ACTA ARITHMETICA
132.4 (2008)

An odd square as a sum of an odd number of odd squares

by

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1. Introduction. Let $s_k(n)$ be the number of representations of n as sum of k positive odd squares. The generating function for $s_k(n)$ is

$$S_k(q) := \sum_{n=0}^{\infty} s_k(n)q^n = \Big(\sum_{j=0}^{\infty} q^{(2j+1)^2}\Big)^k,$$

and clearly $s_k(n) = 0$ if $n \not\equiv k \pmod 8$. The goal of this article is to prove that for any odd positive integer n and every positive integer k,

$$s_{8k+1}(n^2) = -\frac{(2^{4k}-1)B_{4k}}{8k}\sum_{d\mid n}\mu(d)d^{4k-1}s_{16k}\left(\frac{8n}{d}\right) + O(n^{6k-1}), \tag{1.1}$$

where B_k is the kth Bernoulli number [1] given by

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k\,\frac{x^k}{k!},$$

and

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1,\\ (-1)^j & \text{if } n = p_1\cdots p_j, \text{ for distinct primes } p_1,\ldots,p_j,\\ 0 & \text{otherwise.}\end{cases}$$

2000 *Mathematics Subject Classification*: 11F03, 11F11, 11F20.

Key words and phrases: Dirichlet series, Eisenstein series, Hecke operator, modular form, newform, oldform, sum of squares.

The first author is supported by National University of Singapore Academic Research Fund R146000103112.

[1] Note that $B_{4k} < 0$.

When $k = 1$ or 2, the error term in (1.1) is zero, and we obtain the identities

$$s_9(n^2) = \frac{1}{16}\sum_{d|n}\mu(d)d^3 s_{16}\left(\frac{8n}{d}\right),$$

$$s_{17}(n^2) = \frac{17}{32}\sum_{d|n}\mu(d)d^7 s_{32}\left(\frac{8n}{d}\right).$$

When $n = p$ is prime, these simplify further to

$$s_9(p^2) = \frac{1}{16}\, s_{16}(8p), \tag{1.2}$$

$$\frac{1}{17}\, s_{17}(p^2) = \frac{1}{32}\, s_{32}(8p). \tag{1.3}$$

The method we shall use in proving (1.1) is motivated by the work of A. Hurwitz [5]. In Section 2, we illustrate the main idea by proving (1.1) in the case when $k = 1$ and deducing (1.2).

In Section 3, we prove (1.1) and give a precise formula for the error term. We also give an asymptotic formula for $s_{8k+1}(n^2)$ in terms of the divisors of n.

2. Proof of (1.1) for $k = 1$.

Let

$$T_k(q) = \sum_{n=0}^{\infty} t_k(n)q^n = \Big(\sum_{j=0}^{\infty} q^{(2j+1)^2/8}\Big)^k. \tag{2.1}$$

Note that $T_k(q^8) = S_k(q)$ and for all positive integers n,

$$t_{8k}(n) = s_{8k}(8n). \tag{2.2}$$

Next, recall that for $|q| < 1$,

$$\sum_{n=0}^{\infty} t_8(n)q^n = \sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^{2n}}. \tag{2.3}$$

Identity (2.3) is the classical sum of eight triangular numbers formula of Legendre [7, p. 133] and Jacobi [6, p. 170]. It was rediscovered by Ramanujan [9, p. 191], and many proofs have since been given. For example, see [3, eq. (3.71)]. Equating coefficients of q^n on both sides of (2.3), we deduce that

$$t_8(n) = \sum_{d|n}\varepsilon(d)\left(\frac{n}{d}\right)^3, \tag{2.4}$$

where

$$\varepsilon(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{2.5}$$

From (2.4), we see that the corresponding Dirichlet series for $T_8(q)$ (for $\mathrm{Re}\, s > 4$) is

$$\zeta_{T_8}(s) := \sum_{n=1}^{\infty} \frac{t_8(n)}{n^s} = \zeta(s-3)L(s),$$

where $\zeta(s)$ is the Riemann zeta function, and

$$L(s) = \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n^s}.$$

Hence $\zeta_{T_8}(s)$ has the Euler product

$$\zeta_{T_8}(s) = \prod_p \frac{1}{1 - \frac{\varepsilon(p)+p^3}{p^s} + \frac{\varepsilon(p)}{p^{2s-3}}}. \tag{2.6}$$

