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Renormalized solutions to a parabolic equation with mixed boundary constraints

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Abstract

We establish the existence and uniqueness of a renormalized solution to the parabolic equation

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = f \quad \text{ in } \Omega \times (0,T)$$

subject to a mixed boundary condition. Here $b(u)$ is a real function of u, $-\operatorname{div}(a(x,t,u,\nabla u))$ is of Leray–Lions type and f is an L^1-function. Then we compare the renormalized solution to two other notions of solution: distributional solution and weak solution.

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1. Introduction

In this paper, we deal with the existence and uniqueness of a renormalized solution to a parabolic equation with L^1-data. The topic of renormalized solutions has been of great interest since its first appearance in the work of DiPerna and Lions in [DL89]. The type of equation under investigation here is much inspired by [BGR16], with a newly added feature being that we consider a mixed boundary condition instead of the classical Dirichlet boundary condition. This extension has been employed in other parts of mathematical research such as [TOB12, CT18, Vit02, HDJKR16]. Closest to our consideration is [GO17], in which the authors established the existence and uniqueness of a renormalized solution to an elliptic equation with a mixed boundary condition (on a perforated domain). Mixed boundary conditions, besides being more general, pose certain technical challenges. Namely, one has to deal with traces of Sobolev functions and their various aspects. On top of that, our analysis becomes more involved in the presence of the time variable. A tool that is widely employed in this regard is the time regularization introduced in [BMR01]. However, here we explore a different technique developed in [BP05], which was observed in [BGR16, proof of Lemma 3.2] to be "simpler and shorter". Following the existence and uniqueness parts, we also compare the renormalized solution to two other notions of solutions: distributional solution and weak solution. We show that they coincide when the data are sufficiently smooth.

Now we formulate our problem precisely. Let $d \in \mathbb{N}$, $T > 0$ and $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is a non-empty open subset of $\partial\Omega$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. Denote

$$Q_T := \Omega \times (0,T), \quad \Sigma_0 := \Gamma_0 \times (0,T) \quad \text{and} \quad \Sigma_1 := \Gamma_1 \times (0,T).$$

Let $\vec{n}$ be the unit outward normal vector on $\partial\Omega$. Consider the problem

$$(P) \qquad \begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \Sigma_0, \\ a(x,t,u,\nabla u) \cdot \vec{n} + \gamma(x,t)\, h(u) = g & \text{on } \Sigma_1, \\ b(u(x,0)) = b(u_0(x)) & \text{in } \Omega \end{cases}$$

subject to the following structure:

(H1) $f \in L^1(Q_T)$, $g \in L^1(\Sigma_1)$, $0 \le \gamma \in L^\infty(\Sigma_1)$ and $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function with $h(0) = 0$.

(H2) $a : Q_T \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function, in the following sense: a is measurable with respect to (x,t) for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^d$ and continuous with respect to (s,ξ) for a.e. $(x,t) \in Q_T$.

Let $p \in (1,\infty)$ be such that

$$p > \frac{2d}{d+2} \tag{1}$$

and p' its Hölder conjugate index.

There exists a function $K \in L^{p'}(Q_T)$ such that for all $k > 0$ and $s \in [-k,k]$ there exists a constant $\nu_k = \nu(k) > 0$ such that

$$|a(x,t,s,\xi)| \leq \nu_k(K(x,t) + |\xi|^{p-1}) \tag{2}$$

for a.e. $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}^d$.

There exists a constant $\alpha > 0$ such that

$$a(x,t,s,\xi) \cdot \xi \geq \alpha\,|\xi|^p \tag{3}$$

for a.e. $(x,t) \in Q_T$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^d$.

We have

$$(a(x,t,s,\xi) - a(x,t,s,\eta)) \cdot (\xi - \eta) > 0 \tag{4}$$

for a.e. $(x,t) \in Q_T$ and for all $\xi, \eta \in \mathbb{R}^d$ with $\xi \neq \eta$.

(H3) $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function with $b(0) = 0$. Moreover, there exists a constant $\beta > 0$ such that $b'(s) \geq \beta$ for all $s \in \mathbb{R}$.

(H4) $u_0 : \Omega \to \mathbb{R}$ is measurable such that $b(u_0) \in L^1(\Omega)$.

(H5) Γ_0 supports the Poincaré inequality: There exists a constant $H_0 > 0$ such that

$$\|v\|_{W^{1,p}(\Omega)} \leq H_0 \|\nabla v\|_{L^p(\Omega)}$$

for all $v \in W^{1,p}_{\Gamma_0}(\Omega)$, where

$$W^{1,p}_{\Gamma_0}(\Omega) := \{v \in W^{1,p}(\Omega) : v|_{\Gamma_0} = 0\}$$

and $v|_{\Gamma_0}$ is understood in the trace sense.

Two remarks are immediate.

Remark 1.1. In (H2), no growth restriction is imposed on the s-variable of a. In (H3), b is allowed to grow to infinity. Hence a and b present two unbounded nonlinearities in our equation.

Remark 1.2. The study of the space $W^{1,p}_{\Gamma_0}(\Omega)$ is interesting in its own right. We list here some essential facts.

- It is well-known that if the boundary of Γ_0 in $\partial\Omega$ is sufficiently smooth, then (H5) holds. Investigations along this line can be found in, for example, [TOB12] and [CT18]. In particular, a sufficient condition for (H5) is that Γ_0 satisfies the corkscrew condition relative to $\partial\Omega$, in the sense of [CT18, Definition 2.1] or [TOB12, (1.4)]. Also see the paragraph preceding [TOB12, Lemma 3.7].
- If $p = 2$ then (H5) always holds regardless of the geometry of Γ_0 (cf. [SS11, Corollary 2.5.8]).

- In view of (H5), we can replace the usual norm $\|\cdot\|_{W^{1,p}(\Omega)}$ of $W^{1,p}_{\Gamma_0}(\Omega)$ with the equivalent one given by $\|\nabla\cdot\|_{L^p(\Omega)}$. We will follow this practice from here onward.

The functional setting for the solutions to (P) is appropriately described by the Sobolev space

$$V := L^p(0,T;W^{1,p}_{\Gamma_0}(\Omega)).$$

In what follows, the dual space of V is denoted by V^*, i.e.,

$$V^* := L^{p'}(0,T;(W^{1,p}_{\Gamma_0}(\Omega))^*).$$

Note that the range for p in (1) is to guarantee the existence of the Gelfand triple

$$W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega),$$

where $\hookrightarrow\hookrightarrow$ and $\hookrightarrow$ denote the compact and continuous embeddings respectively. Hence we may apply the Aubin–Lions embeddings in what follows.

In this paper, we are interested in renormalized solutions to (P) whose definition is given next.

For each $k > 0$, define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(r) := \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } -k < r < k, \\ -k & \text{if } r \leq -k. \end{cases}$$

For each function $v \in V$ and for a.e. $t \in (0,T)$, the values of $v(t)$ on Γ_1 are understood in the trace sense.

Definition 1.3. A *renormalized solution* to (P) is a measurable function $u : Q_T \to \mathbb{R}$ with the following properties:

(R1) $b(u) \in L^\infty(0,T;L^1(\Omega))$.
(R2) For each $k > 0$ we have $T_k(u) \in V$ and

$$\begin{aligned} -\int_{Q_T} B_S(u)\varphi_t \,dx\,dt + \int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla(S'(u)\varphi)\,dx\,dt + \int_{\Sigma_1} \gamma h(u)S'(u)\varphi\,d\sigma(x)\,dt \\ = \int_{Q_T} fS'(u)\varphi\,dx\,dt + \int_{\Sigma_1} gS'(u)\varphi\,d\sigma(x)\,dt + \int_\Omega B_S(u_0)\varphi(0)\,dx \end{aligned} \tag{5}$$

for all $S \in W^{2,\infty}(\mathbb{R})$ such that $\operatorname{supp} S'$ is compact and $\varphi \in C_c^\infty([0,T)\times\Omega_1)$, where $\Omega_1 := \Omega \cup \Gamma_1$, $d\sigma(x)$ denotes the surface measure on Γ_1 and

$$B_S(\ell) := \int_0^\ell b'(s)S'(s)\,ds \tag{6}$$

for all $\ell \in \mathbb{R}$.
(R3) One has

$$\lim_{k\to\infty} \int_{[k\leq|u|\leq k+1]} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt = 0,$$

where

$$[k \leq |u| \leq k+1] := \{(x,t) \in Q_T : k \leq |u(x,t)| \leq k+1\}.$$

Two remarks are in order.

Remark 1.4. Let u be a renormalized solution to (P). Then each term in (5) is well-defined. Indeed, we have the following:

- $B_S(u) \in L^1(Q_T)$ since $|B_S(u)| \le \|S'\|_{L^\infty(\mathbb{R})}|b(u)| \in L^1(Q_T)$. Similarly, $B_S(u_0) \in L^1(\Omega)$.
- $\gamma h(u)S'(u) \in L^\infty(\Sigma_1)$, $gS'(u) \in L^1(\Sigma_1)$ and $fS'(u) \in L^1(Q_T)$.
- Let $k > 0$ be such that $\operatorname{supp} S' \subset [-k, k]$. Using (2), we get

$$\begin{aligned}
&\left|\int_{Q_T} a(x,t,u,\nabla u)\cdot \nabla(S'(u)\varphi)\,dx\,dt\right| \\
&\quad = \left|\int_{Q_T} a(x,t,u,\nabla u)\cdot(\nabla u)S''(u)\varphi + a(x,t,u,\nabla u)\cdot S'(u)\nabla\varphi\,dx\,dt\right| \\
&\quad = \left|\int_{Q_T} a(x,t,T_k(u),\nabla T_k(u))\cdot(\nabla T_k(u))S''(u)\varphi\right. \\
&\qquad\qquad \left. + a(x,t,T_k(u),\nabla T_k(u))\cdot S'(u)\nabla\varphi\,dx\,dt\right| \\
&\quad \le \nu_k \|S\|_{W^{2,\infty}(\mathbb{R})}\left(\int_{Q_T}(K|\nabla T_k(u)| + |\nabla T_k(u)|^p)|\varphi| + (K + |\nabla T_k(u)|^{p-1})|\nabla\varphi|\,dx\,dt\right) \\
&\quad < \infty,
\end{aligned}$$

where we have used the fact that $T_k(u) \in V$ by (R2). See Lemma 2.2 for more properties of $T_k(u)$.

Remark 1.5. Definition 1.3 is in the spirit of [BGR16, Definition 2.1], which at first sight appears to be different from other definitions of renormalized solutions commonly used in the literature (cf. [BWZ10, Definition 3.1], for example). In fact, the two definitions are equivalent in view of Lemmas 3.10 and 5.2 below.

We first discuss the existence and uniqueness of a renormalized solution to (P) for L^1-data. The existence part is as follows.

Theorem 1.6. *Let $d \in \{1, 2, \ldots\}$, $T > 0$ and $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is a nonempty open subset of $\partial\Omega$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. Assume* (H1)–(H5)*. Then there exists a renormalized solution to (P).*

Under extra assumptions on b and a, uniqueness is also achieved.

Theorem 1.7. *Adopt the assumptions from Theorem* 1.6*. Assume further that*

(i) $\lim_{s\to\infty} b(s) = \infty$, $\lim_{s\to-\infty} b(s) = -\infty$ *and b' is locally Lipschitz,*
(ii) *for all $k > 0$ there exist a constant $\zeta_k \ge 0$ and a function $E_k \in L^{p'}(Q_T)$ such that*

$$|a(x,t,s,\xi) - a(x,t,s',\xi)| \le |s - s'|[E_k(x,t) + \zeta_k|\xi|^{p-1}] \tag{7}$$

for a.e. $(x,t) \in Q_T$, for all s, s' with $|s|, |s'| \le k$ and for each $\xi \in \mathbb{R}^d$.

Let u and $\tilde{u}$ be renormalized solutions to (P). Then $u = \tilde{u}$.

For comparison, we also define distributional and weak solutions to (P).

Definition 1.8. Let $1 < q \le p$. A function $u \in L^q(0,T; W^{1,q}_{\Gamma_0}(\Omega))$ is called a *distributional solution* to (P) if the following properties hold:

(W1) We have
$$\begin{cases} b(u)|_{t=0} = b(u_0) & \text{in } \Omega, \\ \dfrac{\partial}{\partial t} b(u) \in V^*. \end{cases}$$

(W2) We have
$$\int_{Q_T} \Big(\frac{\partial}{\partial t} b(u)\Big) \varphi \, dx \, dt + \int_{Q_T} a(x,t,u,\nabla u) \cdot \nabla \varphi \, dx \, dt + \int_{\Sigma_1} \gamma h(u) \varphi \, d\sigma(x) \, dt$$
$$= \int_{Q_T} f \varphi \, dx \, dt + \int_{\Sigma_1} g \varphi \, d\sigma(x) \, dt \qquad (8)$$

for all $\varphi \in C_c^\infty([0,T) \times \Omega_1)$, where $\Omega_1 := \Omega \cup \Gamma_1$.

If $q = p$ then u is also called a *weak solution* to (P).

REMARK 1.9. Let $q \in (1, \infty)$ and
$$\mathcal{W}_q := \big\{ w \in L^q(0,T; W^{1,q}_{\Gamma_0}(\Omega)) : w_t \in L^{q'}(0,T; W^{-1,q'}(\Omega)) + L^1(Q_T) \big\}. \qquad (9)$$
Then an obvious modification of [Por99, proof of Theorem 1.1] verifies that
$$\mathcal{W}_q \hookrightarrow C([0,T]; L^1(\Omega)). \qquad (10)$$
Hence the initial condition in (W1) makes sense.

The next result relates the three notions of solutions to (P).

PROPOSITION 1.10. *Adopt the assumptions from Theorem* 1.6. *Assume further that* $f \in L^{p'}(Q_T)$ *and* $g \in L^{p'}(\Sigma_1)$. *Then a weak solution to* (P) *is also a renormalized solution.*

Moreover, if in addition $p > 2 - \frac{1}{d+1}$, *then a renormalized solution to* (P) *is also a distributional solution.*

REMARK 1.11. The assumptions $f \in L^{p'}(Q_T)$ and $g \in L^{p'}(\Sigma_1)$ guarantee that (8) continues to hold for all test functions $\varphi \in V$ when $u \in V$. Hence we obtain the usual formulation for a weak solution to (P).

In the "moreover" part, we require $p > 2 - \frac{1}{d+1}$ so that a technical regularity estimate is valid. See Lemma 5.1 below.

The paper is outlined as follows. In Section 2 we discuss the space $W^{1,p}_{\Gamma_0}(\Omega)$ and its related embeddings in more details. Theorems 1.6 and 1.7 are proved in Sections 3 and 4 respectively. Proposition 1.10 is proved in Section 5.

Standing assumptions. In the whole paper, $d \in \{1, 2, \ldots\}$ and $\Omega \subset \mathbb{R}^d$ is open and bounded with Lipschitz boundary. We always assume that (H1)–(H5) hold.

If further assumptions are required in certain statements, we will state them explicitly therein.

Notation. We employ the following notation:

- $\Omega_1 = \Omega \cup \Gamma_1$.
- dx is the Lebesgue measure on Ω and $d\sigma(x)$ is the surface measure on $\partial\Omega$.
- $|E|$ is the Lebesgue measure of E for all measurable $E \subset Q_T$.
- $\sigma(E)$ is the surface measure of E for all measurable $E \subset \Sigma_1$.
- $V = L^p(0,T; W^{1,p}_{\Gamma_0}(\Omega))$ and $V^* = L^{p'}(0,T; W^{-1,p'}_{\Gamma_0}(\Omega))$.

2. Function spaces

Recall that

$$W^{1,p}_{\Gamma_0}(\Omega) := \{u \in W^{1,p}(\Omega) : u|_{\Gamma_0} = 0\},$$

where $u|_{\Gamma_0}$ is understood in the trace sense. An equivalent realization of $W^{1,p}_{\Gamma_0}(\Omega)$ is

$$W^{1,p}_{\Gamma_0}(\Omega) = \overline{\{u \in C^1(\overline{\Omega}) : u|_{\Gamma_0} = 0\}}^{W^{1,p}(\Omega)}.$$

Define

$$p^*_\Omega := \begin{cases} \infty & \text{if } d \le p, \\ \dfrac{pd}{d-p} & \text{if } d > p \end{cases} \quad \text{and} \quad p^*_{\Gamma_1} := \begin{cases} \infty & \text{if } d \le p, \\ \dfrac{p(d-1)}{d-p} & \text{if } d > p. \end{cases}$$

The next lemma explains the trace embedding and tells us that the space $W^{1,p}_{\Gamma_0}(\Omega)$ behaves similarly to $W^{1,p}_0(\Omega)$ in several respects. See [AF03, Theorems 4.12 and 6.3] for more details.

Lemma 2.1.

(i) *The embedding*

$$W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Omega)$$

is continuous for all $q \in [1, p^*_\Omega]$ *and is compact if* $q \in [1, p^*_\Omega)$.

(ii) *The embedding*

$$W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1)$$

is continuous for all $q \in [1, p^*_{\Gamma_1}]$ *and is compact if* $q \in [1, p^*_{\Gamma_1})$.

