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A basis for the space of modular forms

by

Shinji Fukuhara (Tokyo)

1. Introduction and statement of results. Modular forms of one variable have been studied for a long time. They appear in many areas of mathematics and in theoretical physics. In this paper we consider the space M_{2k} of modular forms of weight $2k$, and find a simple basis for M_{2k} in terms of Eisenstein series, which is different from the classically known standard basis. A motivation for looking for a new basis will be explained below.

Throughout the paper, we use the following notation:

$$\begin{aligned}
k &\text{ is an integer greater than or equal to 1,}\\
\Gamma &:= SL_2(\mathbb{Z}) \quad \text{(the full modular group)},\\
M_{2k} &:= \text{the } \mathbb{C}\text{-vector space of modular forms of weight } 2k \text{ on } \Gamma,\\
S_{2k} &:= \text{the } \mathbb{C}\text{-vector space of cusp forms of weight } 2k \text{ on } \Gamma,\\
S_{2k}^* &:= \mathrm{Hom}_{\mathbb{C}}(S_{2k}, \mathbb{C}) \quad \text{(the dual space of } S_{2k}),\\
d_k &:= \begin{cases} \lfloor k/6 \rfloor - 1 & \text{if } 2k \equiv 2 \pmod{12},\\ \lfloor k/6 \rfloor & \text{if } 2k \not\equiv 2 \pmod{12}, \end{cases}
\end{aligned}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x \in \mathbb{R}$. We note that

$$\dim_{\mathbb{C}} S_{2k} = d_k \quad \text{and} \quad \dim_{\mathbb{C}} M_{2k} = d_k + 1.$$

Let B_{2n} be the $2n$th Bernoulli number and $\sigma_{2n-1}(m)$ the $(2n-1)$th divisor function, that is,

$$\sigma_{2n-1}(m) := \sum_{0<d|m} d^{2n-1} \qquad (n \geq 1).$$

Then the Eisenstein series of weight $2n$ for Γ is defined by

$$E_{2n}(z) := -\frac{B_{2n}}{4n} + \sum_{m=1}^{\infty} \sigma_{2n-1}(m) e^{2\pi i m z}$$

where $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

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The classically known basis for M_{2k} is the following set (Serre [8, p. 89]):

$$\{E_4^{\alpha}E_6^{\beta} \mid \alpha, \beta \in \mathbb{Z},\ \alpha, \beta \geq 0,\ 4\alpha + 6\beta = 2k\}.$$

However, the Fourier coefficients of these forms are not so simple when we write down the coefficients as sums of products of divisor functions. This motivates us to look for a new simpler basis for M_{2k}, consisting of modular forms whose Fourier coefficients are convolution sums of two divisor functions. Our result is formulated in the following theorem:

THEOREM 1.1.

(1) *If* $2k \equiv 0 \pmod 4$ *then*

$$\{E_{2k}\} \cup \{E_{4i}E_{2k-4i} \mid i = 1, \ldots, d_k\}$$

is a basis for M_{2k}.

(2) *If* $2k \equiv 2 \pmod 4$ *then*

$$\{E_{2k}\} \cup \{E_{4i+2}E_{2k-4i-2} \mid i = 1, \ldots, d_k\}$$

is a basis for M_{2k}.

Note that the nth Fourier coefficient of $E_{4i}E_{2k-4i}$ is

$$\sum_{l=0}^{n} \sigma_{4i-1}(l)\sigma_{2k-4i-1}(n-l)$$

where we set $\sigma_{2n-1}(0) := -B_{2n}/(4n)$ by convention.

We will also find a new basis for the space of cusp forms on Γ

THEOREM 1.2.

(1) *If* $2k \equiv 0 \pmod 4$ *then*

$$\left\{ E_{4i}E_{2k-4i} + \frac{B_{4i}}{4i}\,\frac{B_{2k-4i}}{2k-4i}\,\frac{k}{B_{2k}}E_{2k} \;\middle|\; i = 1, \ldots, d_k \right\}$$

is a basis for S_{2k}.