For positive integers m, k and some arithmetical function χ, let $\mathbf{T}_{m,\chi}$ be the operator on a power series

$$A(q) = \sum_{n=0}^{\infty} a(n)q^n$$

defined by

$$\mathbf{T}_{m,\chi}(A(q)) = \sum_{n=0}^{\infty} b(n)q^n,$$

where

$$b(n) = \sum_{d\mid\gcd(m,n)} \chi(d)d^{k-1}a\left(\frac{mn}{d^2}\right).$$

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

Let $M_k(N) = M(\Gamma_0(N), k, 1)$ be the space of weight k modular forms on $\Gamma_0(N)$ with multiplier 1. When $\chi = \varepsilon$ with ε given by (2.5), $\mathbf{T}_{m,\varepsilon}$ are the Hecke operators on $M_k(2)$.

It is known that if $A(q) \in M_k(2)$, with $a(1) = 1$, and the corresponding Dirichlet series for $A(q)$ has an Euler product, then (see for example [8, Theorem 4.5.16])

$$\mathbf{T}_{m,\varepsilon}(A(q)) = a(m)A(q).$$

Hence

$$a(m)a(n) = \sum_{d\mid\gcd(m,n)} \varepsilon(d)d^{k-1}a\left(\frac{mn}{d^2}\right). \tag{2.7}$$

Using (2.6) and the fact that $T_{8k} \in M_{4k}(2)$ [10, p. 222, Theorem 7.1.4] for $q = e^{2\pi i\tau}$, we deduce from (2.7) that

$$t_8(m)t_8(n) = \sum_{d|\gcd(m,n)} \varepsilon(d)d^3 t_8\left(\frac{mn}{d^2}\right). \tag{2.8}$$

When $m = 2$ and n is any positive integer, we deduce from (2.8) that

$$t_8(2)t_8(n) = t_8(2n). \tag{2.9}$$

We now state and prove a simple lemma.

LEMMA 2.1. *If f satisfies*

$$f(m)f(n) = \sum_{d|\gcd(m,n)} g(d)f\left(\frac{mn}{d^2}\right), \tag{2.10}$$

where g is a completely multiplicative function, then

$$\begin{aligned} f(mn) &= \sum_{d|\gcd(m,n)} \mu(d)g(d)f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right) \\ &= \sum_{d} \mu(d)g(d)f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right), \end{aligned} \tag{2.11}$$

where we use the convention that the coefficient $f(n/d)$ is zero if d is not a divisor of n.

Proof. Let N be any positive integer. Let m and n be any positive integers satisfying $N = \gcd(m,n)$ and write $m = Ns$ and $n = Nt$, with $\gcd(s,t) = 1$. From (2.10), we have

$$f(Ns)f(Nt) = \sum_{d|N} g(d)f\left(\frac{stN^2}{d^2}\right).$$

Since

$$\sum_{d|N} h(d) = \sum_{d|N} h(N/d),$$

we conclude that

$$f(Ns)f(Nt) = \sum_{d|N} g\left(\frac{N}{d}\right)f(std^2),$$

or

$$\frac{f(Ns)f(Nt)}{g(N)} = \sum_{d|N} \frac{1}{g(d)}\, f(std^2).$$

Applying the Möbius inversion formula, we find that

$$\frac{f(stN^2)}{g(N)} = \sum_{d|N} \mu\left(\frac{N}{d}\right)\frac{f(ds)f(dt)}{g(d)}.$$

Simplifying the above, we obtain the first equality of (2.11). The second equality of (2.11) follows immediately. ∎

Applying Lemma 2.1 with $f = t_8$ and using equation (2.8), we conclude that

$$t_8(mn) = \sum_d \varepsilon(d)\mu(d)d^3 t_8\left(\frac{m}{d}\right)t_8\left(\frac{n}{d}\right). \tag{2.12}$$

We are now ready to prove the main result of this section.