Given a constant $k > 0$ and a measurable function u defined on a set D, we write

$$[u \ge k] := \{x \in D : u(x) \ge k\}.$$

A similar convention applies to

$$[u \le k], \quad [u > k] \quad \text{and} \quad [u < k].$$

The following lemma gives a meaning to the gradient of a function by means of truncation.

Lemma 2.2. *Let* $u : Q_T \to \mathbb{R}$ *be a measurable function such that* $T_k(u) \in V$ *for all* $k > 0$. *Then there exists a unique measurable function* $v : \Omega \to \mathbb{R}^d$ *such that*

$$\nabla T_k(u) = v\, \mathbb{1}_{[|u|<k]} \quad \text{a.e. in } Q_T$$

for all $k > 0$, *where*

$$\mathbb{1}_{[|u|<k]} := \begin{cases} 1 & \text{on } [|u| < k], \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if

$$\sup_{k \ge 1} \tfrac{1}{k} \|T_k(u)\|^p_V < \infty \tag{11}$$

then there exists a unique measurable function $w : \Sigma_1 \to \mathbb{R}$ *such that*

$$T_k(u)|_{\Gamma_1} = T_k(w) \quad \text{a.e. in } \Sigma_1$$

for all $k > 0$.

Proof. The first statement is a consequence of [BBG+95, Lemma 2.1].

To prove the second statement, observe that

$$k^p \sigma([T_k(u)|_{\Sigma_1} \geq k]) \leq \|T_k(u)|_{\Sigma_1}\|^p_{L^p(\Sigma_1)} \tag{12}$$

for all $k > 0$.

Next we infer from (11) that there exists a constant $C > 0$ such that

$$\|T_k(u)\|^p_V < Ck$$

for all $k \geq 1$. For a.e. $\tau \in (0, T)$, one has $T_k(u)(\tau) \in W^{1,p}_{\Gamma_0}(\Omega)$ and hence we may apply Lemma 2.1(ii) to obtain

$$\|T_k(u)(\tau)|_{\Gamma_1}\|^p_{L^p(\Gamma_1)} \leq S_0 \|T_k(u)(\tau)\|^p_{W^{1,p}_{\Gamma_0}(\Omega)}.$$

Integrating both sides of this estimate with respect to τ over $(0, T)$, we arrive at

$$\|T_k(u)|_{\Sigma_1}\|^p_{L^p(\Sigma_1)} \leq S_0 \|T_k(u)\|^p_V.$$

Consequently,

$$\|T_k(u)|_{\Sigma_1}\|^p_{L^p(\Sigma_1)} \leq CS_0 k \tag{13}$$

for all $k \geq 1$.

Combining (12) and (13) yields

$$\sigma([T_k(u)|_{\Sigma_1} \geq k]) \leq CS_0 k^{1-p}$$

for all $k \geq 1$. This implies

$$\lim_{k \to \infty} \sigma([T_k(u)|_{\Sigma_1} \geq k]) = 0.$$

Hence, we may decompose

$$\Sigma_1 = \bigcup_{\ell > 0} [T_k(u)|_{\Sigma_1} < \ell] =: \bigcup_{\ell > 0} \Gamma_{1,\ell}. \tag{14}$$

For all $0 < \ell_1 < \ell_2$ we have

$$T_{\ell_1}(T_{\ell_2}(u)|_{\Sigma_1}) = T_{\ell_1}(T_{\ell_2}(u|_{\Sigma_1})) = T_{\ell_1}(u|_{\Sigma_1}) = T_{\ell_1}(u)|_{\Sigma_1} \quad \text{a.e. in } \Sigma_1.$$

Note that $T_{\ell_2}(u)|_{\Sigma_1}$ is well-defined in view of Lemma 2.1(ii). This together with (14) justifies the unique existence of measurable function $w : \Gamma_1 \to \mathbb{R}$ such that

$$T_k(u)|_{\Sigma_1} = T_k(w) \quad \text{a.e. in } \Sigma_1.$$

This finishes our proof. ■

Remark 2.3. Let u, v and w be as in Lemma 2.2. In what follows, we define

$$\nabla u := v \quad \text{and} \quad u|_{\Sigma_1} := w.$$

3. Existence of renormalized solutions

In this section, we prove Theorem 1.6. The procedure is to first consider an approximate problem of (P) and then to perform limiting arguments to achieve the existence of a renormalized solution to (P). We divide this section into three subsections. Subsection 3.1 provides the existence of a weak solution to the approximate problem. The limiting arguments are presented in Subsection 3.2. Finally, Theorem 1.6 is proved in Subsection 3.3.

3.1. Approximation. Let $\epsilon > 0$. Define

- $b_\epsilon(r) = T_{1/\epsilon}(b(r)) + \epsilon\, r$ for all $r \in \mathbb{R}$,
- $a_\epsilon(x,t,s,\xi) = a(x,t,T_{1/\epsilon}(s),\xi)$ for a.e. $(x,t) \in Q_T$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^d$,
- $f_\epsilon = T_{1/\epsilon} f \in L^\infty(Q_T)$, whence $f_\epsilon \to f$ in $L^1(Q_T)$ as $\epsilon \to 0^+$,
- $g_\epsilon = T_{1/\epsilon} g \in L^\infty(Q_T)$, whence $g_\epsilon \to g$ in $L^1(\Sigma_1)$ as $\epsilon \to 0^+$,
- $h_\epsilon \in C^1(\mathbb{R})$ such that $h_\epsilon(0) = 0$, $|h_\epsilon| \le M_\epsilon$ for some constant $M_\epsilon > 0$ depending on ϵ and $h_\epsilon \to h$ uniformly on $[-K,K]$ for all $K > 0$,
- $u_{0\epsilon} \in C_c^\infty(\Omega)$ such that $b_\epsilon(u_{0\epsilon}) \to b(u_0)$ in $L^1(Q_T)$ as $\epsilon \to 0^+$. Hence, there exists a constant $0 < \epsilon_0 \le 1$ such that
$$\|b_\epsilon(u_{0\epsilon})\|_{L^1(\Omega)} \le \|b(u_0)\|_{L^1(\Omega)} + 1 \tag{15}$$
for all $0 < \epsilon < \epsilon_0$.

Note that a_ϵ and b_ϵ so defined satisfy similar types of estimates specified in (H2) and (H3) respectively, with obvious dependence of the constants involved on ϵ.

Consider the approximate problem
$$(P_\epsilon) \qquad \begin{cases} \dfrac{\partial b_\epsilon(u_\epsilon)}{\partial t} - \operatorname{div}(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)) = f_\epsilon & \text{in } Q_T, \\ u_\epsilon(x,t) = 0 & \text{on } \Sigma_0, \\ a(x,t,u_\epsilon,\nabla u_\epsilon)\cdot \vec{n} + \gamma(x,t)\, h_\epsilon(u_\epsilon) = g_\epsilon & \text{on } \Sigma_1, \\ b_\epsilon(u_\epsilon(x,0)) = b_\epsilon(u_{0\epsilon}(x)) & \text{in } \Omega. \end{cases}$$

A weak solution to (P_ϵ) is defined as follows.

Definition 3.1. A function $u_\epsilon \in V$ is called a *weak solution* to (P_ϵ) if the following properties hold:

(W1ϵ) We have
$$\begin{cases} b_\epsilon(u_\epsilon)|_{t=0} = b_\epsilon(u_{0\epsilon}) & \text{in } \Omega, \\ \dfrac{\partial}{\partial t} b_\epsilon(u_\epsilon) \in V^*. \end{cases}$$

(W2ϵ) We have
$$\int_{Q_T} \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}\varphi\,dx\,dt + \int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt + \int_{\Sigma_1} \gamma\, h_\epsilon(u_\epsilon)\varphi\,d\sigma(x)\,dt = \int_{Q_T} f_\epsilon\varphi\,dx\,dt + \int_{\Sigma_1} g_\epsilon\varphi\,d\sigma(x)\,dt \tag{16}$$
for all $\varphi \in V$.

REMARK 3.2. The initial condition in (W1ϵ) makes sense in view of (10).

We aim to show that (P_ϵ) has a weak solution. In the course of proof, we make use of the following classical result whose proof is presented for the sake of clarity.

LEMMA 3.3. *Let* $(u_n)_{n\in\mathbb{N}} \subset V$ *be such that*

$$\lim_{n\to\infty} u_n = u \qquad \textit{weakly in } V$$

and

$$\lim_{n\to\infty} \int_{Q_T} (a_\epsilon(x,t,u_n,\nabla u_n) - a_\epsilon(x,t,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx\, dt = 0.$$

Then

$$\lim_{n\to\infty} u_n = u \qquad \textit{in } V$$

up to a subsequence.

Proof. Using the fact that $u_n \to u$ weakly in V as $n \to \infty$ and the compact embedding in Lemma 2.1(ii), we infer that

$$\lim_{n\to\infty} u_n = u \qquad \begin{cases} \text{a.e. in } Q_T \text{ (up to a subsequence)},\\ \text{in } L^p(Q_T). \end{cases} \tag{17}$$

Next we set

$$\begin{aligned} L &:= (a_\epsilon(x,t,u_n,\nabla u_n) - a_\epsilon(x,t,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\\ &= a_\epsilon(x,t,u_n,\nabla u_n)\cdot \nabla u_n + a_\epsilon(x,t,u_n,\nabla u)\cdot\nabla u\\ &\quad - a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u - a_\epsilon(x,t,u_n,\nabla u)\cdot\nabla u_n. \end{aligned} \tag{18}$$

Then $L \geq 0$ by (4), whence $L \to 0$ in $L^1(Q_T)$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} L = 0 \qquad \text{a.e. in } Q_T. \tag{19}$$

Furthermore, using (18) and then referring to (3) and (2) yields

$$\begin{aligned} L &\geq \alpha(|\nabla u_n|^p + |\nabla u|^p) - \nu_\epsilon(K + |\nabla u_n|^{p-1})|\nabla u| - \nu_\epsilon(K+|\nabla u|^{p-1})|\nabla u_n|\\ &= |\nabla u_n|^p\left(\alpha + \frac{\alpha|\nabla u|^p - \nu_\epsilon K|\nabla u|}{|\nabla u_n|^p} - \frac{\nu_\epsilon(K+|\nabla u|^{p-1})}{|\nabla u_n|^{p-1}} - \frac{\nu_\epsilon|\nabla u|}{|\nabla u_n|}\right). \end{aligned} \tag{20}$$

This implies $\{(\nabla u_n)(x,t)\}_{n\in\mathbb{N}}$ is bounded for a.e. $(x,t) \in Q_T$. Indeed, suppose the opposite. Then there exists a subset $D \subset Q_T$ of positive measure such that

$$\lim_{n\to\infty} |(\nabla u_n)(x,t)| = \infty \qquad \text{for a.e. } (x,t) \in D.$$

In turn, (20) gives

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_D L\, dx\, dt\\ &\geq \lim_{n\to\infty} \int_D |\nabla u_n|^p\left(\alpha + \frac{\alpha|\nabla u|^p - \nu_\epsilon K|\nabla u|}{|\nabla u_n|^p} - \frac{\nu_\epsilon(K+|\nabla u|^{p-1})}{|\nabla u_n|^{p-1}} - \frac{\nu_\epsilon|\nabla u|}{|\nabla u_n|}\right) dx\, dt,\\ &= \infty, \end{aligned}$$

which is impossible.

Consequently, there exists a function ξ^* on Q_T which is finite a.e. and (up to a subsequence) satisfies

$$\lim_{n\to\infty} \nabla u_n = \xi^* \quad \text{a.e. in } Q_T.$$

But then

$$(a_\epsilon(x,t,u,\xi^*) - a_\epsilon(x,t,u,\nabla u)) \cdot (\xi^* - \nabla u) = 0$$

by virtue of (19). This in turn implies $\xi^* = \nabla u$ due to (4) and so we have

$$\lim_{n\to\infty} \nabla u_n = \nabla u \quad \text{a.e. in } Q_T.$$

Next we note that $\{a_\epsilon(x,t,u_n,\nabla u_n)\}_{n\in\mathbb{N}}$ is bounded in $(L^p(Q_T))^d$ by (2) and the fact that $u_n \to u$ weakly in V as $n \to \infty$. Moreover,

$$\lim_{n\to\infty} a_\epsilon(x,t,u_n,\nabla u_n) = a_\epsilon(x,t,u,\nabla u) \quad \text{a.e. in } Q_T.$$

This in combination with (2) and the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} a_\epsilon(x,t,u_n,\nabla u_n) \cdot \nabla u_n = a_\epsilon(x,t,u,\nabla u) \cdot \nabla u \quad \text{in } L^1(Q_T).$$

Now it follows from (3) that

$$a_\epsilon(x,t,u_n,\nabla u_n) \cdot \nabla u_n + a_\epsilon(x,t,u,\nabla u) \cdot \nabla u \geq \alpha(|\nabla u_n|^p + |\nabla u|^p) \geq 2^{-p}\alpha|\nabla u_n - \nabla u|^p.$$

Hence by Fatou's lemma one has

$$\begin{aligned}
2\alpha \int_{Q_T} & a_\epsilon(x,t,u,\nabla u) \cdot \nabla u \, dx\, dt \\
&\leq \liminf_{n\to\infty} \int_{Q_T} (a_\epsilon(x,t,u_n,\nabla u_n) \cdot \nabla u_n + a_\epsilon(x,t,u,\nabla u) \cdot \nabla u - 2^{-p}\alpha|\nabla u_n - \nabla u|^p) \, dx\, dt \\
&= 2\alpha \int_{Q_T} a_\epsilon(x,t,u,\nabla u) \cdot \nabla u \, dx\, dt - 2^{-p}\alpha \limsup_{n\to\infty} \int_{Q_T} |\nabla u_n - \nabla u|^p \, dx\, dt,
\end{aligned}$$

or equivalently

$$\limsup_{n\to\infty} \int_{Q_T} |\nabla u_n - \nabla u|^p \, dx\, dt \leq 0.$$

This implies

$$\lim_{n\to\infty} \int_{Q_T} |\nabla u_n - \nabla u|^p \, dx\, dt = 0. \tag{21}$$

Combining (17) and (21) yields the claim. ∎

Now we show that (P_ϵ) has a weak solution. To this end, recall the following definition.

Definition 3.4. Let L be a densely defined maximal monotone linear operator from $D(L) \subset V$ to V^*. A bounded operator $\Theta : V \to V^*$ is called *pseudo-monotone* with respect to $D(L)$ if whenever $(u_n)_{n\in\mathbb{N}} \subset D(L)$ satisfies

$$\begin{cases}
\lim\limits_{n\to\infty} u_n = u \quad \text{weakly in } V, \\
\lim\limits_{n\to\infty} Lu_n = Lu \quad \text{weakly in } V^*, \\
\limsup\limits_{n\to\infty} \langle \Theta u_n, u_n - u \rangle_{V^*\times V} \leq 0,
\end{cases}$$

then

$$\begin{cases} \lim_{n\to\infty} \langle \Theta u_n, u_n - u\rangle = 0, \\ \lim_{n\to\infty} \Theta u_n = \Theta u \quad \text{weakly in } V^*. \end{cases}$$

LEMMA 3.5. *Let $0 < \epsilon \le 1$. Then (P_ϵ) has a weak solution u_ϵ.*

Proof. Define

- $L_\epsilon : D(L_\epsilon) \to V^*$ by

$$\langle L_\epsilon u, v\rangle_{V^*\times V} = \int_{Q_T} \frac{\partial b_\epsilon(u)}{\partial t} v\, dx\, dt$$

  for all $u \in D(L_\epsilon)$ and $v \in V$, where

$$D(L_\epsilon) = \{u \in \mathcal{W}_p : u(0) = 0\}$$

  and $\mathcal{W}_p$ is given by (9);
- $A_\epsilon : V \to V^*$ by

$$\langle A_\epsilon u, v\rangle_{V^*\times V} = \int_{Q_T} a_\epsilon(x,t,u,\nabla u)\cdot \nabla v\, dx\, dt + \int_{\Sigma_1} \gamma h_\epsilon(u) v\, d\sigma(x)\, dt$$

  for all $u, v \in V$.

Since $b_\epsilon \in W^{1,\infty}(\mathbb{R})$, it is straightforward to verify that L_ϵ is maximally monotone (cf. [Zei90, Proposition 32.10]). In view of [Zei90, Theorem 32.A], we will show that A possesses the following properties.