(2) *If* $2k \equiv 2 \pmod 4$ *then*

$$\left\{ E_{4i+2}E_{2k-4i-2} + \frac{B_{4i+2}}{4i+2}\,\frac{B_{2k-4i-2}}{2k-4i-2}\,\frac{k}{B_{2k}}E_{2k} \;\middle|\; i = 1, 2, \ldots, d_k \right\}$$

is a basis for S_{2k}.

We note that, for $\Gamma = \Gamma_0(2)$, similar but slightly different formulas were given in [4, Theorem 1.6].

EXAMPLE 1.3. For M_{36}, we have the basis

$$\{E_{36},\ E_4E_{32},\ E_8E_{28},\ E_{12}E_{24}\},$$

and for S_{36}, we have the basis

$$\Big\{E_4E_{32} - \frac{1479565184909325423}{28631015449722183381 8240}E_{36},\quad E_8E_{28} - \frac{651138973032093}{1221028600061681350 10720}E_{36},\quad E_{12}E_{24} - \frac{114819293577343}{1149451061437375891652640}E_{36}\Big\}.$$

2. Preliminaries. Let f be an element of S_{2k}. We write f as a Fourier series

$$f(z) = \sum_{l=1}^{\infty} a_l e^{2\pi i l z}.$$

Let $L(f,s)$ be the L-series of f, that is, the analytic continuation of

$$\sum_{l=1}^{\infty} \frac{a_l}{l^s} \qquad (\Re(s) \gg 0).$$

Then nth period of f, $r_n(f)$, is defined by

$$r_n(f) := \int_0^{i\infty} f(z) z^n \, dz = \frac{n!}{(-2\pi i)^{n+1}} L(f, n+1) \qquad (n = 0, 1, \ldots, w).$$

Each r_n can be regarded as a linear map from S_{2k} to $\mathbb{C}$, that is,

$$r_n \in S_{2k}^* = \mathrm{Hom}_{\mathbb{C}}(S_{2k}, \mathbb{C}).$$

Here we recall the result of Eichler [2], Shimura [9] and Manin [6]:

Theorem 2.1 (Eichler–Shimura–Manin). *The maps*

$$r^+ : S_{2k} \to \mathbb{C}^k, \qquad f \mapsto (r_0(f), r_2(f), \ldots, r_{2k-2}(f)),$$

and

$$r^- : S_{2k} \to \mathbb{C}^{k-1}, \qquad f \mapsto (r_1(f), r_3(f), \ldots, r_{2k-3}(f)),$$

are both injective. In other words,

(1) *the even periods $r_0, r_2, \ldots, r_{2k-2}$ span the vector space S_{2k}^*;*
(2) *the odd periods $r_1, r_3, \ldots, r_{2k-3}$ also span S_{2k}^*.*

However, these periods are not linearly independent. A natural question was raised in [3]: which periods form a basis for S_{2k}^*? A satisfactory answer was obtained in the same paper [3]. To state it, we need the following notation and convention:

Definition 2.2. For an integer i such that $1 \le i \le d_k$, let

$$4i \pm 1 := \begin{cases} 4i+1 & \text{if } 2k \equiv 2 \pmod 4, \\ 4i-1 & \text{if } 2k \equiv 0 \pmod 4. \end{cases}$$

THEOREM 2.3 ([3]). *The set* $\{r_{4i\pm1} \mid i = 1, \ldots, d_k\}$ *is a basis for* S_{2k}^*.

Next we will display a basis for S_{2k}. For $f, g \in S_{2k}$, let (f, g) denote the Petersson scalar product. Then there is a cusp form R_n which is characterized by the formula

$$r_n(f) = (R_n, f) \quad \text{for any } f \in S_{2k}.$$

Passing to the dual space, we obtain a basis for S_{2k}.

THEOREM 2.4 ([3]). *The set* $\{R_{4i\pm1} \mid i = 1, \ldots, d_k\}$ *is a basis for* S_{2k}.

This theorem will be needed to prove Theorem 1.1. Finally some remarks on the Petersson scalar product are in order.