THEOREM 2.2. *For all odd positive integers n,*

$$s_9(n^2) = \frac{1}{16}\sum_{d|n} \mu(d)d^3 s_{16}\left(\frac{8n}{d}\right).$$

Proof. Observe that

$$s_9(n^2) = \sum_{\substack{0<i<n \\ i\,\text{odd}}} s_8(n^2 - i^2) = \sum_{\substack{0<i<n \\ i\,\text{odd}}} s_8\left(4\left(\frac{n+i}{2}\right)\left(\frac{n-i}{2}\right)\right).$$

Since $n-i$ and $n+i$ are both even, let $n-i = 2j$ so that $n+i = 2(n-j)$. Then

$$s_9(n^2) = \sum_{j=1}^{(n-1)/2} s_8(4j(n-j)) = \frac{1}{2}\sum_{j=1}^{n-1} s_8(4j(n-j)).$$

Now apply (2.2), (2.9), (2.12) and then (2.1) to get

$$\begin{aligned} s_9(n^2) &= \frac{1}{2}\sum_{j=1}^{n-1} t_8\left(\frac{j(n-j)}{2}\right) = \frac{1}{16}\sum_{j=1}^{n-1} t_8(j(n-j)) \\ &= \frac{1}{16}\sum_{j=1}^{n-1}\sum_{d\,\text{odd}} \mu(d)d^3 t_8\left(\frac{j}{d}\right)t_8\left(\frac{n-j}{d}\right) \\ &= \frac{1}{16}\sum_{d|n} \mu(d)d^3 [q^{n/d}](T_{16}(q)), \end{aligned}$$

where $[q^n]f(q)$ denotes the coefficient of q^n in the Taylor series expansion of $f(q)$ about $q = 0$. By (2.1) and (2.2), we finally obtain

$$s_9(n^2) = \frac{1}{16}\sum_{d|n} \mu(d)d^3 t_{16}\left(\frac{n}{d}\right) = \frac{1}{16}\sum_{d|n} \mu(d)d^3 s_{16}\left(\frac{8n}{d}\right). \ ∎$$

From Theorem 2.2, we immediately obtain (1.2) by letting n be an odd prime.

3. The main result. Let $M'_{4k}(2)$ denote the subspace of $M_{4k}(2)$ that consists of forms which vanish at $q = 0$. The space $M'_{4k}(2)$ has a basis consisting of the functions [10, p. 222]

$$
\begin{aligned}
F_{k,0}(q) &= \sum_{n=0}^{\infty} f_{k,0}(n)q^n = \sum_{j=1}^{\infty} \frac{j^{4k-1}q^j}{1-q^{2j}}, \\
F_{k,r}(q) &= \sum_{n=r}^{\infty} f_{k,r}(n)q^n = (\eta(2\tau))^{24r-8k}(\eta(\tau))^{16k-24r}, \qquad 0 < r < k.
\end{aligned}
\tag{3.1}
$$

Since $T_{8k}(q) \in M'_{4k}(2)$ [10, p. 222, Theorem 7.1.4], we have the following lemma.

Lemma 3.1. *Let k be a positive integer. Then there exist unique rational numbers $a_{k,0}$, $a_{k,1}, \ldots, a_{k,k-1}$ such that*

$$T_{8k}(q) = \sum_{r=0}^{k-1} a_{k,r}F_{k,r}(q). \tag{3.2}$$

Remarks. The equality in (3.2) is equivalent to the theorem for sums of $8k$ triangular numbers, discovered by Ramanujan [9, p. 191, eqs. 12.6, 12.61]. Ramanujan's formula showed further that

$$a_{k,0} = \frac{-8k}{2^{4k}(2^{4k}-1)B_{4k}}. \tag{3.3}$$

For another proof of Lemma 3.1, including the value of $a_{k,0}$, see [3, Theorem 3.6].