- A_ϵ *is coercive.* Indeed,

$$\begin{aligned} \langle A_\epsilon u, u\rangle_{V^*\times V} &= \int_{Q_T} a_\epsilon(x,t,u,\nabla u)\cdot\nabla u\, dx\, dt + \int_{\Sigma_1} \gamma h_\epsilon(u) u\, d\sigma(x)\, dt \\ &\ge (\alpha-\delta)\int_{Q_T} |\nabla u|^p\, dx\, dt - \frac{1}{\delta}\frac{p-1}{p^{\frac{p}{p-1}}}\|\gamma\|^{p'}_{L^\infty(Q_T)} M_\epsilon^{p'}\sigma(\Sigma_1) \end{aligned}$$

for all $u \in V$ and $\delta > 0$, where $\sigma(\Sigma_1)$ denotes the surface measure of Σ_1 and we have used Young's inequality and (H5) in the last step. Hence by choosing a sufficiently small $\delta > 0$, we obtain

$$\frac{\langle A_\epsilon u, u\rangle_{V^*\times V}}{\|u\|_V} \to \infty \quad \text{as } \|u\|_V \to \infty.$$

- A_ϵ *is pseudo-monotone with respect to $D(L_\epsilon)$.* Indeed, let $(u_n)_{n\in\mathbb{N}} \subset D(L_\epsilon)$ be such that

$$\begin{cases} \lim_{n\to\infty} u_n = u \quad \text{weakly in } V, \\ \lim_{n\to\infty} L_\epsilon u_n = L_\epsilon u \quad \text{weakly in } V^*, \\ \limsup_{n\to\infty} \langle A_\epsilon u_n, u_n - u\rangle_{V^*\times V} \le 0. \end{cases} \tag{22}$$

We will show that

$$\begin{cases} \lim_{n\to\infty} \langle A_\epsilon u_n, u_n - u\rangle_{V^*\times V} = 0, \\ \lim_{n\to\infty} A_\epsilon u_n = A_\epsilon u \quad \text{weakly in } V^*. \end{cases}$$

Using Aubin–Lions embedding, by passing to a subsequence if necessary, we may assume that $u_n \to u$ in $L^1(Q_T)$. It follows from (2) that $(a_\epsilon(x,t,u_n,\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p'}(Q_T))^d$. Therefore, there exist $\mathfrak{a}_\epsilon \in (L^{p'}(Q_T))^d$ such that

$$\lim_{n\to\infty} a_\epsilon(x,t,u_n,\nabla u_n) = \mathfrak{a}_\epsilon \quad \text{weakly in } (L^{p'}(Q_T))^d. \tag{23}$$

Then

$$\begin{aligned}\lim_{n\to\infty} \langle A_\epsilon u_n, v\rangle_{V^*\times V} &= \lim_{n\to\infty} \int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla v\,dx\,dt + \lim_{n\to\infty}\int_{\Sigma_1} \gamma h_\epsilon(u_n) v\,d\sigma(x)\,dt \\ &= \int_{Q_T} \mathfrak{a}_\epsilon\cdot\nabla v\,dx\,dt + \int_{\Sigma_1}\gamma h_\epsilon(u)v\,d\sigma(x)\,dt \end{aligned} \tag{24}$$

for all $v \in V$, where we have used (23) together with the fact that $\gamma, h_\epsilon(u_n) \in L^\infty(\Sigma_1)$ and $u_n \to u$ in $L^1(Q_T)$ in the last step.

Next observe that

$$\begin{aligned}&\left(\limsup_{n\to\infty}\int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt\right) + \int_{\Sigma_1}\gamma h_\epsilon(u)u\,d\sigma(x)\,dt \\ &\qquad = \limsup_{n\to\infty}\left(\int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt + \int_{\Sigma_1}\gamma h_\epsilon(u_n)u_n\,d\sigma(x)\,dt\right) \\ &\qquad = \limsup_{n\to\infty}\langle A_\epsilon u_n, u_n\rangle_{V^*\times V} \le \limsup_{n\to\infty}\langle A_\epsilon u_n, u\rangle_{V^*\times V} \\ &\qquad = \int_{Q_T}\mathfrak{a}_\epsilon\cdot\nabla u\,dx\,dt + \int_{\Sigma_1}\gamma h_\epsilon(u)u\,d\sigma(x)\,dt,\end{aligned}$$

where we have used (22) and (24) in the last two steps respectively. Hence

$$\limsup_{n\to\infty}\int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \le \int_{Q_T}\mathfrak{a}_\epsilon\cdot\nabla u\,dx\,dt. \tag{25}$$

On the other hand, (4) gives

$$\int_{Q_T}(a_\epsilon(x,t,u_n,\nabla u_n) - a_\epsilon(x,t,u_n,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx\,dt \ge 0,$$

which leads to

$$\begin{aligned}\int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \ge{}& \int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u\,dx\,dt \\ &+ \int_{Q_T} a_\epsilon(x,t,u_n,\nabla u)\cdot(\nabla u_n - \nabla u)\,dx\,dt.\end{aligned}$$

In view of (2) and (23) we deduce that

$$\liminf_{n\to\infty}\int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \ge \int_{Q_T}\mathfrak{a}_\epsilon\cdot\nabla u\,dx\,dt. \tag{26}$$

Consequently, (25) and (26) together imply

$$\lim_{n\to\infty}\int_{Q_T} a_\epsilon(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt = \int_{Q_T}\mathfrak{a}_\epsilon\cdot\nabla u\,dx\,dt. \tag{27}$$

Hence

$$\lim_{n\to\infty} \langle A_\epsilon u_n, u_n\rangle_{V^*\times V} = \langle A_\epsilon u, u\rangle_{V^*\times V}$$

and then

$$\lim_{n\to\infty} \langle A_\epsilon u_n, u_n - u\rangle_{V^*\times V} = 0$$

as required.

To finish, it follows from (23) and (27) that

$$\lim_{n\to\infty} \int_{Q_T} (a_\epsilon(x,t,u_n,\nabla u_n) - a_\epsilon(x,t,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx\, dt = 0.$$

This in combination with (22) and Lemma 3.3 yields

$$\lim_{n\to\infty} u_n = u \quad \text{ in } V$$

and

$$\lim_{n\to\infty} \nabla u_n = \nabla u \quad \text{ a.e. in } Q_T.$$

Hence we obtain

$$\lim_{n\to\infty} a_\epsilon(x,t,u_n,\nabla u_n) = a_\epsilon(x,t,u,\nabla u) \quad \text{ weakly in } (L^{p'}(Q_T))^d,$$

whence

$$\lim_{n\to\infty} A_\epsilon u_n = A_\epsilon u \quad \text{ weakly in } V^*$$

as required.

- A_ϵ *is demicontinuous.* Indeed, let $(u_n)_{n\in\mathbb{N}} \subset V$ be such that

$$\lim_{n\to\infty} \|u_n - u\|_V = 0.$$

By passing to a subsequence if necessary, we may assume that $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. in Q_T. Hence,

$$\lim_{n\to\infty} a_\epsilon(x,t,u_n,\nabla u_n) = a_\epsilon(x,t,u,\nabla u) \quad \text{ a.e. in } Q_T.$$

In addition, it follows from (2) that $(a_\epsilon(x,t,u_n,\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p'}(Q_T))^d$. Consequently,

$$\lim_{n\to\infty} a_\epsilon(x,t,u_n,\nabla u_n) = a_\epsilon(x,t,u,\nabla u) \quad \text{ weakly in } (L^{p'}(Q_T))^d. \tag{28}$$

The continuity of h_ϵ guarantees that

$$\lim_{n\to\infty} \gamma h_\epsilon(u_n) = \gamma h_\epsilon(u) \quad \text{ weakly in } L^{p'}(\Sigma_1). \tag{29}$$

Combining (28) and (29) yields

$$\lim_{n\to\infty} \langle A_\epsilon u_n, v\rangle_{V^*\times V} = \langle A_\epsilon u, v\rangle_{V^*\times V}$$

for all $v \in V$.

• A_ϵ *is bounded.* Indeed,

$$
\begin{aligned}
|\langle A_\epsilon u, v\rangle_{V^*\times V}| &= \left|\int_{Q_T} a_\epsilon(x,t,u,\nabla u)\cdot\nabla v\,dx\,dt + \int_{\Sigma_1}\gamma h_\epsilon(u)v\,d\sigma(x)\,dt\right| \\
&\le \left(\int_{Q_T}|a_\epsilon(x,t,u,\nabla u)|^{p'}\,dx\,dt\right)^{1/p'}\left(\int_{Q_T}|\nabla v|^p\,dx\,dt\right)^{1/p} \\
&\quad + \left(\int_{\Sigma_1}|\gamma h_\epsilon(u)|^{p'}\,d\sigma(x)\,dt\right)^{1/p'}\left(\int_{\Sigma_1}|v|^p\,d\sigma(x)\,dt\right)^{1/p} \\
&\le 2\nu_\epsilon\left(\int_{Q_T}|K|^{p'}+|\nabla u|^p\,dx\,dt\right)^{1/p'}\left(\int_{Q_T}|\nabla v|^p\,dx\,dt\right)^{1/p} \\
&\quad + \|\gamma\|_{L^\infty(\Sigma_1)}M_\epsilon\sigma(\Sigma_1)^{1/p'}\left(\int_{\Sigma_1}|v|^p\,d\sigma(x)\,dt\right)^{1/p} \\
&\le \left[2\nu_\epsilon H_0\left(\int_{Q_T}|K|^{p'}+|\nabla u|^p\,dx\,dt\right)^{1/p'} + \|\gamma\|_{L^\infty(\Sigma_1)}M_\epsilon\sigma(\Sigma_1)^{1/p'}S_0\right]\|v\|_V
\end{aligned}
$$

for all $u, v \in V$, where H_0 and S_0 are the constants in the embeddings in (H5) and Lemma 2.1(ii) respectively.

With the above properties in mind, we use [Zei90, Theorem 32.A] to conclude that there exists a weak solution to (P_ϵ) as claimed. ■

Next we present some a priori estimates for weak solutions to (P_ϵ). Set

$$Q_\tau := \Omega\times(0,\tau) \quad\text{and}\quad \Sigma_{1\tau} := \Gamma_1\times(0,\tau)$$

for each $\tau\in(0,T)$.

LEMMA 3.6. *Let $0<\epsilon\le\epsilon_0$, where ϵ_0 is given by (15). Let u_ϵ be a weak solution to (P_ϵ).*

(i) *There exists a constant $C = C(d,\Omega,p,\alpha,\beta) > 0$ such that*

$$\|T_k(u_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|T_k(u_\epsilon)\|_V^p \le Ck(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1)$$

for all $k>0$.

(ii) *There exists a constant $C = C(d,\Omega,p,\alpha,\beta) > 0$ such that*

$$|[|u_\epsilon|>k]| \le Ck^{1-p}(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1)$$

for all $k>0$. As a consequence, u_ϵ is finite a.e. in Q_T.

(iii) *There exists a constant $C = C(d,\Omega,p,\alpha,\beta) > 0$ such that*

$$\sigma([|u_\epsilon|_{\Sigma_1}|>k]) \le Ck^{1-p}(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1).$$

As a consequence, $u_\epsilon|_{\Sigma_1}$ is finite a.e. in Σ_1.

(iv) *We have*

$$\|b_\epsilon(u_\epsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1 + |\Omega|/2.$$

(v) *We have*

$$\int_{[k\le|u_\epsilon|\le k+1]} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt$$
$$\le \int_{[|u_\epsilon|>k]} |f|\,dx\,dt + \int_{[|u_\epsilon|_{\Sigma_1}|>k]} |g|\,d\sigma(x)\,dt + \int_{[|u_{0\epsilon}|>k]} |b(u_0)|\,dx\,dt + \big|[|u_{0\epsilon}|>k]\big|$$

for all $k>0$.

Proof. (i) Let $\tau\in(0,T)$ and $k>0$. Using $T_k(u_\epsilon)\,\mathbb{1}_{(0,\tau)}$ as a test function in (16) gives

$$\int_\Omega B_{\epsilon k}(u_\epsilon(\tau))\,dx + \int_{Q_\tau} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(u_\epsilon)\,dx\,dt + \int_{\Sigma_{1\tau}} \gamma\, h_\epsilon(u_\epsilon)\,T_k(u_\epsilon)\,d\sigma(x)\,dt$$
$$= \int_{Q_\tau} f_\epsilon\, T_k(u_\epsilon)\,dx\,dt + \int_{\Sigma_{1\tau}} g_\epsilon T_k(u_\epsilon)\,d\sigma(x)\,dt + \int_\Omega B_{\epsilon k}(u_{0\epsilon})\,dx,$$

where

$$B_{\epsilon k}(r) := \int_0^r T_k(s) b'_\epsilon(s)\,ds$$

for all $r\in\mathbb{R}$. Concerning the left hand side, we note that

$$\int_{\Sigma_{1\tau}} \gamma h_\epsilon(u_\epsilon) T_k(u_\epsilon)\,d\sigma(x)\,dt \ge 0,$$

$$\begin{aligned}\int_\Omega B_{\epsilon k}(u_\epsilon(\tau))\,dx &\ge \beta\int_\Omega\Big(\int_0^{u_\epsilon(\tau)} T_k(s)\,ds\Big)\,dx\\ &= \begin{cases}\beta\displaystyle\int_\Omega \frac{u_\epsilon(\tau)^2}{2}\,dx & \text{if } |u_\epsilon(\tau)|\le k,\\ \beta\displaystyle\int_\Omega\Big(k|u_\epsilon(\tau)|-\frac{k^2}{2}\Big)\,dx & \text{otherwise}\end{cases}\\ &\ge \frac{\beta}{2}\int_\Omega T_k(u_\epsilon(\tau))^2\,dx \end{aligned}\tag{30}$$

and

$$\begin{aligned}&\int_{Q_\tau} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(u_\epsilon)\,dx\,dt\\ &\qquad= \int_{\{(x,t)\in Q_\tau:\,|u_\epsilon(x,t)|\le k\}} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_k(u_\epsilon)\,dx\,dt\\ &\qquad= \int_{\{(x,t)\in Q_\tau:\,|u_\epsilon(x,t)|\le k\}} a_\epsilon(x,t,u_\epsilon,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\,dx\,dt\\ &\qquad\ge \alpha\int_{\{(x,t)\in Q_\tau:\,|u_\epsilon(x,t)|\le k\}} |\nabla T_k(u_\epsilon)|^p\,dx\,dt = \alpha\int_{Q_\tau}|\nabla T_k(u_\epsilon)|^p\,dx\,dt\\ &\qquad\ge \alpha H_0^{-p}\int_0^\tau \|T_k(u_\epsilon)\|^p_{W^{1,p}_{\Gamma_0}(\Omega)}\,dt,\end{aligned}$$

where we have used (H5) in the last step.

For the right hand side, we have

$$\int_{Q_\tau} f_\epsilon T_k(u_\epsilon)\,dx\,dt \le k\|f\|_{L^1(Q_T)}, \qquad \int_{\Sigma_{1\tau}} g_\epsilon T_k(u_\epsilon)\,d\sigma(x)\,dt \le k\|g\|_{L^1(\Sigma_1)}$$

and (30) gives

$$0 \le \int_\Omega B_{\epsilon k}(u_{0\epsilon})\,dx \le k\int_\Omega b_\epsilon(u_{0\epsilon})\,dx \le k(\|b(u_0)\|_{L^1(\Omega)} + 1).$$

Combining the estimates for the two sides and taking the supremum over all $\tau \in (0,T)$, we arrive at

$$\begin{aligned}\frac{\beta}{2}\sup_{\tau\in(0,T)}\int_\Omega T_k(u_\epsilon(\tau))^2\,dx + \alpha H_0^{-p}\|T_k(u_\epsilon)\|_V^p \\ \le k(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1)\end{aligned}$$

for all $k > 0$.

(ii) It follows from (i) that

$$\int_0^T\int_\Omega |\nabla T_k(u_\epsilon)|^p\,dx\,dt \le Ck(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1),$$

where $C = C(d,\Omega,p,\alpha,\beta) > 0$. Using (H5), we infer that

$$\begin{aligned}\big|[|u_\epsilon| > k]\big| &\le k^{-p}\|T_k(u_\epsilon)\|^p_{L^p(Q_T)} \le H_0^p k^{-p}\|\nabla T_k(u_\epsilon)\|^p_{L^p(Q_T)} \\ &\le CH_0^p k^{1-p}(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1)\end{aligned}$$

for all $k > 0$.

Letting $k \to \infty$ yields

$$\big|[|u_\epsilon| = \infty]\big| = 0,$$

whence u_ϵ is finite a.e. in Q_T.

(iii) It follows from (i), Lemma 2.2 and Remark 2.3 that

$$T_k(u_\epsilon)|_{\Sigma_1} = T_k(u_\epsilon|_{\Sigma_1}) \quad \text{a.e. in } \Sigma_1.$$

Then similar to (ii), we have

$$\begin{aligned}\sigma([|u_\epsilon|_{\Sigma_1}| > k|]) &\le k^{-p}\|T_k(u_\epsilon|_{\Sigma_1})\|^p_{L^p(\Sigma_1)} = k^{-p}\|T_k(u_\epsilon)|_{\Sigma_1}\|^p_{L^p(\Sigma_1)} \\ &\le S_p k^{-p}\int_0^T \|T_k(u_\epsilon)\|^p_{W^{1,p}_{\Gamma_0}(\Omega)}\,dt \\ &\le S_p(1 + H_0^p)k^{-p}\|\nabla T_k(u_\epsilon)\|^p_{L^p(Q_T)} \\ &\le C(1 + H_0^p)k^{1-p}(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1)\end{aligned}$$

for all $k > 0$, where we have used Lemma 2.1(ii) and (H5) in the third and fourth steps respectively. Here S_p denotes the corresponding constant in the embedding in Lemma 2.1(ii) and $C = C(d,\Omega,p,\alpha,\beta) > 0$.

