gamma'\le T} w(\gamma-\gamma') \ll T \end{aligned} \tag{5.7}$$

by (5.6). We can now remove $W(\rho-\rho')$ from the terms in (5.3) with $|\beta-\beta'| < 1/\log T$ with an error absorbed into the error term on the right-hand side. On taking real parts we obtain (5.1). ∎

REMARK. In applications we only need the bound in (5.7) to be $o(T\log T)$, which is obtained if we sum over all pairs of nontrivial zeros ρ,ρ' with $|\beta-\beta'| < \frac{(1-\varepsilon)\log\log T}{\log T}$ for any $\varepsilon > 0$.

6. Assumptions on zeros. In Theorem 2 we assume that all the zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function with $T^{3/8} < \gamma \le T$ lie within the thin box

$$\frac{1}{2} - \frac{1}{2\log T} < \beta < \frac{1}{2} + \frac{1}{2\log T}, \tag{6.1}$$

and in Theorem 3 we assume the strong zero-density hypothesis

$$N(\sigma,T) = o(T^{2(1-\sigma)}(\log T)^{-1}) \quad \text{for} \quad \frac{1}{2} + \frac{1}{2\log T} \le \sigma \le \frac{25}{32} + \eta, \tag{6.2}$$

for any fixed $\eta > 0$.

We now prove that either of these assumptions implies that, for any sufficiently large T,

$$\mathcal{S}(T) = 2\pi \operatorname{Re} \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\beta-\beta'|\ge 1/\log T}} K(-i(\rho-\rho')\log T) W(\rho-\rho') = o(T\log T). \tag{6.3}$$

To do this, we first prove that the density hypothesis (6.2) implies (6.3), and next show that the essentially stronger hypothesis (6.1) implies (6.2), and thus (6.3) by the first step. If we ignore the condition that $T^{3/8} < \gamma \le T$ in (6.1) then it is trivial that (6.3) holds since there are no terms in the sum and $\mathcal{S}(T) = 0$. The proof of (6.3) using the density hypothesis (6.2) needs to deal with the complication that while the condition $|\beta-\beta'| \ge \frac{1}{\log T}$ does exclude pairs of zeros where both are within the thin box in (6.1), it does not exclude pairs where one zero is in the box and the other is not. We use standard results and methods for applying zero-density results to explicit formulas, see [Ivi85, Chapters 11 and 12] and [IK04, Chapter 10].

By Lemma 6(b), we have

$$K(z) \ll \frac{e^{|\operatorname{Im} z|}}{|z|^2},$$

and therefore

$$K(-i(\rho-\rho')\log T) \ll \frac{T^{|\beta-\beta'|}}{((\beta-\beta')\log T)^2 + ((\gamma-\gamma')\log T)^2}.$$

By (5.5) we have $W(z) \ll 1$, and therefore

$$\begin{aligned}\mathcal{S}(T) &\ll \sum_{\substack{0<\gamma,\gamma'\le T\\ |\beta-\beta'|\ge 1/\log T}} \frac{T^{|\beta-\beta'|}}{((\beta-\beta')\log T)^2+((\gamma-\gamma')\log T)^2}\\ &\ll \sum_{\substack{0<\gamma,\gamma'\le T\\ |\beta-\beta'|\ge 1/\log T}} \frac{T^{|\beta-\beta'|}}{1+(\gamma-\gamma')^2} \ll \sum_{\substack{0<\gamma,\gamma'\le T\\ |\beta-\beta'|\ge 1/\log T}} T^{|\beta-\beta'|}w(\gamma-\gamma').\end{aligned}$$

As in (2.1), we now write $\delta=\beta-1/2$ so that $\rho=1/2+\delta+i\gamma$ with $|\delta|<1/2$, and thus

$$\mathcal{S}(T) \ll \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ |\delta-\delta'|\ge 1/\log T}} T^{|\delta-\delta'|}w(\gamma-\gamma').$$