Our proof of (1.1) given in Section 2 relies heavily on the fact that the coefficients t_8 of T_8 satisfy (2.7). The modular form $F_{k,0}$ satisfies (2.7) by the same argument we gave for T_8. However, in general, the cusp forms

$$F_{k,r} = \sum_{n=r}^{\infty} f_{k,r}(n)q^n, \qquad 1 \le r \le k-1,$$

do not have coefficients $f_{k,r}$ that satisfy (2.7).

In order to prove (1.1) using the ideas in Section 2, we need the following lemma:

Lemma 3.2. *The space $M'_{4k}(2)$ has a basis of modular forms*

$$\{E_{k,r} \mid 0 \le r \le k-1\},$$

with $E_{k,0} = F_{k,0}$, such that the coefficients of the series expansion of each $E_{k,r}$ at $q = 0$ satisfy (2.7).

Proof. The space $M'_{4k}(2)$ can be written as

$$M'_{4k}(2) = \mathbb{C}F_{k,0} \oplus S_{4k}(2),$$

where $\mathbb{C}$ is the field of complex numbers and $S_{4k}(2)$ is the space of cusp forms on $\Gamma_0(2)$ of weight $4k$ and multiplier 1. It is known that the space of cusp forms can be further written as [1, Theorem 5]

$$S_{4k}(2) = S_{4k}^{\mathrm{new}}(2) \oplus S_{4k}^{\mathrm{old}}(2),$$

where $S_{4k}^{\mathrm{new}}(2)$ and $S_{4k}^{\mathrm{old}}(2)$ are the spaces of newforms and oldforms, respectively. Furthermore, it is known [1, Theorem 5] that the space of newforms can be expressed as a direct sum of one-dimensional subspaces generated by eigenforms of $\mathbf{T}_{m,\varepsilon}$, for all positive integers m. As a result, the eigenforms that generate the space of newforms satisfy (2.7).

It remains to show that there is a basis of eigenforms of $\mathbf{T}_{m,\varepsilon}$ for the space of oldforms.

Using the notation in Section 2, we denote by $M_k(1)$ the space of modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$. The corresponding Hecke operators on $M_k(1)$ are $\mathbf{T}_{m,u}$ with $u(n) = 1$ for all integers n.

It is known that

$$S_{4k}^{\mathrm{old}}(2) = \bigoplus O_i, \tag{3.4}$$

where O_i is a two-dimensional space generated by $f_i(q)$ and $f_i(q^2)$, where $f_i(q)$ is an eigenform of the Hecke operators $\mathbf{T}_{m,u}$. Note that when m is odd and $f(q) \in M_k(1)$, the actions of $\mathbf{T}_{m,u}$ and $\mathbf{T}_{m,\varepsilon}$ on $f(q)$ are identical. Hence if $f_i(q)$ is an eigenform for $\mathbf{T}_{m,u}$ for odd m then $f_i(q)$ is also an eigenform for $\mathbf{T}_{m,\varepsilon}$ for odd m.

Using $f_i(q)$ and $f_i(q^2)$, we proceed to construct $e_i(q)$ and $e_i^*(q)$ which are eigenforms for $\mathbf{T}_{2,\varepsilon}$ and generate O_i. From now on, we drop the subscript and normalize

$$f(q) = \sum_{n=1}^{\infty} a(n)q^n$$

so that $a(1) = 1$. Note that

$$f(q^2) = \sum_{n=1}^{\infty} a(n/2)q^n,$$

where $a(k) = 0$ when $k \notin \mathbb{Z}^+$. We also let

$$e(q) = \sum_{n=1}^{\infty} \alpha(n)q^n \quad \text{and} \quad e^*(q) = \sum_{n=1}^{\infty} \alpha^*(n)q^n$$

so that $\alpha(1) = \alpha^*(1) = 1$. Note that $e(q)$ is a linear combination of $f(q)$ and $f(q^2)$ and we may write

$$e(q) = c_1 f(q) + c_2 f(q^2).$$

Since both $f(q)$ and $e(q)$ are normalized, by comparing coefficients of q, we deduce that $c_1 = 1$.