(iv) Let $\tau \in (0,T)$. Let $T_1(b_\epsilon(u_\epsilon))$ be a test function in (16). Then

$$\begin{aligned}
\int_\Omega X_\epsilon(u_\epsilon(\tau))\,dx + \int_{Q_\tau} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_1(b_\epsilon(u_\epsilon))\,dx\,dt \\
+ \int_{\Sigma_{1\tau}} \gamma h_\epsilon(u_\epsilon) T_1(b_\epsilon(u_\epsilon))\,d\sigma(x)\,dt \\
= \int_{Q_\tau} f_\epsilon T_1(b_\epsilon(u_\epsilon))\,dx\,dt + \int_{\Sigma_{1\tau}} g_\epsilon T_1(b_\epsilon(u_\epsilon))\,d\sigma(x)\,dt + \int_\Omega X_\epsilon(u_{0\epsilon})\,dx \\
\le \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1,
\end{aligned}$$

where

$$X_\epsilon(r) := \int_0^{b_\epsilon(r)} T_1(s)\,ds$$

for all $r \in \mathbb{R}$.

Note that

$$\begin{aligned}
\int_\Omega X_\epsilon(u_\epsilon(\tau))\,dx &= \int_\Omega \left(\int_0^{b_\epsilon(u_\epsilon(\tau))} T_1(s)\,ds\right)dx \\
&= \begin{cases} \displaystyle\int_\Omega \frac{b_\epsilon(u_\epsilon(\tau))^2}{2}\,dx & \text{if } |b_\epsilon(u_\epsilon(\tau))| \le 1, \\ \displaystyle\int_\Omega \left(|b_\epsilon(u_\epsilon(\tau))| - \frac{1}{2}\right)dx & \text{otherwise.} \end{cases}
\end{aligned}$$

With this in mind, we have

$$\begin{aligned}
\int_\Omega |b_\epsilon(u_\epsilon(\tau))|\,dx &\le \begin{cases} \displaystyle\frac{|\Omega|}{2} + \int_\Omega \frac{b_\epsilon(u_\epsilon(\tau))^2}{2}\,dx & \text{if } |b_\epsilon(u_\epsilon(\tau))| \le 1, \\ \displaystyle\frac{|\Omega|}{2} + \int_\Omega X_\epsilon(u_\epsilon(\tau))\,dx & \text{otherwise} \end{cases} \\
&= \frac{|\Omega|}{2} + \int_\Omega X_\epsilon(u_\epsilon(\tau))\,dx.
\end{aligned}$$

We proceed as follows. Concerning the left hand side, we note that

$$\int_{\Sigma_{1\tau}} \gamma h_\epsilon(u_\epsilon) T_1(u_\epsilon)\,d\sigma(x)\,dt \ge 0$$

and

$$\begin{aligned}
&\int_{Q_\tau} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_1(b_\epsilon(u_\epsilon))\,dx\,dt \\
&\quad= \int_{\{(x,t)\in Q_\tau : |b_\epsilon(u_\epsilon(x,t))|\le 1\}} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla T_1(b_\epsilon(u_\epsilon))\,dx\,dt \\
&\quad= \int_{\{(x,t)\in Q_\tau : |b_\epsilon(u_\epsilon(x,t))|\le 1\}} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot(\nabla u_\epsilon) T_1'(b_\epsilon(u_\epsilon))\, b_\epsilon'(u_\epsilon)\,dx\,dt \\
&\quad\ge 0.
\end{aligned}$$

Hence

$$\sup_{0<\tau<T}\int_\Omega |b_\epsilon(u_\epsilon(\tau))|\,dx \le \frac{|\Omega|}{2} + \sup_{0<\tau<T}\int_\Omega X_\epsilon(u_\epsilon(\tau))\,dx$$
$$\le \frac{|\Omega|}{2} + \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1$$

as required.

(v) Let $k>0$. Using

$$\varphi = T_{k+1}(u_\epsilon) - T_k(u_\epsilon)$$

as a test function in (16) gives

$$\int_\Omega H_{\epsilon k}(u_\epsilon(T))\,dx + \int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt + \int_{\Sigma_1}\gamma h_\epsilon(u_\epsilon)\varphi\,d\sigma(x)\,dt$$
$$= \int_{Q_T} f_\epsilon\varphi\,dx\,dt + \int_{\Sigma_1} g_\epsilon\varphi\,d\sigma(x)\,dt + \int_\Omega H_{\epsilon k}(u_{0\epsilon})\,dx,$$

where

$$H_{\epsilon k}(r) := \int_0^r [T_{k+1}(s) - T_k(s)]b'_\epsilon(s)\,ds$$

for all $r\in\mathbb{R}$. Concerning the left hand side, we note that

$$\int_{\Sigma_{1\tau}}\gamma h_\epsilon(u_\epsilon)\varphi\,d\sigma(x)\,dt \ge 0 \quad\text{and}\quad \int_\Omega H_{\epsilon k}(u_\epsilon(T))\,dx \ge 0,$$

whereas

$$\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt = \int_{[k\le|u_\epsilon|\le k+1]} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt.$$

For the right hand side, we have

$$\int_{Q_T} f_\epsilon\varphi\,dx\,dt \le \int_{[|u_\epsilon|>k]}|f|\,dx\,dt.$$

To deal with the boundary term, we note that

$$T_k(u_\epsilon)|_{\Sigma_1} = T_k(u_\epsilon|_{\Sigma_1}),$$

whence

$$\int_{\Sigma_1} g_\epsilon\varphi\,d\sigma(x)\,dt \le \int_{[|u_\epsilon|_{\Sigma_1}|>k]}|g|\,d\sigma(x)\,dt.$$

Also,

$$\int_\Omega H_{\epsilon k}(u_{0\epsilon})\,dx \le \int_{[|u_{0\epsilon}|>k]} b_\epsilon(u_{0\epsilon})\,dx \le \int_{[|u_{0\epsilon}|>k]}|b(u_0)|\,dx + \big|[|u_{0\epsilon}|>k]\big|.$$

Combining the estimates for the two sides, we arrive at the claim. ∎

3.2. Limiting procedure. The estimates in Lemma 3.6 enable us to perform several limiting procedures.

LEMMA 3.7. *Let $k>0$ and let u_ϵ be a weak solution to (P_ϵ) for each $0<\epsilon\le\epsilon_0$, where ϵ_0 is given by* (15). *Then there exists a $u\in V$ such that there exists a subsequence of $\{u_\epsilon\}_{0<\epsilon\le\epsilon_0}$, still denoted by $\{u_\epsilon\}_{0<\epsilon\le\epsilon_0}$, which satisfies the following properties as $\epsilon\to0^+$:*

(i) *u is finite a.e. in Q_T.*
(ii) *$u_\epsilon \to u$ a.e. in Q_T.*
(iii) *$T_k(u) \in V$.*
(iv) *$T_k(u_\epsilon) \to T_k(u)$ in V weakly, in $L^p(Q_T)$ and a.e. in Q_T.*
(v) *$u|_{\Sigma_1}$ is finite a.e. in Σ_1.*
(vi) *$u_\epsilon|_{\Sigma_1} \to u|_{\Sigma_1}$ a.e. in Σ_1.*
(vii) *$T_k(u_\epsilon)|_{\Sigma_1} \to T_k(u)|_{\Sigma_1}$ in $L^p(\Sigma_1)$ and a.e. in Σ_1.*
(viii) *$b(u) \in L^\infty(0,T;L^1(\Omega))$.*

Proof. Lemma 2.1 and (1) together assert that the embeddings

$$W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Omega) \quad \text{and} \quad W^{1,p}_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Gamma_1)$$

are compact. Hence in view of Lemma 3.6(i) and the Aubin–Lions embedding, there exists a function $v_k \in V$ such that

$$\lim_{\epsilon\to 0^+} T_k(u_\epsilon) = v_k \quad \begin{cases} \text{in } V \text{ weakly,} \\ \text{in } L^1(Q_T), \\ \text{a.e. in } Q_T \end{cases} \tag{31}$$

and

$$\lim_{\epsilon\to 0^+} T_k(u_\epsilon)|_{\Sigma_1} = v_k|_{\Sigma_1} \quad \begin{cases} \text{in } L^1(\Sigma_1), \\ \text{a.e. in } \Sigma_1. \end{cases} \tag{32}$$

We will show that there exists a constant $0 < \epsilon_1 \le \epsilon_0$ such that $\{u_\epsilon\}_{0<\epsilon\le\epsilon_1}$ and $\{u_\epsilon|_{\Sigma_1}\}_{0<\epsilon\le\epsilon_1}$ are Cauchy sequences in measure. Indeed, recall from Lemma 3.6(ii, iii) that

$$\big|[|u_\epsilon| > k]\big| \le Ck^{1-p}(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)}) \tag{33}$$

and

$$\sigma(\{x \in \Sigma_1 : |u_\epsilon|_{\Sigma_1}| > k\}) \le Ck^{1-p}(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1) \tag{34}$$

for all $0 < \epsilon \le \epsilon_0$, where $C = C(d,\Omega,p,\alpha,\beta) > 0$.

Let $\eta > 0$ and $\omega > 0$. Let k be sufficiently large such that

$$\big|[|u_\epsilon| > k]\big| < \eta/3 \quad \text{and} \quad \sigma(\{x \in \Sigma_1 : |u_\epsilon|_{\Sigma_1}| > k\}) < \eta/3$$

for all $0 < \epsilon \le \epsilon_0$.

Due to (31) and (32), there exists a constant $0 < \epsilon_1 \le \epsilon_0$ such that

$$\big|[|T_k(u_\epsilon) - T_k(u_{\epsilon'})| \ge \omega]\big| < \eta/3$$

and

$$\big|[|T_k(u_\epsilon)|_{\Sigma_1} - T_k(u_{\epsilon'})|_{\Sigma_1}| \ge \omega]\big| < \eta/3$$

for all $0 < \epsilon, \epsilon' < \epsilon_1$.

Note that

$$[|u_\epsilon - u_{\epsilon'}| \ge \omega] \subset [|u_\epsilon| \ge k] \cup [|u_{\epsilon'}| \ge k] \cup [|T_k(u_\epsilon) - T_k(u_{\epsilon'})| \ge \omega],$$

whence

$$\big|[|u_\epsilon - u_{\epsilon'}| \ge \omega]\big| \le \big|[|u_\epsilon| \ge k]\big| + \big|[|u_{\epsilon'}| \ge k]\big| + \big|[|T_k(u_\epsilon) - T_k(u_{\epsilon'})| \ge \omega]\big| < \eta$$

for all $0 < \epsilon, \epsilon' < \epsilon_1$. Hence $\{u_\epsilon\}_{0<\epsilon\le\epsilon_1}$ is a Cauchy sequence in measure.

Regarding $\{u_\epsilon|_{\Sigma_1}\}_{0<\epsilon\le\epsilon_1}$, we first note that

$$T_k(u_\epsilon)|_{\Sigma_1} = T_k(u_\epsilon|_{\Sigma_1}). \tag{35}$$

Then arguments similar to the above show that $\{u_\epsilon|_{\Sigma_1}\}_{0<\epsilon\le\epsilon_1}$ is also a Cauchy sequence in measure.

Consequently, there exists a measurable function $u : Q_T \to \overline{\mathbb{R}}$ such that

$$\lim_{\epsilon\to0^+} u_\epsilon = u \quad \text{ a.e. in } Q_T.$$

Moreover, u is finite a.e. in Q_T due to (33). By continuity, we deduce that

$$\lim_{\epsilon\to0^+} T_k(u_\epsilon) = T_k u \quad \text{ a.e. in } Q_T, \tag{36}$$

whence $T_k u = v_k \in V$. At this point, (i)–(iv) are justified, except the L^p-convergence. But this follows easily from the Lebesgue dominated convergence theorem.

Likewise, there exists a measurable function $w : \Sigma_1 \to \overline{\mathbb{R}}$ such that

$$\lim_{\epsilon\to0^+} u_\epsilon|_{\Sigma_1} = w \quad \text{ a.e. in } \Sigma_1. \tag{37}$$

Moreover, w is finite a.e. in Σ_1 due to (34).

By virtue of Lemma 2.2 and Remark 2.3, $u|_{\Sigma_1}$ is well-defined. Furthermore, (32), (35), (36) and (37) together imply

$$v_k|_{\Sigma_1} = T_k(w) = T_k(u)|_{\Sigma_1} = T_k(u|_{\Sigma_1})$$

for all $k > 0$. Hence $w = u|_{\Sigma_1}$. At this point, (v)–(vii) are also justified.

It remains to prove (viii). To this end, we note that

$$\lim_{\epsilon\to0^+} b_\epsilon(u_\epsilon) = b(u) \quad \text{ a.e. in } Q_T.$$

Using Fatou's lemma and Lemma 3.6(iv) yields

$$\begin{aligned}\|b(u)\|_{L^\infty(0,T;L^1(\Omega))} &\le \liminf_{\epsilon\to0^+} \|b_\epsilon(u_\epsilon)\|_{L^\infty(0,T;L^1(\Omega))}\\ &\le \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + 1 + |\Omega|/2,\end{aligned}$$

which implies $b(u) \in L^\infty(0,T;L^1(\Omega))$ as required.

The proof is complete. ∎

Lemma 3.8. *Adopt the assumptions and notation from Lemma* 3.7. *Then*

$$\lim_{\epsilon\to0^+} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) = a(x,t,T_k(u),\nabla T_k(u)) \quad \textit{weakly in } (L^{p'}(Q_T))^d$$

and

$$\lim_{\epsilon\to0^+} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon) = a(x,t,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) \quad \textit{in } (L^1(Q_\tau))^d$$

for all $\tau \in (0,T)$, *where* $Q_\tau := (0,\tau)\times\Omega$.

Proof. We borrow the technique from [BGR16, proof of Lemma 3.2].

First, in view of (2) and Lemma 3.6(i), there exists a function $\mathfrak{a}_k \in (L^{p'}(Q_T))^d$ such that

$$\lim_{\epsilon\to0^+} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) = \mathfrak{a}_k \quad \text{ weakly in } (L^{p'}(Q_T))^d. \tag{38}$$

Hereafter, the proof is divided into two steps as follows.

Step 1: We prove that

$$\lim_{\epsilon\to 0^+}\int_{Q_T} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)\xi\,dx\,dt \le \int_{Q_T}\mathfrak{a}_k\cdot\nabla T_k(u)\xi\,dx\,dt \tag{39}$$

for all $\xi\in C_c^\infty([0,T);\mathbb{R})$.

Indeed, let $n>0$ and define

$$S(r)=\int_0^r (1-|T_{n+1}(s)-T_n(s)|)\,ds$$

for each $r\in\mathbb{R}$. Also let

$$\phi\in C_c^\infty([0,T)\times\Omega_1),$$

where $\Omega_1=\Omega\cup\Gamma_1$. Then $\varphi:=S(u_\epsilon)\,\phi\in V$ and by choosing φ as a test function in (16) we obtain

$$\begin{aligned}
-\int_0^T\int_\Omega \phi_t B_{\epsilon S}(u_\epsilon)\,dx\,dt+\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(S'(u_\epsilon)\phi)\,dx\,dt&\\
+\int_{\Sigma_1}\gamma h_\epsilon(u_\epsilon)S'(u_\epsilon)\phi\,d\sigma(x)\,dt&\\
=\int_{Q_T} f_\epsilon\,S'(u_\epsilon)\phi\,dx\,dt+\int_{\Sigma_1} g_\epsilon S'(u_\epsilon)\phi\,d\sigma(x)\,dt-\int_\Omega\phi(0)B_{\epsilon S}(u_{0\epsilon})\,dx&,
\end{aligned}$$

where

$$B_{\epsilon S}(r):=\int_0^r b_\epsilon'(s)S'(s)\,ds$$

for all $r\in\mathbb{R}$.

As a consequence,

$$\begin{aligned}
-\|\phi\|_{L^\infty(Q_T)}&\int_{[n<|u_\epsilon|<n+1]} a(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt\\
&=-\|\phi\|_{L^\infty(Q_T)}\int_{[n<|u_\epsilon|<n+1]} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt\\
&\le-\int_{\Sigma_1}\gamma h_\epsilon(u_\epsilon)S'(u_\epsilon)\phi\,d\sigma(x)\,dt-\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot(\nabla\phi)S'(u_\epsilon)\,dx\,dt\\
&\quad+\int_0^T\int_\Omega\phi_t\,B_{\epsilon S}(u_\epsilon)dx\,dt-\int_\Omega\phi(0)B_{\epsilon S}(u_{0\epsilon})\,dx\\
&\quad+\int_{Q_T} f_\epsilon S'(u_\epsilon)\phi\,dx\,dt+\int_{\Sigma_1} g_\epsilon S'(u_\epsilon)\phi\,d\sigma(x)\,dt\\
&\le\|\phi\|_{L^\infty(Q_T)}\int_{[n<|u_\epsilon|<n+1]} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt\\
&=\|\phi\|_{L^\infty(Q_T)}\int_{[n<|u_\epsilon|<n+1]} a(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt
\end{aligned}\tag{40}$$

provided that $\epsilon<\frac{1}{n+1}$.