REMARK 2.5. Let f and g be modular forms in M_{2k}, at least one of them a cusp form. Then the Petersson scalar product (f, g) is defined by

$$(f, g) = \int_{\Gamma/\mathbb{H}} f(z)\overline{g(z)}y^{2k-2}\,dx\,dy$$

where $z = x+iy$. We note that the Petersson scalar product of an Eisenstein series and a cusp form is always zero (refer to [1, p. 183]).

However, there is a natural extension of the Petersson scalar product from the space of cusp forms to the space of all modular forms (Zagier [10, pp. 434–435]). This extended scalar product is always non-degenerate, and it is positive definite if and only if $2k \equiv 2 \pmod 4$.

The Petersson scalar products considered in this article are those extended ones in the above sense.

3. Proofs of Theorems 1.1 and 1.2. We need the following standard lemma:

LEMMA 3.1. *Let* V *be a* $\mathbb{C}$*-vector space of dimension* n *and*

$$B : V \times V \to \mathbb{C}$$

be a non-degenerate bilinear (*or sesquilinear*) *form. Let*

$$\{u_i \in V \mid i = 1, \ldots, n\} \quad \textit{and} \quad \{v_i \in V \mid i = 1, \ldots, n\}$$

be two sets of vectors in V. *Then the determinant* $|B(u_i, v_j)|_{i,j=1,\ldots,n}$ *is not zero if and only if both the above sets are sets of linearly independent vectors.*

The proof of this lemma is quite standard and we omit it.

Proof of Theorem 1.1. First we assume that $2k \equiv 0 \pmod 4$. We consider two sets of modular forms:

$$\{E_{2k}\} \cup \{E_{4i}E_{2k-4i} \mid i = 1, \ldots, d_k\} \quad \text{and} \quad \{E_{2k}\} \cup \{R_{4i-1} \mid i = 1, \ldots, d_k\}.$$

To verify that $E_{2k},\ E_{4i}E_{2k-4i}$ $(i=1,\dots,d_k)$ are linearly independent, by Lemma 3.1 it is sufficient to show that

(3.1)
$$\begin{vmatrix} (E_{2k},E_{2k}) & (R_{4-1},E_{2k}) & \cdots & (R_{4d_k-1},E_{2k}) \\ (E_{2k},E_4E_{2k-4}) & (R_{4-1},E_4E_{2k-4}) & \cdots & (R_{4d_k-1},E_4E_{2k-4}) \\ \cdots & \cdots & \cdots & \cdots \\ (E_{2k},E_{4d_k}E_{2k-4d_k}) & (R_{4-1},E_{4d_k}E_{2k-4d_k}) & \cdots & (R_{4d_k-1},E_{4d_k}E_{2k-4d_k}) \end{vmatrix} \neq 0.$$

Since $(E_{2k},E_{2k}) \neq 0$ and $(R_{4i-1},E_{2k}) = 0$ as mentioned in Remark 2.5, (3.1) is equivalent to

(3.2)
$$\begin{vmatrix} (R_{4-1},E_4E_{2k-4}) & (R_{8-1},E_4E_{2k-4}) & \cdots & (R_{4d_k-1},E_4E_{2k-4}) \\ (R_{4-1},E_8E_{2k-8}) & (R_{8-1},E_8E_{2k-8}) & \cdots & (R_{4d_k-1},E_8E_{2k-8}) \\ \cdots & \cdots & \cdots & \cdots \\ (R_{4-1},E_{4d_k}E_{2k-4d_k}) & (R_{8-1},E_{4d_k}E_{2k-4d_k}) & \cdots & (R_{4d_k-1},E_{4d_k}E_{2k-4d_k}) \end{vmatrix} \neq 0.$$