Since $|ab|\le \frac{1}{2}(a^2+b^2)$, we have $T^{|\delta-\delta'|}\le T^{|\delta|}T^{|\delta'|}\le \frac{1}{2}(T^{2|\delta|}+T^{2|\delta'|})$. Thus

$$\mathcal{S}(T) \ll \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ |\delta-\delta'|\ge 1/\log T}} (T^{2|\delta|}+T^{2|\delta'|})w(\gamma-\gamma').$$

From the condition in the sum above we obtain $|\delta|+|\delta'|\ge|\delta-\delta'|\ge 1/\log T$, which implies that at least one of the conditions $|\delta|\ge\frac{1}{2\log T}$ or $|\delta'|\ge\frac{1}{2\log T}$ is true for each term (ρ,ρ') in the sum. Since all the terms are positive, we increase the number of terms by enforcing each of these conditions one at a time. Thus

$$\mathcal{S}(T) \ll \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ |\delta|\ge 1/(2\log T)}} (T^{2|\delta|}+T^{2|\delta'|})w(\gamma-\gamma') + \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ |\delta'|\ge 1/(2\log T)}} (T^{2|\delta|}+T^{2|\delta'|})w(\gamma-\gamma').$$

These sums are actually equal since switching ρ and ρ' converts one sum into the other. Using the first sum, we either have $|\delta'|<|\delta|$, in which case $T^{2|\delta'|}\ll T^{2|\delta|}$, or else $\frac{1}{2\log T}\le|\delta|\le|\delta'|$ and thus $T^{2|\delta|}\ll T^{2|\delta'|}$. Hence

$$\mathcal{S}(T) \ll \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ |\delta|\ge 1/(2\log T)\\ |\delta'|<|\delta|}} T^{2|\delta|}w(\gamma-\gamma') + \sum_{\substack{\rho,\rho'\\ 0<\gamma,\gamma'\le T\\ |\delta'|\ge 1/(2\log T)\\ |\delta|\le|\delta'|}} T^{2|\delta'|}w(\gamma-\gamma'),$$

and on dropping the condition $|\delta'|<|\delta|$ in the first sum and $|\delta|\le|\delta'|$ in the second sum, the two sums become the same and provide the upper bound

$$\mathcal{S}(T) \ll \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\le T \\ |\delta|\ge 1/(2\log T)}} T^{2|\delta|} w(\gamma-\gamma') \ll \sum_{\substack{\rho \\ 0<\gamma\le T \\ |\delta|\ge 1/(2\log T)}} T^{2|\delta|} \sum_{0<\gamma'\le T} w(\gamma-\gamma')$$
$$\ll \log T \sum_{\substack{0<\gamma\le T \\ |\beta-1/2|\ge 1/(2\log T)}} T^{|2\beta-1|},$$

where we used the estimate in (2.14). By the symmetry of the zeros on either side of the critical line, we conclude that

$$\mathcal{S}(T) \ll \log T \sum_{\substack{0<\gamma\le T \\ 1/2+1/(2\log T)\le\beta<1}} T^{2\beta-1}. \tag{6.4}$$

Applying Bourgain's [Bou00] zero-density estimate

$$N(\sigma,T) = o(T^{2(1-\sigma)}(\log T)^{-1}) \quad \text{for } 25/32+\eta\le\sigma\le 1, \tag{6.5}$$

which is the hypothesis (6.2) in the remaining range, we obtain

$$\begin{aligned} \mathcal{S}(T) &\ll \log T \int_{1/2+1/(2\log T)}^{1} T^{2u-1}\, d(-N(u,T)) \\ &= (T^{1/\log T}\log T) N\left(\frac{1}{2}+\frac{1}{2\log T}, T\right) \\ &\quad + 2\log^2 T \int_{1/2+1/(2\log T)}^{1} N(u,T) T^{2u-1}\, du \\ &= o(T\log T), \end{aligned}$$

which proves (6.3).