Observe that

$$\begin{aligned}
&a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot(\nabla\phi)S'(u_\epsilon)\\
&\qquad\qquad = a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\cdot(\nabla\phi)\mathbb{1}_{[|u_\epsilon|\le k]}\\
&\qquad\qquad\quad + a(x,t,T_{n+1}(u_\epsilon),\nabla T_{n+1}(u_\epsilon))\cdot(\nabla\phi)S'(u_\epsilon)\mathbb{1}_{[|u_\epsilon|>k]} \quad \text{a.e. in } Q_T
\end{aligned}$$

for all $n>k$ and $\epsilon<\frac{1}{n+1}$. This and (38) imply

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot(\nabla\phi)S'(u_\epsilon)\,dx\,dt\\
&\qquad\qquad = \int_{[|u|\le k]}\mathfrak{a}_k\cdot(\nabla\phi)\,dx\,dt + \int_{[|u|>k]}\mathfrak{a}_{n+1}\cdot(\nabla\phi)S'(u)\,dx\,dt
\end{aligned}$$

for all $n>k$ and $\epsilon<\frac{1}{n+1}$.

Letting $\epsilon\to 0^+$ in (40) gives

$$\begin{aligned}
&-\|\phi\|_{L^\infty(Q_T)}\omega(n)\\
&\quad\le \int_0^T\int_\Omega \phi_t B_S(u)\,dx\,dt - \int_\Omega \phi(0)B_S(u_0)\,dx\\
&\qquad - \int_{\Sigma_1}\gamma h(u)S'(u)\phi\,d\sigma(x)\,dt - \int_{[|u|\le k]}\mathfrak{a}_k\cdot\nabla\phi\,dx\,dt - \int_{[|u|>k]}\mathfrak{a}_{n+1}\cdot(\nabla\phi)S'(u)\,dx\,dt\\
&\qquad + \int_{Q_T} fS'(u)\phi\,dx\,dt + \int_{\Sigma_1} gS'(u)\phi\,d\sigma(x)\,dt\\
&\quad =: \int_0^T\int_\Omega \phi_t B_S(u)\,dx\,dt - \int_\Omega \phi(0)B_S(u_0)\,dx + \mathfrak{A}\\
&\quad \le \|\phi\|_{L^\infty(Q_T)}\,\omega(n),
\end{aligned} \tag{41}$$

where $\omega(n)$ is such that

$$\lim_{n\to\infty}\omega(n)=0.$$

Let $\{u_{0j}\}_{j\in\mathbb{N}}\subset C_c^\infty(\Omega_1)$ be such that $u_{0j}\to u_0$ pointwise in Ω as $j\to\infty$. Set $u(t):=u_{0j}$ if $t<0$. Let $h>0$ and $\xi\in C_c^\infty([0,T);\mathbb{R})$ be such that $0\le\xi\le 1$. Then

$$\phi := \xi\frac{1}{h}\int_{t-h}^t T_k(u(s))\,ds \in V\cap L^\infty(Q_T) \tag{42}$$

and $\phi_t\in L^\infty(Q_T)$. Hence we may substitute (42) into (41) to arrive at

$$\begin{aligned}
-\|\phi\|_{L^\infty(Q_T)}\omega(n) &\le -\int_0^T\int_\Omega \frac{\partial}{\partial t}\Big(\xi\frac{1}{h}\int_{t-h}^t T_k(u(s))\,ds\Big)(B_S(u)-B_S(u_0))\,dx\,dt + \mathfrak{A}\\
&\le \|\phi\|_{L^\infty(Q_T)}\omega(n).
\end{aligned} \tag{43}$$

An application of [BP05, Lemma 2.3] with

$$w=u,\quad F(\lambda)=T_k(\lambda),\quad B=B_S,\quad \beta=B_S(u),\quad \beta_0=B_S(u_0)\quad\text{and}\quad u_0=u_{0j}$$

yields

$$
\begin{aligned}
-\int_0^T\int_\Omega \frac{\partial}{\partial t}\bigg(\xi\frac{1}{h}\int_{t-h}^t T_k(u(s))\,ds\bigg)(B_S(u)-B_S(u_0))\,dx\,dt\\
\le -\int_{Q_T}\xi_t\bigg(B_S(u)\frac{1}{h}\int_{t-h}^t T_k(u(s))\,ds-\frac{1}{h}\int_{t-h}^t\int_0^{u(s)}T_k'(r)B_S(r)\,dr\,ds\bigg)\,dx\,dt\\
-\int_\Omega \xi(0)\bigg(B_S(u_0)T_k(u_{0j})-\int_0^{u_{0j}}T_k'(r)B_S(r)\,dr\bigg)\,dx.
\end{aligned}
$$

In turn we obtain

$$
\begin{aligned}
-\liminf_{h\to 0^+}\int_0^T\int_\Omega \frac{\partial}{\partial t}\bigg(\xi\frac{1}{h}\int_{t-h}^t T_k(u(s))\,ds\bigg)(B_S(u)-B_S(u_0))\,dx\,dt\\
\le -\int_{Q_T}\xi_t(B_S(u)T_k(u)-\int_0^u T_k'(r)B_S(r)\,dr\,ds)\,dx\,dt\\
-\int_\Omega \xi(0)\bigg(B_S(u_0)T_k(u_{0j})-\int_0^{u_{0j}}T_k'(r)B_S(r)\,dr\bigg)\,dx.
\end{aligned}
$$

Since

$$
\lim_{h\to 0^+}\frac{1}{h}\int_{t-h}^t T_k(u(s))\,ds=T_k(u)\quad\text{ in } V,
$$

letting $h\to 0^+$ and $j\to\infty$ in (43) yields

$$
\begin{aligned}
-k\omega(n)\le -\int_{Q_T}\xi_t\bigg(B_S(u)T_k(u)-\int_0^u T_k'(r)B_S(r)\,dr\,ds\bigg)\,dx\,dt\\
-\int_\Omega \xi(0)\bigg(B_S(u_0)\,T_k(u_0)-\int_0^{u_0}T_k'(r)B_S(r)\,dr\bigg)\,dx\\
-\int_{\Sigma_1}\gamma h(u)S'(u)T_k(u)\xi d\sigma(x)\,dt-\int_{Q_T}\mathfrak{a}_k\cdot(\nabla T_k(u))\xi\,dx\,dt\\
-\int_{[|u|>k]}\mathfrak{a}_{n+1}\cdot(\nabla T_k(u))\xi\,dx\,dt\\
+\int_{Q_T}fS'(u)T_k(u)\xi\,dx\,dt+\int_{\Sigma_1}gS'(u)T_k(u)\xi\,d\sigma(x)\,dt.
\end{aligned}
\tag{44}
$$

Next, observe that

$$
\mathbb{1}_{[|u|>k]}\nabla T_k(u)=0\quad\text{ a.e. in } Q_T
$$

as well as

$$
\lim_{n\to\infty}B_S(r)=b(r)\quad\text{ and }\quad\lim_{n\to\infty}S'(r)=1
$$

for all $r\in\mathbb{R}$. Therefore, letting $n\to\infty$ in (44) yields

$$
\begin{aligned}
0\le -\int_{Q_T}\xi_t\bigg(b(u)T_k(u)-\int_0^u T_k'(r)b(r)\,dr\,ds\bigg)\,dx\,dt\\
-\int_\Omega \xi(0)\bigg(b(u_0)T_k(u_0)-\int_0^{u_0}T_k'(r)b(r)\,dr\bigg)\,dx\\
-\int_{\Sigma_1}\gamma h(u)T_k(u)\xi\,d\sigma(x)\,dt-\int_{Q_T}\mathfrak{a}_k\cdot(\nabla T_k(u))\xi\,dx\,dt\\
+\int_{Q_T}fT_k(u)\xi\,dx\,dt+\int_{\Sigma_1}gT_k(u)\xi\,d\sigma(x)\,dt.
\end{aligned}
$$

Using the relation

$$b(z)T_k(z) - \int_0^z T_k'(r)b(r)\,dr\,ds = \int_0^z T_k(r)b'(r)\,dr$$

for all $z \in \mathbb{R}$, we infer that

$$\begin{aligned} 0 \le & -\int_{Q_T} \xi_t \Big(\int_0^u T_k(r)b'(r)\,dr \Big)\,dx\,dt - \int_\Omega \xi(0) \Big(\int_0^{u_0} T_k(r)b'(r)\,dr \Big)\,dx \\ & - \int_{\Sigma_1} \gamma h(u) T_k(u)\xi\,d\sigma(x)\,dt - \int_{Q_T} \mathfrak{a}_k \cdot (\nabla T_k(u))\xi\,dx\,dt \\ & + \int_{Q_T} f T_k(u)\xi\,dx\,dt + \int_{\Sigma_1} g T_k(u)\xi\,d\sigma(x)\,dt. \end{aligned} \tag{45}$$

On the other hand, let $\xi \in C_c^\infty([0,T),\mathbb{R})$. We take

$$\varphi = T_k(u_\epsilon)\,\xi$$

as a test function in (16) to obtain

$$\begin{aligned} -\int_0^T \int_\Omega \xi_t \Big(\int_0^{u_\epsilon} T_k(s)b_\epsilon'(s)\,ds \Big)\,dx\,dt + \int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) \cdot (\nabla T_k(u_\epsilon))\xi\,dx\,dt \\ + \int_{\Sigma_1} \gamma h_\epsilon(u_\epsilon) T_k(u_\epsilon)\xi\,d\sigma(x)\,dt \\ = \int_{Q_T} f_\epsilon T_k(u_\epsilon)\xi\,dx\,dt + \int_{\Sigma_1} g_\epsilon T_k(u_\epsilon)\xi\,d\sigma(x)\,dt + \int_\Omega \xi(0) \Big(\int_0^{u_{0\epsilon}} T_k(s)b_\epsilon'(s)\,ds \Big)\,dx. \end{aligned}$$

Consequently,

$$\begin{aligned} \lim_{\epsilon\to 0^+} & \int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon) \cdot (\nabla T_k(u_\epsilon))\xi\,dx\,dt \\ & = \lim_{\epsilon\to 0^+} \int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) \cdot (\nabla T_k(u_\epsilon))\xi\,dx\,dt \\ & = \int_0^T \int_\Omega \xi_t \Big(\int_0^u T_k(s)b'(s)\,ds \Big)\,dx\,dt + \int_\Omega \xi(0) \Big(\int_0^{u_0} T_k(s)b'(s)\,ds \Big)\,dx \\ & \quad - \int_{\Sigma_1} \gamma h(u) T_k(u)\xi\,d\sigma(x)\,dt + \int_{Q_T} f T_k(u)\xi\,dx\,dt + \int_{\Sigma_1} g T_k(u)\xi\,d\sigma(x)\,dt. \end{aligned} \tag{46}$$

Combining (45) and (46), we arrive at (39) as required.

Step 2: Recall from Lemma 3.7(iv) and (38) that

$$\lim_{\epsilon\to 0^+} T_k(u_\epsilon) = T_k(u) \quad \begin{cases} \text{in } L^p(Q_T), \\ \text{a.e. in } Q_T \end{cases}$$

and

$$\lim_{\epsilon\to 0^+} a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) = \mathfrak{a}_k \quad \text{weakly in } (L^{p'}(Q_T))^d.$$

Furthermore, a is monotone in view of (4). Keeping these facts and (39) in mind, the lemma follows by an application of Minty's trick (cf. [Gal21, Lemma 3.6]). ∎

Lemma 3.9. *Let u be as in Lemma 3.7. Then*

$$\lim_{k\to\infty}\int_{[k\le|u|\le k+1]} a(x,t,u,\nabla u)\cdot\nabla u\,dx=0.$$

Proof. Let $\{\xi_j\}_{j\in\mathbb{N}}\subset C_c^\infty([0,T);\mathbb{R})$ be such that

$$0\le\xi_j\le 1\quad\text{and}\quad\lim_{j\to\infty}\xi_j=1\quad\text{pointwise in }[0,T).$$

Then

$$\begin{aligned}
&\int_{[k\le|u|\le k+1]} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt\\
&=\int_{[k\le|u|\le k+1]} a(x,t,T_{k+1}(u),\nabla T_{k+1}(u))\cdot\nabla T_{k+1}(u)\,dx\,dt\\
&\le\liminf_{j\to\infty}\int_{[k\le|u|\le k+1]} a(x,t,T_{k+1}(u),\nabla T_{k+1}(u))\cdot\nabla T_{k+1}(u)\xi_j\,dx\,dt\\
&=\lim_{\epsilon\to0^+}\liminf_{j\to\infty}\int_{[k\le|u^\epsilon|\le k+1]} a_\epsilon(x,t,T_{k+1}(u_\epsilon),\nabla T_{k+1}(u_\epsilon))\cdot\nabla T_{k+1}(u_\epsilon)\xi_j\,dx\,dt\\
&\le\lim_{\epsilon\to0^+}\int_{[k\le|u|\le k+1]} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt\\
&\le\lim_{\epsilon\to0^+}\left(\int_{[|u_\epsilon|>k]}|f|\,dx\,dt+\int_{[|u_\epsilon|_{\Sigma_1}|>k]}|g|\,d\sigma(x)\,dt+\int_{[|u_{0\epsilon}|>k]}|b(u_0)|\,dx\,dt+\big|[|u_{0\epsilon}|>k]\big|\right)\\
&=\int_{[|u|>k]}|f|\,dx\,dt+\int_{[|u|_{\Sigma_1}|>k]}|g|\,d\sigma(x)\,dt+\int_{[|u_0|>k]}|b(u_0)|\,dx\,dt+\big|[|u_0|>k]\big| \qquad (47)
\end{aligned}$$

for all $k>0$, where we have used Fatou's lemma in the second step, Lemma 3.8 in the third step and Lemma 3.6(v) in the fifth step.

Thus

$$\lim_{k\to\infty}\int_{[k\le|u|\le k+1]} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt=0$$

as claimed. ■

3.3. Existence of renormalized solutions. In this subsection, we prove Theorem 1.6. We start with an integration by parts formula which is essentially [BP05, Lemma 2.4] with $W_0^{1,p}(\Omega)$ replaced by $W_{\Gamma_0}^{1,p}(\Omega)$. We emphasize that such a replacement causes no harm to [BP05, proof of Lemma 2.4].

Lemma 3.10. *Let $F:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function and B be a maximal monotone graph in $\mathbb{R}$. Define*

$$G(\lambda):=\int_0^\lambda F'(r)\,B(r)\,dr.$$

Let $w\in L^p(0,T;W_{\Gamma_0}^{1,p}(\Omega))$, $w_0\in L^\infty(\Omega)$ and $\beta\in L^\infty(Q_T)$, $\beta_0\in L^\infty(\Omega)$ be such that

$$\begin{aligned}
&\beta\in B(w)\quad\text{a.e. in }Q_T, &&\beta_0\in B(w_0)\quad\text{a.e. in }\Omega,\\
&\beta_t\in L^{p'}(0,T;W^{-1,p'}(\Omega)), &&\beta(0)=\beta_0\quad\text{in }W^{-1,p'}(\Omega).
\end{aligned}$$

Then

$$-\int_0^T\int_\Omega \beta_t F(w)\xi\,dx\,dt$$
$$=\int_0^T\int_\Omega \xi_t(\beta F(w)-G(w))\,dx\,dt+\int_\Omega \xi(0)(\beta_0 F(w_0)-G(w_0))\,dx$$

for all $\xi\in W^{1,\infty}(Q_T)$ *such that* $\xi F(w)\in L^p(0,T;W^{1,p}_{\Gamma_0}(\Omega))$ *and* $\xi(T)=0$.

In particular,

$$-\int_0^T\int_\Omega \beta_t\mathcal{F}(w)\xi\,dx\,dt=\int_0^T\int_\Omega \xi_t\mathcal{F}(\beta)\,dx\,dt+\int_\Omega \xi(0)F(\beta_0)\,dx$$

for all $\xi\in W^{1,\infty}(Q_T)$ *such that* $\xi F(w)\in L^p(0,T;W^{1,p}_{\Gamma_0}(\Omega))$ *and* $\xi(T)=0$*, where*

$$\mathcal{F}(s):=\int_0^s F(B^{-1})(r)\,dr$$

for all $s\in\mathbb{R}$.

We are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. We will verify (R1)–(R3) in Definition 1.3. (R1) follows from Lemma 3.7(viii) and (R3) is clear due to Lemma 3.9. It remains to verify (R2).

Lemma 3.7(iii) asserts that $T_k(u)\in V$ for each $k>0$. Let us now show that (5) in (R2) holds.