Now let $\{f_i \mid i=1,\dots,d_k\}$ be a basis for S_{2k} such that each f_i is a normalized Hecke eigenform. Then, since $\{R_{4i-1} \mid i=1,\dots,d_k\}$ is also a basis for S_{2k} by Theorem 2.4, we know that (3.2) is equivalent to

$$\begin{vmatrix} (f_1,E_4E_{2k-4}) & (f_2,E_4E_{2k-4}) & \cdots & (f_{d_k},E_4E_{2k-4}) \\ (f_1,E_8E_{2k-8}) & (f_2,E_8E_{2k-8}) & \cdots & (f_{d_k},E_8E_{2k-8}) \\ \cdots & \cdots & \cdots & \cdots \\ (f_1,E_{4d_k}E_{2k-4d_k}) & (f_2,E_{4d_k}E_{2k-4d_k}) & \cdots & (f_{d_k},E_{4d_k}E_{2k-4d_k}) \end{vmatrix} \neq 0. \tag{3.3}$$

To show (3.3), we use the following Rankin identity ([7]; also refer to Kohnen–Zagier [5] noting that their definition of $r_n(f)$ differs from ours by a factor of i^{n+1}): for a normalized eigenform f in S_{2k},

$$(f,E_{2n}E_{2k-2n}) = \frac{1}{(2i)^{2k-1}} r_{2k-2}(f) r_{2n-1}(f) \tag{3.4}$$

where $n=2,\dots,k-2$. From this identity, (3.3) is equivalent to

(3.5)
$$\frac{r_{2k-2}(f_1)r_{2k-2}(f_2)\cdots r_{2k-2}(f_{d_k})}{(2i)^{(2k-1)d_k}} \begin{vmatrix} r_{4-1}(f_1) & r_{4-1}(f_2) & \cdots & r_{4-1}(f_{d_k}) \\ r_{8-1}(f_1) & r_{8-1}(f_2) & \cdots & r_{8-1}(f_{d_k}) \\ \cdots & \cdots & \cdots & \cdots \\ r_{4d_k-1}(f_1) & r_{4d_k-1}(f_2) & \cdots & r_{4d_k-1}(f_{d_k}) \end{vmatrix} \neq 0.$$

Finally, (3.5) is equivalent to

$$\begin{vmatrix} (R_{4-1}, f_1) & (R_{4-1}, f_2) & \cdots & (R_{4-1}, f_{d_k}) \\ (R_{8-1}, f_1) & (R_{8-1}, f_2) & \cdots & (R_{8-1}, f_{d_k}) \\ \cdots & \cdots & \cdots & \cdots \\ (R_{4d_k-1}, f_1) & (R_{4d_k-1}, f_2) & \cdots & (R_{4d_k-1}, f_{d_k}) \end{vmatrix} \neq 0. \tag{3.6}$$

Now (3.6) holds, since both $\{f_i \mid i = 1, \ldots, d_k\}$ and $\{R_{4i-1} \mid i = 1, \ldots, d_k\}$ are bases for S_{2k}. This implies assertion (1) of Theorem 1.1.

A similar argument proves assertion (2). ∎

Proof of Theorem 1.2. In Theorem 1.1 we proved that the set

$$\{E_{2k}\} \cup \{E_{4i}E_{2k-4i} \mid i = 1, \ldots, d_k\}$$

is a basis for M_{2k} and, in particular, its members are linearly independent. Hence the elements of $\{E_{2k}\} \cup \{E_{4i}E_{2k-4i} + \frac{B_{4i}}{4i}\frac{B_{2k-4i}}{2k-4i}\frac{k}{B_{2k}}E_{2k} \mid i = 1, \ldots, d_k\}$ are linearly independent; in particular, $E_{4i}E_{2k-4i} + \frac{B_{4i}}{4i}\frac{B_{2k-4i}}{2k-4i}\frac{k}{B_{2k}}E_{2k}$, $i = 1, \ldots, d_k$, are linearly independent. Moreover, since the latter elements are all in S_{2k}, they form a basis for S_{2k}. This completes the proof. ∎

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Shinji Fukuhara
Department of Mathematics
Tsuda College
Tsuda-machi 2-1-1, Kodaira-shi
Tokyo 187-8577, Japan
E-mail: fukuhara@tsuda.ac.jp

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