We now prove that the assumption (6.1) implies (6.2). Let $0<\eta<\frac{1}{32}$ be arbitrary and fixed, and suppose that

$$\frac{1}{2}+\frac{1}{2\log T}\le\sigma\le\frac{25}{32}+\eta.$$

Then by our hypothesis we have

$$\begin{aligned} N(\sigma,T) &= \#\{\rho=\beta+i\gamma : \beta\ge\sigma \text{ and } T^{3/8}<\gamma\le T\} \\ &\quad + \#\{\rho=\beta+i\gamma : \beta\ge\sigma \text{ and } 0<\gamma\le T^{3/8}\} \\ &= \#\{\rho=\beta+i\gamma : \beta\ge\sigma \text{ and } 0<\gamma\le T^{3/8}\} \\ &\le \#\{\rho=\beta+i\gamma : 0<\gamma\le T^{3/8}\}. \end{aligned}$$

The number on the last line is $N(T^{3/8})$, which is $O(T^{3/8}\log T)$ by Lemma 2.

Thus

$$N(\sigma, T) \ll T^{3/8+\varepsilon} = T^{2(1-26/32)+\varepsilon} = o(T^{2(1-\sigma)}(\log T)^{-1})$$

as $T \to \infty$, since

$$1 - \frac{26}{32} < 1 - \frac{25}{32} - \eta \leq 1 - \sigma.$$

Hence

$$N(\sigma, T) = o(T^{2(1-\sigma)}(\log T)^{-1}) \qquad \text{for } \frac{1}{2} + \frac{1}{2\log T} \leq \sigma \leq \frac{25}{32} + \eta \text{ and fixed } 0 < \eta < \frac{1}{32},$$

as $T \to \infty$. The estimate $N(\sigma, T) = o(T^{2(1-\sigma)}(\log T)^{-1})$ also holds for $\sigma \geq \frac{25}{32} + \varepsilon$, for any $\varepsilon > 0$, by [Bou00, p. 146]. Therefore

$$N(\sigma, T) = o(T^{2(1-\sigma)}(\log T)^{-1}) \qquad \text{for } \frac{1}{2} + \frac{1}{2\log T} \leq \sigma \leq \frac{25}{32} + \eta \text{ and fixed } \eta > 0,$$

as $T \to \infty$, which is (6.2).

7. Proof of Theorems 2 and 3. Let m_ρ denote the multiplicity of a zero ρ of $\zeta(s)$. Then

$$\sum_{\substack{\rho \\ 0<\gamma\leq T}} m_\rho = \sum_{\substack{\rho,\rho' \\ 0<\gamma,\gamma'\leq T \\ \rho=\rho'}} 1 = \frac{1}{K(0)} \sum_{\substack{\rho=\rho' \\ 0<\gamma,\gamma'\leq T}} \operatorname{Re} K(-i(\rho-\rho')\log T).$$

Next note trivially that if $\rho = \rho'$, then the zeros are within the range $|\beta - \beta'| < 1/\log T$. By Lemma 6(c) we also have $\operatorname{Re} K(-i(\rho - \rho')\log T) > 0$ in the same range. Therefore we may upper bound the sum on the right-hand side above by extending the sum to all zeros with $|\beta - \beta'| < 1/\log T$ and obtain

$$\sum_{\substack{\rho \\ 0<\gamma\leq T}} m_\rho \leq \frac{1}{K(0)} \sum_{\substack{\rho,\rho' \\ |\beta-\beta'|<1/\log T \\ 0<\gamma,\gamma'\leq T}} \operatorname{Re} K(-i(\rho-\rho')\log T).$$

We proved in the previous section that the assumption on zeros used in either one of Theorem 2 or Theorem 3 implies that $\mathcal{S}(T) = o(T\log T)$.