Let $0<\epsilon\le 1$, $S\in W^{2,\infty}(\mathbb{R})$ with $\operatorname{supp} S'\subset[-k,k]$ for some $k>0$ and $\varphi\in C_c^\infty([0,T)\times\Omega_1)$. Taking $S_n'(u_\epsilon)\,\varphi$ as a test function in (16), we obtain

$$\begin{aligned}\mathfrak{L}^1_\epsilon+\mathfrak{L}^2_\epsilon+\mathfrak{L}^3_\epsilon &:=\int_{Q_T}\Big(\frac{\partial}{\partial t}b_\epsilon(u_\epsilon)\Big)S'(u_\epsilon)\varphi\,dx\,dt+\int_{Q_T}a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\cdot\nabla(S'(u_\epsilon)\varphi)\,dx\,dt\\ &\quad+\int_{\Sigma_1}\gamma h_\epsilon(u_\epsilon)S'(u_\epsilon)\varphi\,d\sigma(x)\,dt\\ &=\int_{Q_T}f_\epsilon S'(u_\epsilon)\varphi\,dx\,dt+\int_{\Sigma_1}g_\epsilon S'(u_\epsilon)\varphi\,d\sigma(x)\,dt\\ &=:\mathfrak{R}^1_\epsilon+\mathfrak{R}^2_\epsilon.\end{aligned}\tag{48}$$

We aim to pass to the limit when $\epsilon\to0^+$ in each term above. First we deal with $\mathfrak{L}^1_\epsilon$. Applying Lemma 3.10 with

$$\begin{cases} w=u_\epsilon, \quad w_0=u_{0\epsilon},\\ F=S'(u_\epsilon), \quad \xi=\varphi,\\ B=B_{\epsilon S}, \quad \beta=B_{\epsilon S}(u_\epsilon), \quad \beta_0=B_{\epsilon S}(u_{0\epsilon}),\end{cases}$$

we have

$$\begin{aligned}
\mathfrak{L}_\epsilon^1 &= \int_{Q_T} \left(\frac{\partial}{\partial t} b_\epsilon(u_\epsilon)\right) S'(b_\epsilon^{-1}(b_\epsilon(u_\epsilon)))\varphi\, dx\, dt \\
&= -\int_0^T \int_\Omega \varphi_t \left(\int_0^{b_\epsilon(u_\epsilon)} S'(b_\epsilon^{-1}(s))\, ds\right) dx\, dt - \int_\Omega \varphi(0) \left(\int_0^{b_\epsilon(u_{0\epsilon})} S'(b_\epsilon^{-1}(s))\, ds\right) dx \\
&= -\int_0^T \int_\Omega \varphi_t \left(\int_0^{u_\epsilon} S'(s) b_\epsilon'(s)\, ds\right) dx\, dt - \int_\Omega \varphi(0) \left(\int_0^{u_{0\epsilon}} S'(s) b_\epsilon'(s)\, ds\right) dx \\
&= -\int_{Q_T} B_{\epsilon S}(u_\epsilon)\varphi_t\, dx\, dt - \int_\Omega B_{\epsilon S}(u_{0\epsilon})\varphi(0)\, dx,
\end{aligned}$$

where

$$B_{\epsilon S}(\ell) = \int_0^\ell b_\epsilon'(s) S'(s)\, ds, \qquad \ell \in \mathbb{R}.$$

Therefore,

$$\lim_{\epsilon\to 0^+} \mathfrak{L}_\epsilon^1 = -\int_{Q_T} B_S(u)\varphi_t\, dx\, dt - \int_\Omega B_S(u_0)\varphi(0)\, dx,$$

due to the definition of u_ϵ, Lemma 3.6(iv) and the Lebesgue dominated convergence theorem.

Next we use Lemma 3.8 and the fact that $\operatorname{supp} S' \subset [-k, k]$ to obtain

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \mathfrak{L}_\epsilon^2 &= \lim_{\epsilon\to 0^+} \int_{Q_T} a_\epsilon(x, t, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \cdot \nabla(S'(u_\epsilon)\varphi)\, dx\, dt \\
&= \int_{Q_T} a(x, t, T_k(u), \nabla T_k(u)) \cdot \nabla(S'(u)\varphi)\, dx\, dt \\
&= \int_{Q_T} a(x, t, u, \nabla u) \cdot \nabla(S'(u)\varphi)\, dx\, dt.
\end{aligned}$$

Also,

$$\lim_{\epsilon\to 0^+} \mathfrak{L}_\epsilon^3 = \int_{\Sigma_1} \gamma h(u) S'(u)\varphi\, d\sigma(x)\, dt$$

by the Lebesgue dominated convergence theorem.

Concerning the right hand terms, we have

$$\lim_{\epsilon\to 0^+} \mathfrak{R}_\epsilon^1 = \int_\Omega f S'(u)\varphi\, dx$$

and

$$\lim_{\epsilon\to 0^+} \mathfrak{R}_\epsilon^2 = \int_{\Sigma_1} g S'(u)\varphi\, d\sigma(x)\, dt,$$

again by the Lebesgue dominated convergence theorem.

In sum,

$$\begin{aligned}
-\int_{Q_T} B_S(u)\varphi_t\, dx\, dt + \int_{Q_T} a(x, t, u, \nabla u) \cdot \nabla(S'(u)\varphi)\, dx\, dt + \int_{\Sigma_1} \gamma h(u) S'(u)\varphi\, d\sigma(x)\, dt \\
= \int_\Omega f S'(u)\varphi\, dx + \int_{\Sigma_1} g S'(u)\varphi\, d\sigma(x)\, dt + \int_\Omega B_S(u_0)\varphi(0)\, dx,
\end{aligned}$$

which is (5).

At this point, (R2) is verified. Hence u is a renormalized solution for (P). ∎

4. Uniqueness

In this section, we prove Theorem 1.7. Inspired by [BGR16], we rely on two crucial lemmas.

Lemma 4.1. *Let u be a renormalized solution to (P). Let $H \in W^{2,\infty}(\mathbb{R})$ with* $\operatorname{supp} H'$ *compact. Then*

$$\int_{Q_T} \frac{\partial H(b(u))}{\partial t}\varphi\,dx\,dt + \int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla(H'(b(u))\varphi)\,dx\,dt + \int_{\Sigma_1} \gamma h(u)H'(b(u))\varphi\,d\sigma(x)\,dt = \int_{Q_T} fH'(b(u))\varphi\,dx\,dt + \int_{\Sigma_1} gH'(b(u))\varphi\,d\sigma(x)\,dt \tag{49}$$

and

$$H(b(u))|_{t=0} = H(b(u_0)) \qquad \text{in } \Omega.$$

Proof. First we show (49). Set

$$S(\ell) = \int_0^\ell H'(b(s))\,ds, \qquad \ell \in \mathbb{R}.$$

Then $S \in W^{2,\infty}(\mathbb{R})$ and $\operatorname{supp} S'$ is compact. Next, let $\varphi \in C_c^\infty([0,T)\times\Omega_1)$. Using $S'(u)\varphi$ as a test function in (5) and noting that

$$B_S(\ell) = \int_0^\ell b'(s)S'(s)\,ds = \int_0^\ell b'(s)\,H'(b(s))\,ds = H(b(\ell)), \qquad \ell \in \mathbb{R},$$

we arrive at (49).

Secondly, we show the initial condition. Observe that

$$H(b(u)) \in V \quad \text{and} \quad \frac{\partial H(b(u))}{\partial t} \in V^* + L^1(Q_T).$$

Therefore,

$$H(b(u)) \in C([0,T]; W^{-1,s}(\Omega)) \qquad \text{for all } 1 < s < \min\left\{p', \frac{d}{d-1}\right\}. \tag{50}$$

Let $\psi \in C_c^\infty(\Omega)$ and set

$$\varphi = \psi(x)\left(1 - \frac{t}{\sigma}\right)^+.$$

Using $S'(u)\varphi$ as a test function in (5) gives

$$\int_{Q_T} \frac{\partial H(b(u))}{\partial t}\varphi\,dx\,dt + \int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla(H'(b(u))\varphi)\,dx\,dt + \int_{\Sigma_1} \gamma h(u)H'(b(u))\varphi\,d\sigma(x)\,dt = \int_{Q_T} fH'(b(u))\varphi\,dx\,dt + \int_{\Sigma_1} gH'(b(u))\varphi\,d\sigma(x)\,dt.$$

We focus on the first term of the left hand side. Integration by parts gives

$$\int_{Q_T} \frac{\partial H(b(u))}{\partial t}\varphi\,dx\,dt = -\int_{Q_T} H(b(u))\partial_t\varphi\,dx\,dt = \frac{1}{\sigma}\int_0^\sigma\int_\Omega H(b(u))\psi\,dx\,dt.$$

We treat this last term in two different ways. First,

$$\lim_{\sigma\to 0^+}\frac{1}{\sigma}\int_0^\sigma\int_\Omega \psi H(b(u))\,dx\,dt = \int_\Omega \psi H(b(u_0))\,dx\,dt = \langle H(b(u_0)),\psi\rangle_{W^{-1,s}(\Omega)\times W_0^{1,s'}(\Omega)}$$

by the Lebesgue differentiation theorem. Second,

$$\begin{aligned}\lim_{\sigma\to 0^+}\frac{1}{\sigma}\int_0^\sigma\int_\Omega \psi H(b(u))\,dx\,dt &= \lim_{\sigma\to 0^+}\frac{1}{\sigma}\int_0^\sigma \langle H(b(u(t))),\psi\rangle_{W^{-1,s}(\Omega)\times W_0^{1,s'}(\Omega)}\,dt\\ &= \langle H(b(u(t)))|_{t=0},\psi\rangle_{W^{-1,s}(\Omega)\times W_0^{1,s'}(\Omega)}\end{aligned}$$

by (50).

Hence

$$\langle H(b(u_0)),\psi\rangle_{W^{-1,s}(\Omega)\times W_0^{1,s'}(\Omega)} = \langle H(b(u(t)))|_{t=0},\psi\rangle_{W^{-1,s}(\Omega)\times W_0^{1,s'}(\Omega)}$$

for all $\psi\in C_c^\infty(\Omega)$. By density,

$$H(b(u_0)) = H(b(u(t)))|_{t=0}$$

as required. ■

LEMMA 4.2. *Let u and v be renormalized solutions to (P). Define*

$$A(u,s,k) := \{(x,t)\in Q_T : |b(u)-b(\pm s)|\le k\}$$

and

$$\begin{aligned}W(u,v,s,k) := &\int_{A(u,s,k)} b'(u)\,a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt\\ &+\int_{A(v,s,k)} b'(v)\,a(x,t,v,\nabla v)\cdot\nabla v\,dx\,dt\end{aligned}$$

for all $0<k<s$. Then

$$\liminf_{s\to\infty}\limsup_{k\to 0^+}\frac{W(u,v,s,k)}{k} = 0.$$

Proof. We argue by contradiction. For each $s>0$ define

$$\begin{aligned}F(s) &:= \int_{[0<b(u)<s]} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt + \int_{[0<b(v)<s]} a(x,t,v,\nabla v)\cdot\nabla v\,dx\,dt,\\ G(s) &:= \int_{[-s<b(u)<0]} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt + \int_{[-s<b(v)<0]} a(x,t,v,\nabla v)\cdot\nabla v\,dx\,dt.\end{aligned}$$

Then F and G are increasing, hence are differentiable a.e. with F' and G' being measurable. Moreover,

$$F(s)-F(\eta)\ge\int_\eta^s F'(z)\,dz \quad\text{and}\quad G(s)-G(\eta)\ge\int_\eta^s G'(z)\,dz \tag{51}$$

for all $0<\eta<s$.

At the same time,

$$F'(s) = \frac{1}{2} \limsup_{k\to 0^+} \frac{1}{k} \Big[\int_{[s-k\le b(u)<s+k]} a(x,t,u,\nabla u)\cdot \nabla u\, dx\, dt + \int_{[s-k\le b(v)<s+k]} a(x,t,v,\nabla v)\cdot \nabla v\, dx\, dt \Big]$$

and

$$G'(s) = \frac{1}{2} \limsup_{k\to 0^+} \frac{1}{k} \Big[\int_{[-s-k\le b(u)<-s+k]} a(x,t,u,\nabla u)\cdot \nabla u\, dx\, dt + \int_{[-s-k\le b(v)<-s+k]} a(x,t,v,\nabla v)\cdot \nabla v\, dx\, dt \Big]$$

for a.e. $s > 0$.

Now suppose that, contrary to the statement, there exist constants $\epsilon_0, s_0 > 0$ such that

$$\limsup_{k\to 0^+} \frac{1}{k} W(u,v,s,k) \ge \epsilon_0$$

for all $s \ge s_0$. Using (H3) we deduce that

$$\begin{aligned}
\epsilon_0/2 &\le \limsup_{k\to 0^+} \frac{1}{k} W(u,v,s,k) \\
&\le b'(s) \limsup_{k\to 0^+} \frac{1}{k} \Big[\int_{[b(s)-k\le b(u)<b(s)+k]} a(x,t,u,\nabla u)\cdot \nabla u\, dx\, dt \\
&\qquad + \int_{[b(s)-k\le b(v)<b(s)+k]} a(x,t,v,\nabla v)\cdot \nabla v\, dx\, dt \Big] \\
&\quad + b'(-s) \limsup_{k\to 0^+} \frac{1}{k} \Big[\int_{[b(-s)-k\le b(u)<b(-s)+k]} a(x,t,u,\nabla u)\cdot \nabla u\, dx\, dt \\
&\qquad + \int_{[-s-k\le b(v)<-s+k]} a(x,t,v,\nabla v)\cdot \nabla v\, dx\, dt \Big] \\
&= 2b'(s) F'(b(s)) + 2b'(-s) G'(-b(-s))
\end{aligned}$$

for a.e. $s > s_0$.

Let $n > s_0$. Integrating this last estimate over the interval $(n, n+1)$ and then referring to (51), we arrive at

$$F(b(n+1)) - F(b(n)) + G(-b(-n-1)) - G(-b(-n)) \ge \epsilon_0/2,$$

or equivalently

$$\int_{[n\le |u|<n+1]} a(x,t,u,\nabla u)\cdot \nabla u\, dx\, dt + \int_{[n\le |v|<n+1]} a(x,t,v,\nabla v)\cdot \nabla v\, dx\, dt \ge \epsilon_0/2.$$

But this contradicts (R3) in Definition 1.3. ■

Now we prove Theorem 1.7.

Proof of Theorem 1.7. We divide the proof into two steps.

Step 1: Let $H \in W^{2,\infty}(\mathbb{R})$ be such that $\operatorname{supp} H'$ is compact and $n, \delta, k > 0$. For each $r \in \mathbb{R}$ define

$$\widetilde{T}_n(r) := \begin{cases} b(n) & \text{if } r > b(n), \\ r & \text{if } b(-n) \le r \le b(n), \\ b(-n) & \text{if } r < b(-n) \end{cases}$$

and a smooth approximation $\widetilde{T}_n^\delta(r)$ of $\widetilde{T}_n(r)$ by

$$\big(\widetilde{T}_n^\delta\big)'(r) = \begin{cases} 0 & \text{if } r < b(-n) - \delta, \\ \dfrac{r + \delta - b(-n)}{\delta} & \text{if } b(-n) - \delta \le r \le b(-n), \\ 1 & \text{if } b(-n) \le r \le b(n), \\ \dfrac{b(n) + \delta - r}{\delta} & \text{if } b(n) \le r \le b(n) + \delta, \\ 0 & \text{if } r \ge b(n) + \delta. \end{cases}$$

Since $\widetilde{T}_n^\delta \in W^{2,\infty}(\mathbb{R})$ and $\operatorname{supp}\big(\widetilde{T}_n^\delta\big)' \subset [b(-n) - \delta, b(n) + \delta]$, we may use

$$\varphi = \frac{1}{k} T_k\big(\widetilde{T}_n^\delta(b(u)) - \widetilde{T}_n^\delta(b(v))\big) \mathbb{1}_{[0,t]}$$

for each $t \in (0, T)$ as a test function in (49) to obtain

$$\begin{aligned} \int_0^T \int_{Q_t} \frac{\partial \widetilde{T}_n^\delta(b(u))}{\partial t} \varphi \, dx \, ds \, dt + \int_0^T \int_{Q_t} a(x,t,u,\nabla u) \cdot \nabla\big(\big(\widetilde{T}_n^\delta\big)'(b(u))\varphi\big) \, dx \, ds \, dt \\ + \int_0^T \int_{\Sigma_{1,t}} \gamma h(u) \big(\widetilde{T}_n^\delta\big)'(b(u)) \, \varphi \, d\sigma(x) \, dt \\ = \int_0^T \int_{Q_t} f \big(\widetilde{T}_n^\delta\big)'(b(u)) \varphi \, dx \, ds \, dt + \int_0^T \int_{\Sigma_{1,t}} g \big(\widetilde{T}_n^\delta\big)'(b(u)) \varphi \, d\sigma(x) \, ds \, dt, \end{aligned} \tag{52}$$

where we denote

$$Q_t := \Omega \times (0, t) \quad \text{and} \quad \Sigma_{1,t} := \Gamma_1 \times (0, t).$$

In the same way,

$$\begin{aligned} \int_0^T \int_{Q_t} \frac{\partial \widetilde{T}_n^\delta(b(v))}{\partial t} \varphi \, dx \, ds \, dt + \int_0^T \int_{Q_t} a(x,t,v,\nabla v) \cdot \nabla\big(\big(\widetilde{T}_n^\delta\big)'(b(v))\varphi\big) \, dx \, ds \, dt \\ + \int_0^T \int_{\Sigma_{1,t}} \gamma h(u) \big(\widetilde{T}_n^\delta\big)'(b(v)) \varphi \, d\sigma(x) \, dt \\ = \int_0^T \int_{Q_t} f \big(\widetilde{T}_n^\delta\big)'(b(v)) \varphi \, dx \, ds \, dt + \int_0^T \int_{\Sigma_{1,t}} g \big(\widetilde{T}_n^\delta\big)'(b(v)) \varphi \, d\sigma(x) \, ds \, dt. \end{aligned} \tag{53}$$