Next, we want $e(q)$ to satisfy the relation

$$\mathbf{T}_{2,\varepsilon}(e(q)) = \alpha(2)e(q).$$

This leads to the relation

$$a(2n) + c_2 a(n) = (a(2) + c_2)(a(n) + c_2 a(n/2)). \tag{3.5}$$

When $n = 2$, (3.5) gives

$$a(4) + c_2 a(2) = (a(2) + c_2)(a(2) + c_2). \tag{3.6}$$

Since $f(q)$ is an eigenform of $\mathbf{T}_{2,u}$, we find that

$$a(4) + 2^{4k-1} = a^2(2). \tag{3.7}$$

Substituting (3.7) into (3.6), we conclude that

$$c_2^2 + a(2)c_2 + 2^{4k-1} = 0.$$

In order to obtain two distinct values of c_2 that correspond to two eigenforms $e(q)$ and $e^*(q)$ of $\mathbf{T}_{2,\varepsilon}$, we must show that

$$a^2(2) \neq 4 \cdot 2^{4k-1}. \tag{3.8}$$

To establish (3.8), we follow an argument by J.-P. Serre [11]. First, note that if

$$a^2(2) = 4 \cdot 2^{4k-1},$$

then

$$a^2(2) \equiv -1 \pmod{3}. \tag{3.9}$$

On the other hand, from [4, (5)], we find that $(a(p) - 1 - p)/3$ is an algebraic integer. When $p = 2$, this says that

$$a(2) \equiv 0 \pmod{3}.$$

This clearly contradicts (3.9), and (3.8) must hold. Hence, we conclude that each O_i in (3.4) is spanned by two eigenforms of $\mathbf{T}_{m,\varepsilon}$, and this completes our proof of Lemma 3.2. ∎

Since $E_{k,0} = F_{k,0}$, we deduce from Lemmas 3.1 and 3.2 that

$$T_{8k}(q) = \sum_{r=0}^{k-1} a'_{k,r} E_{k,r}(q), \tag{3.10}$$

where (see (3.3))

$$a'_{k,0} = a_{k,0} = \frac{-8k}{2^{4k}(2^{4k} - 1)B_{4k}}.$$

We are now ready to prove the generalization of Theorem 2.2.

THEOREM 3.3. *Let n be an odd positive integer and $E_{k,r}$ be the basis given in Lemma 3.2. Then*

$$s_{8k+1}(n^2) = \sum_{d|n} \mu(d) d^{4k-1} [q^{n/d}] \Big\{ \sum_{r=0}^{k-1} c_{k,r} E_{k,r}^2(q) \Big\},$$

for some complex numbers $c_{k,r}$, $1 \le r \le k-1$, and

$$c_{k,0} = \frac{a_{k,0}}{2^{4k}} = \frac{-8k}{2^{8k}(2^{4k}-1)B_{4k}}. \tag{3.11}$$

Proof. The proof is similar to that for Theorem 2.2. We write

$$s_{8k+1}(n^2) = \frac{1}{2} \sum_{j=1}^{n-1} t_{8k}\bigg(\frac{j(n-j)}{2}\bigg).$$

From (3.10), we can rewrite the above as

$$s_{8k+1}(n^2) = \frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=0}^{k-1} a'_{k,r} e_{k,r}\bigg(\frac{j(n-j)}{2}\bigg),$$

where $e_{k,r}(n)$ are given by

$$E_{k,r}(q) = \sum_{n=0}^{\infty} e_{k,r}(n) q^n. \tag{3.12}$$

Since each $e_{k,r}(n)$ satisfies (2.7), we find that

$$s_{8k+1}(n^2) = \frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=0}^{k-1} b_{k,r} e_{k,r}(j(n-j)),$$

with

$$b_{k,r} = \frac{a'_{k,r}}{e_{k,r}(2)}.$$

Hence, by Lemma 2.1, we deduce that

$$s_{8k+1}(n^2) = \sum_{d|n} \mu(d) d^{4k-1} [q^{n/d}] \sum_{r=0}^{k-1} c_{k,r} E_{k,r}^2(q),$$