Therefore equation (5.1) of Lemma 7 gives

$$\frac{1}{K(0)} \sum_{\substack{\rho,\rho' \\ |\beta-\beta'|<1/\log T \\ 0<\gamma,\gamma'\leq T}} \operatorname{Re} K(-i(\rho-\rho')\log T) \sim \frac{1}{2\pi K(0)}\left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha\right) \frac{T}{2\pi}\log T,$$

which implies

$$\sum_{\substack{\rho \\ 0<\gamma\leq T}} m_\rho \leq \frac{1}{2\pi K(0)}\left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + o(1)\right) \frac{T}{2\pi}\log T \tag{7.1}$$

as $T \to \infty$. Following Montgomery's [Mon73] argument, we see that the number of zeros which are simple satisfies

$$\sum_{\substack{\rho\colon \text{simple} \\ 0<\gamma\leq T}} 1 \geq \sum_{\substack{\rho \\ 0<\gamma\leq T}} (2-m_\rho).$$

Hence, the proportion of simple zeros of $\zeta(s)$ is

$$\frac{1}{N(T)} \sum_{\substack{\rho\colon \text{simple} \\ 0<\gamma\leq T}} 1 \geq 2 - \frac{1}{N(T)} \sum_{\substack{\rho \\ 0<\gamma\leq T}} m_\rho,$$

which, since $N(T) \sim \frac{T}{2\pi}\log T$ by Lemma 2, shows by (7.1) that

$$\frac{1}{N(T)} \sum_{\substack{\rho\colon \text{simple} \\ 0<\gamma\leq T}} 1 \geq 2 - \frac{1}{2\pi K(0)}\left(\widehat{K}(0) + 2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha + o(1)\right). \tag{7.2}$$

Suppose first we take the Fejér kernel $j(\alpha) = j_F(\alpha)$. Then $\widehat{K}(0) = 1$ and computation gives

$$2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha = 2\int_0^1 \alpha(1-\alpha)\operatorname{sech}(\alpha)\, d\alpha = 0.2913876354\ldots.$$

Further, applying (4.2), we deduce upon computation that

$$\pi K(0) = \int_0^1 j(u)\operatorname{sech}(u)\, du = \int_0^1 (1-u)\operatorname{sech}(u)\, du = 0.4640648392\ldots.$$

Substituting these into (7.2) we have

$$\frac{1}{N(T)} \sum_{\substack{\rho\colon \text{simple} \\ 0<\gamma\leq T}} 1 \geq 2 - \frac{1.291387636 + o(1)}{2 \times 0.464064839} = 0.608612927\ldots + o(1).$$

Thus, subject to the hypotheses in Theorem 2 or Theorem 3, at least 60.8% of the zeros ρ of $\zeta(s)$ are simple.

We improve the above proportion to 61.7% using the Montgomery–Taylor kernel $j(\alpha) = j_M(\alpha)$. Computation gives

$$\widehat{K}(0) = j_M(0) = 1.0061271908\ldots,$$

$$2\int_0^1 \alpha \widehat{K}\left(\frac{\alpha}{2\pi}\right) d\alpha = 2\int_0^1 \alpha j_M(\alpha)\operatorname{sech}(\alpha)\, d\alpha = 0.2832624869\ldots,$$

$$\pi K(0) = \int_0^1 j_M(u)\operatorname{sech}(u)\, du = 0.4663199124\ldots.$$

Hence substituting these values into (7.2) as before, we have

$$\frac{1}{N(T)} \sum_{\substack{\rho:\,\text{simple}\\ 0<\gamma\le T}} 1 \ge 2 - \frac{1.289389678 + o(1)}{2\times 0.466319912} = 0.617483786\ldots + o(1).$$

Acknowledgements and funding. The authors wish to thank the referee for correcting some errors in the paper as well as suggesting a number of improvements in the arguments and exposition. The authors also thank the American Institute of Mathematics for its hospitality and for providing a pleasant research environment where most of the research was conducted.

The first author was supported by NSF DMS-1854398 FRG. The third author was supported by JSPS KAKENHI Grant Number 22K13895. The fourth author was partially supported by NSF DMS-1902193, NSF DMS-1854398 FRG, and NSF CAREER DMS-2239681.