Subtracting (53) from (52), we obtain

$$\begin{aligned} &\mathfrak{L}^1_{\delta,k,n} + \mathfrak{L}^2_{\delta,k,n} + \mathfrak{L}^3_{\delta,k,n} + \mathfrak{L}^4_{\delta,k,n} \\ &\quad := \int_0^T \int_{Q_t} \left(\frac{\partial \widetilde{T}_n^\delta(b(u))}{\partial t} - \frac{\partial \widetilde{T}_n^\delta(b(v))}{\partial t} \right) \varphi \, dx \, ds \, dt \\ &\quad + \int_0^T \int_{Q_t} \big[a(x,t,u,\nabla u) \big(\widetilde{T}_n^\delta\big)'(b(u)) - a(x,t,v,\nabla v) \big(\widetilde{T}_n^\delta\big)'(b(v)) \big] \cdot \nabla\varphi \, dx \, ds \, dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_{Q_t}\big[a(x,t,u,\nabla u)\big(\widetilde{T}_n^\delta\big)''(b(u))b'(u)\cdot\nabla u\\
&\qquad\qquad\qquad - a(x,t,v,\nabla v)\big(\widetilde{T}_n^\delta\big)''(b(v))b'(v)\cdot\nabla v\big]\varphi\,dx\,ds\,dt\\
&+\int_0^T\int_{\Sigma_{1,t}}\gamma h(u)\big[\big(\widetilde{T}_n^\delta\big)'(b(u))-\big(\widetilde{T}_n^\delta\big)'(b(v))\big]\varphi\,d\sigma(x)\,ds\,dt\\
&=\int_0^T\int_{Q_t} f\big[\big(\widetilde{T}_n^\delta\big)'(b(u))-\big(\widetilde{T}_n^\delta\big)'(b(v))\big]\varphi\,dx\,ds\,dt\\
&+\int_0^T\int_{\Sigma_{1,t}} g\big[\big(\widetilde{T}_n^\delta\big)'(b(u))-\big(\widetilde{T}_n^\delta\big)'(b(v))\big]\varphi\,d\sigma(x)\,ds\,dt\\
&=: \mathfrak{R}^1_{\delta,k,n}+\mathfrak{R}^2_{\delta,k,n}.
\end{aligned}
\tag{54}
$$

Next we estimate each term separately. By Lemma 4.1,

$$\widetilde{T}_n^\delta(b(u))|_{t=0}=\widetilde{T}_n^\delta(b(v))|_{t=0}=\widetilde{T}_n^\delta(b(u_0)).$$

This in combination with [BMP89, Lemma 2.4] gives

$$
\begin{aligned}
\mathfrak{L}^1_{\delta,k,n}&=\frac{1}{k}\int_0^T\int_{Q_t}\bigg(\frac{\partial\widetilde{T}_n^\delta(b(u))}{\partial t}-\frac{\partial\widetilde{T}_n^\delta(b(v))}{\partial t}\bigg)T_k\big(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))\big)\,dx\,ds\,dt\\
&=\frac{1}{k}\int_{Q_T}\overline{T}_k\big(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))\big)\,dx\,dt,
\end{aligned}
$$

where

$$\overline{T}_k(z):=\int_0^z T_k(s)\,ds,\qquad z\in\mathbb{R}.$$

Using the definition of $\widetilde{T}_n^\delta$, we deduce further that

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{L}^1_{\delta,k,n}=\int_{Q_T}|b(u)-b(v)|\,dx\,dt.$$

In a series of technical lemmas below, we verify that

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{L}^2_{\delta,k,n}\geq 0$$

and

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{E}=0,$$

where $\mathfrak{E}$ is either $\mathfrak{L}^3_{\delta,k,n}$, $\mathfrak{L}^4_{\delta,k,n}$, $\mathfrak{R}^1_{\delta,k,n}$ or $\mathfrak{R}^2_{\delta,k,n}$.

Putting these estimates in (54), we arrive at

$$\int_{Q_T}|b(u)-b(v)|\,dx\,dt\leq 0,$$

which in turn yields

$$b(u)=b(v)\qquad\text{a.e. in } Q_T.$$

Since b is strictly increasing, we also have $u=v$ a.e. in Q_T as required. ∎

LEMMA 4.3. *Let $\mathfrak{L}^2_{\delta,k,n}$ be defined by* (54). *Then*

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{L}^2_{\delta,k,n}\geq 0.$$

Proof. It follows from the definition of $\widetilde{T}_n^\delta$ that

$$\lim_{\delta\to 0^+}\big(\widetilde{T}_n^\delta\big)'(b(u)) = \mathbb{1}_{[b(-n)\le b(u)\le b(n)]} = \mathbb{1}_{[|u|\le n]} \quad \text{a.e. in } Q_T$$

and

$$\lim_{\delta\to 0^+}\big(\widetilde{T}_n^\delta\big)'(b(u)) = \mathbb{1}_{[b(-n)\le b(u)\le b(n)]} = \mathbb{1}_{[|u|\le n]} \quad \text{in } L^q(Q_T) \text{ for all } 1<q<\infty.$$

Furthermore,

$$\lim_{\delta\to 0^+}\widetilde{T}_n^\delta(b(u)) = T_n(b(u)) \quad \text{a.e. in } Q_T$$

and

$$\lim_{\delta\to 0^+}\widetilde{T}_n^\delta(b(u)) = T_n(b(u)) \quad \text{in } V.$$

For all sufficiently small δ, we have $\operatorname{supp}\big(\widetilde{T}_n^\delta\big)' \subset [b(-n)-\delta, b(n)+\delta]$ and hence

$$\big(\widetilde{T}_n^\delta\big)'(b(u))a(x,t,u,\nabla u) = \big(\widetilde{T}_n^\delta\big)'(b(u))\, a(x,t,T_{n+1}(u),T_{n+1}(\nabla u)) \quad \text{a.e. in } Q_T.$$

It follows that

$$\begin{aligned}
\lim_{\delta\to 0^+}\mathfrak{L}^2_{\delta,k,n} &= \frac{1}{k}\int_{Q_T}(T-t)\\
&\quad\times\big[\mathbb{1}_{[|u|\le n]}a(x,t,T_{n+1}(u),T_{n+1}(\nabla u)) - \mathbb{1}_{[|v|\le n]}a(x,t,T_{n+1}(v),T_{n+1}(\nabla v))\big]\\
&\quad\times\nabla T_k\big(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))\big)\,dx\,dt\\
&= \frac{1}{k}\bigg(\int_{[|u|\le n,|v|\le n]} + \int_{[|u|\le n,|v|>n]} + \int_{[|u|>n,|v|\le n]}\bigg)\dots\,dx\,dt\\
&=: \mathfrak{L}^{2,1}_{k,n} + \mathfrak{L}^{2,2}_{k,n} + \mathfrak{L}^{2,3}_{k,n}.
\end{aligned}$$

We estimate each term separately.

Term $\mathfrak{L}^{2,1}_{k,n}$: For short, we denote

$$\mathcal{D} := [|u|\le n, |v|\le n]\cap[|b(u)-b(v)|\le K].$$

Then

$$\begin{aligned}
\mathfrak{L}^{2,1}_{k,n} &= \frac{1}{k}\int_{\mathcal{D}}(T-t)[a(x,t,u,\nabla u)-a(x,t,b,\nabla v)]\cdot\nabla[b(u)-b(v)]\,dx\,dt\\
&= \frac{1}{k}\int_{\mathcal{D}}(T-t)[a(x,t,u,\nabla u)-a(x,t,u,\nabla v)]\cdot\nabla[b(u)-b(v)]\,dx\,dt\\
&\quad+\frac{1}{k}\int_{\mathcal{D}}(T-t)[a(x,t,u,\nabla v)-a(x,t,v,\nabla v)]\cdot\nabla[b(u)-b(v)]\,dx\,dt\\
&=: \mathfrak{L}^{2,1a}_{k,n} + \mathfrak{L}^{2,1b}_{k,n}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathfrak{L}^{2,1a}_{k,n} &= \frac{1}{k}\int_{\mathcal{D}} (T-t)[a(x,t,u,\nabla u) - a(x,t,u,\nabla v)]\cdot(\nabla u - \nabla v) b'(u)\,dx\,dt \\
&\quad + \frac{1}{k}\int_{\mathcal{D}} (T-t)[a(x,t,u,\nabla u) - a(x,t,u,\nabla v)]\cdot(\nabla v)[b'(u) - b'(v)]\,dx\,dt \\
&\geq \frac{1}{k}\int_{\mathcal{D}} (T-t)[a(x,t,u,\nabla u) - a(x,t,u,\nabla v)]\cdot(\nabla v)[b'(u) - b'(v)]\,dx\,dt \\
&=: I,
\end{aligned}$$

where we have used (H2) and (H3) in the second step.

By hypothesis, b' is locally Lipschitz and hence

$$|b'(r) - b'(s)| \leq C(n)|r-s| \leq \frac{C(n)}{\beta}|b(r) - b(s)|$$

for all $r, s \in [-n, n]$. Consequently,

$$\begin{aligned}
|I| &\leq \frac{C(n)T}{\beta k}\int_{\mathcal{D}} |a(x,t,u,\nabla u) - a(x,t,u,\nabla v)|\,|\nabla v|\,|b(u) - b(v)|\,dx\,dt \\
&\leq \frac{C(n)T}{\beta}\int_{\mathcal{D}} |a(x,t,u,\nabla u) - a(x,t,u,\nabla v)|\,|\nabla v|\,dx\,dt \\
&= \frac{C(n)T}{\beta}\int_{\mathcal{D}} \big|a(x,t,T_n(u),\nabla T_n(u)) - a(x,t,T_n(u),\nabla T_n(v))\big|\,|\nabla T_n(v)|\,dx\,dt.
\end{aligned}$$

But

$$\lim_{k\to 0^+} \big|a(x,t,T_n(u),\nabla T_n(u)) - a(x,t,T_n(u),\nabla T_n(v))\big|\,|\nabla T_n(v)|\mathbb{1}_{\mathcal{D}} = 0 \qquad \text{a.e. in } Q_T$$

and

$$\begin{aligned}
&\big|a(x,t,T_n(u),\nabla T_n(u)) - a(x,t,T_n(u),\nabla T_n(v))\big|\,|\nabla T_n(v)|\mathbb{1}_{\mathcal{D}} \\
&\qquad \leq \big|a(x,t,T_n(u),\nabla T_n(u)) - a(x,t,T_n(u),\nabla T_n(v))\big|\,|\nabla T_n(v)| \in L^1(Q_T).
\end{aligned}$$

By virtue of the Lebesgue dominated convergence theorem, we may conclude that

$$\lim_{k\to 0^+} |I| \geq 0.$$

This implies

$$\limsup_{k\to 0^+} \mathfrak{L}^{2,1a}_{k,n} \geq 0.$$

Next we deal with $\mathfrak{L}^{2,1b}_{k,n}$. In view of (7),

$$\mathfrak{L}^{2,1b}_{k,n} \leq \frac{T}{k}\int_{\mathcal{D}} |T_n(u) - T_n(v)|[E_n(x,t) + \zeta_n|\nabla T_n(v)|^{p-1}]|\nabla T_n(b(u)) - \nabla T_n(b(v))|\,dx\,dt.$$

Since u and v are renormalized solutions to (P), we have

$$[E_n(x,t) + \zeta_n|\nabla T_n(v)|^{p-1}]|\nabla T_n(b(u)) - \nabla T_n(b(v))| \in L^1(Q_T).$$

Next recall that b' is positive continuous function. Hence for all $n > 0$ there exists a constant $\alpha_n > 0$ such that

$$|u-v| = |T_n(u) - T_n(v)| \leq \alpha_n |b(u) - b(v)| \qquad \text{a.e. in } [|u| \leq n, |v| \leq n].$$

Therefore,

$$\mathfrak{L}^{2,1b}_{k,n} \leq T\alpha_n \int_{\mathcal{D}\cap[b(u)\neq b(v)]} [E_n(x,t) + \zeta_n |\nabla T_n(v)|^{p-1}] \big|\nabla T_n(b(u)) - \nabla T_n(b(v))\big| \, dx\, dt.$$

But

$$\lim_{k\to 0^+} \mathbb{1}_{\mathcal{D}\cap[b(u)\neq b(v)]} = 0 \qquad \text{a.e. in } Q_T.$$

Using the Lebesgue dominated convergence theorem, we obtain

$$\limsup_{k\to 0^+} \mathfrak{L}^{2,1b}_{k,n} = 0.$$

Thus,

$$\limsup_{k\to 0^+} \mathfrak{L}^{2}_{k,n} \geq 0.$$

Terms $\mathfrak{L}^{2,2}_{k,n}$ *and* $\mathfrak{L}^{2,3}_{k,n}$: We have

$$\begin{aligned} |\mathfrak{L}^{2,2}_{k,n}| &= \frac{1}{k} \int_{[|u|\leq n, |v|>n]} (T-t) a(x,t,T_n(u), \nabla T_n(u)) \nabla T_k(b(u) - b(n \operatorname{sign}(v)))\, dx\, dt \\ &\leq \frac{T}{k} \int_{[b(-n)<b(u)<b(-n)+k]\cup[b(n)-k<b(u)<b(n)]} a(x,t,T_n(u),\nabla T_n(u)) \cdot \nabla b(u)\, dx\, dt \end{aligned}$$

provided that $n \geq 1$.

Similarly,

$$|\mathfrak{L}^{2,3}_{k,n}| \leq \frac{T}{k} \int_{[b(-n)<b(v)<b(-n)+k]\cup[b(n)-k<b(v)<b(n)]} a(x,t,T_n(v),\nabla T_n(v)) \cdot \nabla b(v)\, dx\, dt$$

provided that $n \geq 1$.

By recalling the definition of W in Lemma 4.2, we infer that

$$|\mathfrak{L}^{2,2}_{k,n}| + |\mathfrak{L}^{2,3}_{k,n}| \leq \frac{T}{k} W(u,v,n,k)$$

provided that $n \geq 1$.

Hence

$$\liminf_{n\to\infty} \limsup_{k\to 0^+} (|\mathfrak{L}^{2,2}_{k,n}| + |\mathfrak{L}^{2,3}_{k,n}|) = 0.$$

This verifies our claim. ■

Lemma 4.4. *Let* $\mathfrak{L}^{3}_{\delta,k,n}$ *be defined by* (54). *Then*

$$\liminf_{n\to\infty} \limsup_{k\to 0^+} \limsup_{\delta\to 0^+} \mathfrak{L}^{3}_{\delta,k,n} = 0.$$

Proof. We have

$$
\begin{aligned}
&|\mathfrak{L}^3_{\delta,k,n}| \\
&\quad = \frac{1}{k}\left|\int_0^T \int_{Q_t} a(x,t,u,\nabla u)\cdot \big(\widetilde{T}_n^\delta\big)''(b(u))b'(u)(\nabla u)T_k\big(\widetilde{T}_n^\delta(b(u)) - \widetilde{T}_n^\delta(b(v))\big)\,dx\,ds\,dt\right| \\
&\qquad + \frac{1}{k}\left|\int_0^T \int_{Q_t} a(x,t,v,\nabla v)\cdot \big(\widetilde{T}_n^\delta\big)''(b(v))b'(v)(\nabla v)T_k\big(\widetilde{T}_n^\delta(b(u)) - \widetilde{T}_n^\delta(b(v))\big)\,dx\,ds\,dt\right| \\
&\quad \le \frac{T}{\sigma}\int_{[b(-n)-\sigma<b(u)<b(-n)]\cup[b(n)<b(u)<b(n)+\sigma]} b'(u)a(x,t,u,\nabla u)\cdot \nabla u\,dx\,dt \\
&\quad \le \frac{T}{\sigma}\int_{[b(-n)-\sigma<b(v)<b(-n)]\cup[b(n)<b(v)<b(n)+\sigma]} b'(v)\,a(x,t,v,\nabla v)\cdot \nabla v\,dx\,dt \\
&\quad \le \frac{T}{\sigma}W(u,v,n,\sigma).
\end{aligned}
$$

Now an application of Lemma 4.2 justifies the claim. ∎

LEMMA 4.5. *Let $\mathfrak{R}^1_{\delta,k,n}$ be defined by* (54). *Then*

$$
\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{R}^1_{\delta,k,n} = 0.
$$

Proof. We have

$$
\mathfrak{R}^1_{\delta,k,n} = \frac{1}{k}\int_0^T\int_{Q_t} f\big[\big(\widetilde{T}_n^\delta\big)'(b(u)) - \big(\widetilde{T}_n^\delta\big)'(b(v))\big]T_k\big(\widetilde{T}_n^\delta(b(u)) - \widetilde{T}_n^\delta(b(v))\big)\,dx\,ds\,dt.
$$

Observe that

$$
\lim_{\delta\to 0^+} f\big[\big(\widetilde{T}_n^\delta\big)'(b(u)) - \big(\widetilde{T}_n^\delta\big)'(b(v))\big] = f[\mathbb{1}_{[b(-n)\le b(u)\le b(n)]} - \mathbb{1}_{[b(-n)\le b(v)\le b(n)]}] \quad \text{in } L^1(Q_T)
$$

and

$$
\lim_{\delta\to 0^+} T_k\big(\widetilde{T}_n^\delta(b(u)) - \widetilde{T}_n^\delta(b(v))\big) = T_k\big(\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))\big) \quad \begin{cases}\text{a.e. in } Q_T,\\ \text{weak}^* \text{ in } L^\infty(Q_T).\end{cases}
$$

Also,

$$
\lim_{k\to 0^+}\frac{1}{k}T_k(r) = \operatorname{sign}(r) \quad \begin{cases}\text{in } \mathbb{R},\\ \text{weak}^* \text{ in } L^\infty(\mathbb{R}).\end{cases}
$$

It follows that

$$
\begin{aligned}
&\lim_{k\to 0^+}\lim_{\delta\to 0^+}\mathfrak{R}^1_{\delta,k,n} \\
&\quad = \int_{Q_T}(T-t)f[\mathbb{1}_{[b(-n)\le b(u)\le b(n)]} - \mathbb{1}_{[b(-n)\le b(v)\le b(n)]}]\operatorname{sign}\big(\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))\big)\,dx\,dt.
\end{aligned}
$$

Next

$$
\lim_{n\to\infty} f[\mathbb{1}_{[b(-n)\le b(u)\le b(n)]} - \mathbb{1}_{[b(-n)\le b(v)\le b(n)]}]\operatorname{sign}\big(\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))\big) = 0 \quad \text{a.e. in } Q_T
$$

and

$$
\big|f[\mathbb{1}_{[b(-n)\le b(u)\le b(n)]} - \mathbb{1}_{[b(-n)\le b(v)\le b(n)]}]\operatorname{sign}\big(\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))\big)\big| \le 2|f| \in L^1(Q_T).
$$

Hence

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{R}^1_{\delta,k,n}=0$$

in view of the Lebesgue dominated convergence theorem. ∎

LEMMA 4.6. *Let* $\mathfrak{L}^4_{\delta,k,n}$ *and* $\mathfrak{R}^2_{\delta,k,n}$ *be defined by* (54). *Then*

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{L}^4_{\delta,k,n}=0$$

and

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{R}^2_{\delta,k,n}=0.$$

Proof. The proof is similar to that of Lemma 4.5. We will prove the identity concerning $\mathfrak{R}^2_{\delta,k,n}$ only.