where $c_{k,r} = b_{k,r}/2$. Furthermore,

$$c_{k,0} = \frac{b_{k,0}}{2} = \frac{a'_{k,0}}{2e_{k,0}(2)} = \frac{a_{k,0}}{2f_{k,0}(2)} = \frac{a_{k,0}}{2^{4k}},$$

where $f_{k,0}$ is given by (3.1). ∎

COROLLARY 3.4. *For odd positive integers n,*

$$s_{8k+1}(n^2) = \sum_{d|n} \mu(d) d^{4k-1} [q^{n/d}](F(q)),$$

where $F(q) \in M_{8k}(2)$ with a zero of order at least two at $\tau = i\infty$.

As an application of Corollary 3.4, we give a proof of (1.3).

Proof of (1.3). By Corollary 3.4 with $k = 2$, we have

$$s_{17}(n^2) = \sum_{d|n} \mu(d) d^7 [q^{n/d}](F(q)), \tag{3.13}$$

where $F(q) \in M_{16}(2)$ has a zero of order at least two at $\tau = i\infty$. It is known [2, proof of (1.8)] that $F(q)$ is a linear combination of $\mathcal{T}_{2k}\mathcal{T}_{2l}$, where $2k + 2l = 16$, $l \geq k > 1$, and

$$\mathcal{T}_k = \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^{2n}}.$$

Accordingly, let

$$F(q) = c_1 \mathcal{T}_4 \mathcal{T}_{12} + c_2 \mathcal{T}_6 \mathcal{T}_{10} + c_3 \mathcal{T}_8^2,$$

where c_1, c_2 and c_3 are some constants to be determined. If we use this in (3.13), successively let $n = 3$, $n = 5$, and solve, we obtain

$$\begin{aligned} F(q) = {} & \frac{17}{32 \cdot 75600} \left(\mathcal{T}_4 \mathcal{T}_{12} - \frac{25}{4} \mathcal{T}_6 \mathcal{T}_{10} + \frac{21}{4} \mathcal{T}_8^2 \right) \\ & + c \left(\mathcal{T}_4 \mathcal{T}_{12} + \frac{15}{4} \mathcal{T}_6 \mathcal{T}_{10} - 16 \mathcal{T}_8^2 \right), \end{aligned} \tag{3.14}$$

where c is an arbitrary constant. Now [2, (1.9)]

$$\frac{1}{75600} \left(\mathcal{T}_4 \mathcal{T}_{12} - \frac{25}{4} \mathcal{T}_6 \mathcal{T}_{10} + \frac{21}{4} \mathcal{T}_8^2 \right) = T_{32}(q), \tag{3.15}$$

and the methods in [2] can be used to show that

$$\begin{aligned} & \mathcal{T}_4 \mathcal{T}_{12} + \frac{15}{4} \mathcal{T}_{10} \mathcal{T}_6 - 16 \mathcal{T}_8^2 \\ & \qquad = -\frac{45}{4} q^2 \prod_{j=1}^{\infty} (1 - q^{2j})^{24} \left(1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^{2j}}{1 - q^{2j}} \right), \end{aligned} \tag{3.16}$$

which clearly contains only even powers of q. If we substitute the results of (3.14)–(3.16) into (3.13), we obtain

$$s_{17}(n^2) = \frac{17}{32} \sum_{d|n} \mu(d) d^7 [q^{n/d}](T_{32}(q)) = \frac{17}{32} \sum_{d|n} \mu(d) d^7 s_{32}\left(\frac{8n}{d}\right).$$

If we let $n = p$ be prime, we obtain (1.3). ∎

We are now ready to prove (1.1).