References

[Ary22] F. Aryan, *On an extension of the Landau–Gonek formula*, J. Number Theory 233 (2022), 389–404.

[Bou00] J. Bourgain, *On large values estimates for Dirichlet polynomials and the density hypothesis for the Riemann zeta function*, Int. Math. Res. Notices 2000, 133–146.

[BHB13] H. M. Bui and D. R. Heath-Brown, *On simple zeros of the Riemann zeta-function*, Bull. London Math. Soc. 45 (2013), 953–961.

[CG93] A. Y. Cheer and D. A. Goldston, *Simple zeros of the Riemann zeta-function*, Proc. Amer. Math. Soc. 118 (1993), 365–372.

[CGdL20] A. Chirre, F. Gonçalves, and D. de Laat, *Pair correlation estimates for the zeros of the zeta function via semidefinite programming*, Adv. Math. 361 (2020), art. 106926, 22 pp.

[CGG98] J. B. Conrey, A. Ghosh, and S. M. Gonek, *Simple zeros of the Riemann zeta-function*, Proc. London Math. Soc. 76 (1998), 497–522.

[CIS13] J. B. Conrey, H. Iwaniec, and K. Soundararajan, *Critical zeros of Dirichlet L-functions*, J. Reine Angew. Math. 681 (2013), 175–198.

[Gol81] D. A. Goldston, *Large differences between consecutive prime numbers*, Thesis, U.C. Berkeley, 1981, 75 pp.

[GM87] D. A. Goldston and H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, in: Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), A. C. Adolphson et al. (eds.), Progr. Math. 70, Birkhäuser, Boston, MA, 1987, 183–203.

[Ing90] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge Math. Library, Cambridge Univ. Press, Cambridge, 1990 (reprint of the 1932 original).

[Ivi85] A. Ivić, *The Riemann Zeta-Function*, Wiley, New York, 1985.

[IK04] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. 53, Amer. Math. Soc., Providence, RI, 2004.

[Lan09] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Teubner, Berlin, 1909.

[LPZ17] A. Languasco, A. Perelli, and A. Zaccagnini, *An extended pair-correlation conjecture and primes in short intervals*, Trans. Amer. Math. Soc. 369 (2017), 4235–4250.

[Mon73] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, in: Analytic Number Theory (St. Louis, MO, 1972), Proc. Sympos. Pure Math. 24, Amer. Math. Soc., Providence, RI, 1973, 181–193.

[Mon75] H. L. Montgomery, *Distribution of the zeros of the Riemann zeta function*, in: Proc. Int. Congress of Mathematicians (Vancouver, 1975), Vol. 1, Canad. Math. Congr., Montréal, QC, 1975, 379–381.

[MV07] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, Cambridge, 2007.

[PRZZ20] K. Pratt, N. Robles, A. Zaharescu, and D. Zeindler, *More than five-twelfths of the zeros of ζ are on the critical line*, Results Math. Sci. 7 (2020), no. 1, art. 2, 74 pp.

[Sel91] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, in: Collected Papers, Vol. II, Springer, Berlin, 1991, 47–63.

[Tit86] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, New York, 1986.

[Tsa93] K.-M. Tsang, *The large values of the Riemann zeta-function*, Mathematika 40 (1993), 203–214.

Siegfred Alan C. Baluyot
American Institute of Mathematics
Caltech 8-32
Pasadena, CA 91125, USA
E-mail: sbaluyot@aimath.org

Daniel Alan Goldston
Department of Mathematics and Statistics
San José State University
San José, CA 95192-0103, USA
E-mail: daniel.goldston@sjsu.edu

Ade Irma Suriajaya
Faculty of Mathematics
Kyushu University
Fukuoka 819-0395, Japan
E-mail: adeirmasuriajaya@math.kyushu-u.ac.jp

Caroline L. Turnage-Butterbaugh
Mathematics and Statistics Department
Carleton College
Northfield, MN 55057, USA
E-mail: cturnageb@carleton.edu