Recall that

$$\mathfrak{R}^2_{\delta,k,n}=\int_0^T\int_{\Sigma_{1,t}} g\big[\big(\widetilde{T}^\delta_n\big)'(b(u))-\big(\widetilde{T}^\delta_n\big)'(b(v))\big]\varphi\,d\sigma(x)\,ds\,dt.$$

Observe that

$$\begin{aligned}\lim_{\delta\to 0^+} g\big[\big(\widetilde{T}^\delta_n\big)'(b(u|_{\Sigma_1}))-\big(\widetilde{T}^\delta_n\big)'(b(v|_{\Sigma_1}))\big]&\\ =g[\mathbb{1}_{[b(-n)\le b(u|_{\Sigma_1})\le b(n)]}-\mathbb{1}_{[b(-n)\le b(v|_{\Sigma_1})\le b(n)]}]&\quad\text{in } L^1(\Sigma_1)\end{aligned}$$

and

$$\begin{aligned}\lim_{\delta\to 0^+} T_k\big(\widetilde{T}^\delta_n(b(u|_{\Sigma_1}))-\widetilde{T}^\delta_n(b(v|_{\Sigma_1}))\big)&\\ =T_k\big(\widetilde{T}_n(b(u|_{\Sigma_1}))-\widetilde{T}_n(b(v|_{\Sigma_1}))\big)&\quad\begin{cases}\text{a.e. in } \Sigma_1,\\ \text{weak}^* \text{ in } L^\infty(\Sigma_1).\end{cases}\end{aligned}$$

Also,

$$\lim_{k\to 0^+}\frac{1}{k}T_k(r)=\operatorname{sign}(r)\qquad\begin{cases}\text{in } \mathbb{R},\\ \text{weak}^* \text{ in } L^\infty(\mathbb{R}).\end{cases}$$

It follows that

$$\begin{aligned}\lim_{k\to 0^+}\lim_{\delta\to 0^+}\mathfrak{R}^2_{\delta,k,n}=\int_{\Sigma_1}&(T-t)g[\mathbb{1}_{[b(-n)\le b(u|_{\Sigma_1})\le b(n)]}-\mathbb{1}_{[b(-n)\le b(v|_{\Sigma_1})\le b(n)]}]\\ &\times\operatorname{sign}\big(\widetilde{T}_n(b(u|_{\Sigma_1}))-\widetilde{T}_n(b(v|_{\Sigma_1}))\big)\,dx\,dt.\end{aligned}$$

Next

$$\begin{aligned}\lim_{n\to\infty} g[\mathbb{1}_{[b(-n)\le b(u|_{\Sigma_1})\le b(n)]}-\mathbb{1}_{[b(-n)\le b(v|_{\Sigma_1})\le b(n)]}]&\\ \times\operatorname{sign}\big(\widetilde{T}_n(b(u|_{\Sigma_1}))-\widetilde{T}_n(b(v|_{\Sigma_1}))\big)=0&\quad\text{a.e. in } \Sigma_1\end{aligned}$$

and

$$\begin{aligned}\big|g[\mathbb{1}_{[b(-n)\le b(u|_{\Sigma_1})\le b(n)]}-\mathbb{1}_{[b(-n)\le b(v|_{\Sigma_1})\le b(n)]}]&\\ \times\operatorname{sign}\big(\widetilde{T}_n(b(u|_{\Sigma_1}))-\widetilde{T}_n(b(v|_{\Sigma_1}))\big)\big|&\le 2|g|\in L^1(\Sigma_1).\end{aligned}$$

Hence

$$\liminf_{n\to\infty}\limsup_{k\to 0^+}\limsup_{\delta\to 0^+}\mathfrak{R}^1_{\delta,k,n}=0$$

in view of the Lebesgue dominated convergence theorem. ■

5. Renormalized solutions versus weak and distributional solutions

This section is devoted to the proof of Proposition 1.10. We report two preliminary results: one concerning a regularity estimate and one concerning the initial condition in (P).

LEMMA 5.1. *Let $2-\frac{1}{d+1}<p<\infty$ and $\kappa>0$. Suppose $g\in L^p(0,T;W^{1,p}_{\Gamma_0}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$ satisfies*

$$\|g\|_{L^\infty(0,T;L^1(\Omega))}\le\kappa$$

and

$$\sup_{\ell\ge 0}\int_{[\ell\le|u|\le\ell+1]}|\nabla g|^p\,dx\,dt\le\kappa$$

for all $\ell\ge 0$. Set

$$q_0:=\min\left\{\frac{d}{d+1}(p-1),p\right\}.$$

Then for all $1\le q<q_0$ there exists a $C=C(\kappa,q)>0$ such that

$$\|g\|_{L^q(0,T;W^{1,q}_{\Gamma_0}(\Omega))}\le C. \tag{56}$$

Proof. First, if $2-\frac{1}{d+1}<p<d$ then arguments analogous to those used in [BWZ10, proof of Lemma 2.1] justify (56) for all

$$1\le q<\frac{d}{d+1}(p-1). \tag{57}$$

The only difference is that [BWZ10, proof of Lemma 2.1] requires $g\in L^p(0,T;W^{1,p}_0(\Omega))$ in place of $g\in L^p(0,T;W^{1,p}_{\Gamma_0}(\Omega))$. This is required to ensure the validity of the Poincaré inequality, which remains valid for $g\in L^p(0,T;W^{1,p}_{\Gamma_0}(\Omega))$ in view of (H5).

Secondly, if $p\ge d$, then we replace (57) by

$$1\le q<p$$

and then the same arguments as before carry over. ■

LEMMA 5.2. *Let u be a renormalized solution to (P). Let $S\in W^{2,\infty}(\mathbb{R})$ and $\varphi\in C_c^\infty(Q_T)$ be such that* $\operatorname{supp} S'$ *is compact and $S'(u)\varphi\in V$. Then*

$$B_S(u)|_{t=0}=B_S(u_0)\quad\text{in }\Omega.$$

Proof. We simply repeat the second part of the proof of Lemma 4.1 with $H(b(u))$ replaced by $B_S(u)$. ■

Keeping the above auxiliary results in mind, we proceed with the proof of Proposition 1.10.

Proof of Proposition 1.10. First assume $f \in L^{p'}(Q_T)$ and $g \in L^{p'}(\Sigma_1)$. Let u be a weak solution to (P). We aim to show that u is also a renormalized solution. To derive (R1), we argue similarly as for Lemma 3.6(iv). Condition (R3) also follows due to (2) and the fact that $u \in V$. It remains to verify (R2).

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that $\operatorname{supp} S'$ is compact and $\varphi \in C_c^\infty([0,T) \times \Omega_1)$. Observe that (8) holds for all test functions in V by density. Therefore, we may use $S'(u)\varphi \in V$ as a test function in (8) to obtain

$$\int_{Q_T} \Big(\frac{\partial}{\partial t} b(u)\Big) S'(u)\varphi \, dx \, dt + \int_{Q_T} a(x,t,u,\nabla u) \cdot \nabla(S'(u)\varphi) \, dx \, dt + \int_{\Sigma_1} \gamma h(u) S'(u)\varphi \, d\sigma(x) \, dt$$
$$= \int_{Q_T} f S'(u)\varphi \, dx \, dt + \int_{\Sigma_1} g S'(u)\varphi \, d\sigma(x) \, dt. \tag{58}$$

Comparing (58) to (5), we see that (R2) holds if

$$\int_{Q_T} \Big(\frac{\partial}{\partial t} b(u)\Big) S'(u)\varphi \, dx \, dt = -\int_{Q_T} B_S(u)\varphi_t \, dx \, dt + \int_\Omega B_S(u_0)\varphi(0) \, dx. \tag{59}$$

But this is clear in view of Lemma 3.10. Indeed, applying Lemma 3.10 with

$$w = u, \quad w_0 = u_0, \quad F = T_k, \quad B = B_S, \quad \beta = B_S(u) \quad \text{and} \quad \beta_0 = B_S(u_0),$$

we have

$$\begin{aligned}
&\int_{Q_T} \Big(\frac{\partial}{\partial t} b(u)\Big) S'(u)\varphi \, dx \, dt \\
&\quad = \int_{Q_T} \Big(\frac{\partial}{\partial t} b(u)\Big) S'(b^{-1}(b(u)))\varphi \, dx \, dt \\
&\quad = \int_0^T \int_\Omega \varphi_t \Big(\int_0^{b(u)} S'(b^{-1}(s)) ds\Big) \, dx \, dt + \int_\Omega \varphi(0) \Big(\int_0^{b(u_0)} S'(b^{-1}(s)) \, ds\Big) \, dx \\
&\quad = \int_0^T \int_\Omega \varphi_t \Big(\int_0^u S'(s) b'(s) \, ds\Big) \, dx \, dt + \int_\Omega \varphi(0) \Big(\int_0^{u_0} S'(s) b'(s) \, ds\Big) \, dx,
\end{aligned}$$

which is (59). At this point, we may conclude that u is a renormalized solution to (P).

Now we deal with the "moreover" part. Let $p > 2 - \frac{1}{d+1}$ and u be a renormalized solution to (P). We aim to show that u is also a distributional solution. To this end, let $1 \le q < p$. We will verify that $u \in L^q(0,T; W^{1,q}_{\Gamma_0}(\Omega))$ and u satisfies (W1) and (W2).

We first show that $u \in L^q(0,T; W^{1,q}_{\Gamma_0}(\Omega))$. Since u is a renormalized solution to (P), we have $b(u) \in L^\infty(0,T; L^1(\Omega))$. From this, we deduce that

$$\begin{aligned}
\|u\|_{L^\infty(0,T;L^1(\Omega))} &= \operatorname*{ess\,sup}_{0<t<T} \Big(\int_\Omega |u(x,t)| \, dx\Big) = \operatorname*{ess\,sup}_{0<t<T} \Big(\int_\Omega |b^{-1}(b(u(x,t)))| \, dx\Big) \\
&\le \beta^{-1} \operatorname*{ess\,sup}_{0<t<T} \Big(\int_\Omega |b(u(x,t))| \, dx\Big) = \beta^{-1} \|b(u)\|_{L^\infty(0,T;L^1(\Omega))} < \infty,
\end{aligned}$$

where we have used (H3) in the third step.

Consequently, for all $k > 0$ we have $T_k(u) \in V \cap L^\infty(0,T; L^1(\Omega))$ and

$$\alpha \int_{[\ell \le |T_k(u)| \le \ell+1]} |\nabla T_k(u)|^p \, dx \, dt \le \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_1)} + \|b(u_0)\|_{L^1(\Omega)} + |\Omega|$$

for all $\ell \geq 0$ in view of (47) and (3). It follows from (56) that

$$\|T_k(u)\|_{L^q(0,T;W^{1,q}_{\Gamma_0}(\Omega))} \leq C,$$

where the constant $C > 0$ is independent of k. Hence, we also have $u \in L^q(0,T;W^{1,q}_{\Gamma_0}(\Omega))$ as required.

We proceed to verify (W1) and (W2). Recall from (6) that

$$B_S(r) = \int_0^u b'(s)S'(s)\,ds$$

for all $r \in \mathbb{R}$ and $S \in W^{2,\infty}(\mathbb{R})$ such that $\operatorname{supp} S'$ is compact. Now we set

$$S(r) = S_k(r) := \int_0^r (1 - |T_{k+1}(s) - T_k(s)|)\,ds$$

for each $r \in \mathbb{R}$ and $k > 0$. Then

$$B_{S_k}(r) = \int_0^r b'(s)\left(1 - |T_{k+1}(s) - T_k(s)|\right)ds$$

for all $r \in \mathbb{R}$ and $k > 0$. Letting $k \to \infty$ yields

$$\lim_{k\to\infty} B_{S_k}(r) = \int_0^r b'(s)\,ds = b(r)$$

for all $r \in \mathbb{R}$. It follows from Lemma 5.2 that $B_{S_k}(u)|_{t=0} = B_{S_k}(u_0)$. Hence

$$b(u)|_{t=0} = \lim_{k\to\infty} B_{S_k}(u)|_{t=0} = \lim_{k\to\infty} B_{S_k}(u_0) = b(u_0). \tag{60}$$

In addition, we deduce from (5) that

$$\begin{aligned} -\int_{Q_T} B_{S_k}(u)\varphi_t\,dx\,dt + \int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla(S_k'(u)\varphi)\,dx\,dt + \int_{\Sigma_1}\gamma h(u)S_k'(u)\varphi\,d\sigma(x)\,dt \\ = \int_{Q_T} fS_k'(u)\varphi\,dx\,dt + \int_{\Sigma_1} gS_k'(u)\varphi\,d\sigma(x)\,dt - \int_\Omega B_{S_k}(u_0)\varphi(0)\,dx \end{aligned}$$

for all $\varphi \in C_c^\infty([0,T)\times\Omega_1)$. Now we let $k \to \infty$ to arrive at

$$\begin{aligned} -\int_{Q_T} b(u)\varphi_t\,dx\,dt + \int_{Q_T} a(x,t,u,\nabla u)\cdot\nabla\varphi\,dx\,dt + \int_{\Sigma_1}\gamma h(u)\varphi\,d\sigma(x)\,dt \\ = \int_{Q_T} f\varphi\,dx\,dt + \int_{\Sigma_1} g\varphi\,d\sigma(x)\,dt - \int_\Omega b(u_0)\varphi(0)\,dx \end{aligned}$$

for all $\varphi \in C_c^\infty([0,T)\times\Omega_1)$. Since $f \in L^{p'}(Q_T)$ and $g \in L^{p'}(\Sigma_1)$,

$$\frac{\partial}{\partial t}b(u) \in V^*. \tag{61}$$

At this point, (60) and (61) together verify (W1).

Next, Lemma 3.10 implies

$$\int_\Omega b(u_0)\varphi(0)\,dx - \int_{Q_T} b(u)\varphi_t\,dx\,dt = \int_{Q_T}\left(\frac{\partial}{\partial t}b(u)\right)\varphi\,dx\,dt$$

for all $\varphi \in C_c^\infty([0,T)\times\Omega_1)$. Consequently,

$$\int_{Q_T}\Big(\frac{\partial}{\partial t}b(u)\Big)\varphi\,dx\,dt+\int_{Q_T}a(x,t,u,\nabla u)\cdot\nabla\varphi\,dx\,dt+\int_{\Sigma_1}\gamma h(u)\varphi\,d\sigma(x)\,dt$$
$$=\int_{Q_T}f\varphi\,dx\,dt+\int_{\Sigma_1}g\varphi\,d\sigma(x)\,dt \qquad (62)$$

for all $\varphi \in C_c^\infty([0,T)\times\Omega_1)$. By density, (62) also holds for all $\varphi\in V$, which justifies (W2). Hence u is a weak solution to (P) as required. ■

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