THEOREM 3.5. *Let n be an odd positive integer. Then*

$$s_{8k+1}(n^2) = -\frac{(2^{4k} - 1) B_{4k}}{8k} \sum_{d|n} \mu(d) d^{4k-1} s_{16k}\left(\frac{8n}{d}\right) + O(n^{6k-1}).$$

Proof. We begin by writing the result of Theorem 3.3 as

$$s_{8k+1}(n^2) = \sum_{d|n} \mu(d) d^{4k-1} [q^{n/d}] c_{k,0} E_{k,0}^2(q) + \sum_{d|n} \mu(d) d^{4k-1} [q^{n/d}] \Big\{ \sum_{r=1}^{k-1} c_{k,r} E_{k,r}^2(q) \Big\}. \tag{3.17}$$

Now

$$E_{k,0}^2(q) - \frac{1}{a_{k,0}^2} T_{16k}(q) = E_{k,0}^2(q) - \frac{1}{a_{k,0}^2} (T_{8k}(q))^2 = \frac{1}{a_{k,0}^2} (a_{k,0} E_{k,0}(q) - T_{8k}(q))(a_{k,0} E_{k,0}(q) + T_{8k}(q)).$$

By (3.10) this can be written as

$$E_{k,0}^2(q) - \frac{1}{a_{k,0}^2} T_{16k}(q) = -\frac{1}{a_{k,0}^2} \Big(\sum_{r=1}^{k-1} a'_{k,r} E_{k,r}(q) \Big) \Big(2a_{k,0} E_{k,0}(q) + \sum_{r=1}^{k-1} a'_{k,r} E_{k,r}(q) \Big). \tag{3.18}$$

Substituting (3.18) into (3.17), we deduce that

$$s_{8k+1}(n^2) = \frac{c_{k,0}}{a_{k,0}^2} \sum_{d|n} \mu(d) d^{4k-1} s_{16k}\left(\frac{8n}{d}\right) + \sum_{d|n} \mu(d) d^{4k-1} [q^{n/d}] \sum_{\substack{0 \le r,m \le k-1 \\ (r,m) \ne (0,0)}} d_{k,r,m} E_{k,r}(q) E_{k,m}(q),$$

for some numbers $d_{k,r,m}$. From (3.1), (3.12) and Lemma 3.2, we find that

$$e_{k,0}(n) = O(n^{4k-1}).$$

The order of $e_{k,r}(n), 1 \le r \le k-1$, on the other hand, is given by [10, Theorem 4.5.2(i)]

$$e_{k,r}(n) = O(n^{2k}).$$

This implies that

$$s_{8k+1}(n^2) = \frac{c_{k,0}}{a_{k,0}^2} \sum_{d|n} \mu(d) d^{4k-1} s_{16k}\left(\frac{8n}{d}\right) + O(n^{6k-1}).$$

Rewriting $c_{k,0}/a_{k,0}^2$ using (3.3) and (3.11), we conclude our proof of Theorem 3.5. ∎

COROLLARY 3.6. *Let n be an odd positive integer. Then*

$$s_{8k+1}(n^2) = \frac{B_{4k}}{2^{8k-1}(2^{4k}+1)B_{8k}} \sum_{d|n} \mu(d) d^{4k-1} \sigma_{8k-1}\left(\frac{n}{d}\right) + O(n^{6k-1}),$$

where

$$\sigma_k(n) = \sum_{d|n} d^k.$$

Proof. From Lemma 3.2, we may write

$$s_{16k}\left(\frac{8n}{d}\right) = t_{16k}\left(\frac{n}{d}\right) = -\frac{16k}{2^{8k}(2^{8k}-1)B_{8k}} \sigma_{8k-1}\left(\frac{n}{d}\right) + O(n^{4k}). \tag{3.19}$$

Substituting (3.19) into Theorem 3.5, we deduce Corollary 3.6. ∎

It is clear that when p is prime, we have

$$s_{8k+1}(p^2) \sim \frac{B_{4k}}{2^{8k-1}(2^{4k}+1)B_{8k}} p^{8k-1}.$$

Acknowledgements. We are grateful to Profs. A. Knapp, W. Kohnen and W. Li for answering our questions on modular forms during the preparation of this article.

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Received on 8.8.2007
and in revised form on 26.2.2008 (5496)